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DISCREPANCY ESTIMATES FOR INDEX-TRANSFORMED UNIFORMLY DISTRIBUTED SEQUENCES

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Abstract: In this paper we show discrepancy bounds for index-transformed uniformly distributed sequences. From a general result we deduce very tight lower and upper bounds on the discrepancy of index-transformed van der Corput-, Halton-, and (t,s)-sequences indexed by the sum-of-digits function. We also analyze the discrepancy of sequences indexed by other functions, such as, e.g., $\lfloor n^\alpha \rfloor$ with $0<\alpha<1$.

Keywords: discrepancy, uniform distribution, van der Corput-sequence, Halton-sequence, (t,s)-sequence, sum-of-digits function.

1. Introduction

A sequence $(\boldsymbol{y}_n)_{n\geqslant 0}$ in the unit-cube $[0,1)^s$ is said to be *uniformly distributed modulo one* if for all intervals $[\boldsymbol{a},\boldsymbol{b})\subseteq[0,1)^s$ it is true that

$$\lim_{N\to\infty}\frac{\#\{n\,:\,0\leqslant n<N, \boldsymbol{y}_n\in[\boldsymbol{a},\boldsymbol{b})\}}{N}=\mathrm{vol}([\boldsymbol{a},\boldsymbol{b})). \tag{1}$$

A quantitative version of (1) can be stated in terms of discrepancy. For an infinite sequence $(\boldsymbol{y}_n)_{n\geqslant 0}$ in $[0,1)^s$ its *discrepancy* is defined as

$$D_N((\boldsymbol{y}_n)_{n\geqslant 0}):=\sup_{[\boldsymbol{a},\boldsymbol{b})\subseteq[0,1)^s}\left|\frac{\#\{n\,:\,0\leqslant n<N, \boldsymbol{y}_n\in[\boldsymbol{a},\boldsymbol{b})\}}{N}-\mathrm{vol}([\boldsymbol{a},\boldsymbol{b}))\right|,$$

where the supremum is extended over all sub-intervals $[\boldsymbol{a},\boldsymbol{b})$ of $[0,1)^s$. For a given finite sequence $X=(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_M)$ we write $D_M(X)$ for the discrepancy of X with

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2010 Mathematics Subject Classification: primary: 11K06; secondary: 11K31, 11K36, 11K38 the obvious adaptions in the above definition. An infinite sequence is uniformly distributed modulo one if and only if its discrepancy tends to zero as N goes to infinity. However, convergence of the discrepancy to zero cannot take place arbitrarily fast. It follows from a result of Roth [28] that for any infinite sequence $(\boldsymbol{y}_n)_{n\geqslant 0}$ in $[0,1)^s$ we have $ND_N((\boldsymbol{y}_n)_{n\geqslant 0}) \geqslant c_s(\log N)^{s/2}$ for infinitely many values of $N \in \mathbb{N}$ (by $\mathbb{N}$ we denote the set of positive integers, and we put $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). An improvement of this bound can be obtained from [4]. For the special case $s = 1$, Schmidt [29] (see also [2]) showed that for any infinite sequence $(y_n)_{n\geqslant 0}$ in $[0,1)$ we have $ND_N((y_n)_{n\geqslant 0}) \geqslant \frac{\log N}{66 \log 4}$ for infinitely many values of $N \in \mathbb{N}$. This result is best possible with respect to the order of magnitude in N. An excellent introduction to this topic can be found in the book of Kuipers and Niederreiter [20] (see also [6, 9, 21, 24]).

Well known examples of uniformly distributed sequences are $(n\boldsymbol{\alpha})$-sequences (also called Kronecker-sequences, see [9, 20]), van der Corput-sequences and their multivariate analogues called Halton-sequences (see [6, 19, 20, 24]), as well as (digital) (t,s)-sequences (see [6, 24]).

In recent years, also the distribution properties of index-transformed uniformly distributed sequences have been studied, especially for the examples mentioned above. In this paper, we mean by an index-transformed sequence of a sequence $(x_n)_{n\geqslant 0}$ a sequence $(x_{f(n)})_{n\geqslant 0}$, where $f : \mathbb{N}_0 \to \mathbb{N}_0$. Note that $(x_{f(n)})_{n\geqslant 0}$ is in general no subsequence of $(x_n)_{n\geqslant 0}$ since we do *not* require that f is strictly increasing.

For instance, the distribution properties of index-transformed Kronecker-sequences indexed by the sum-of-digits function were studied in [5, 8, 30, 31]. For this special case, very precise results can be found in [8]. In [7] the well-distribution of index-transformed Kronecker-sequences indexed by q-additive functions is considered. Furthermore, in [26] a discrepancy bound for van der Corput-sequences in bases of the form $b = 5^\ell$, $\ell \in \mathbb{N}$, indexed by Fibonacci numbers is shown. The papers [17, 18, 26] deal with index-transformed van der Corput-, Halton-, and (t,s)-sequences.

In this paper we are specifically interested in discrepancy bounds for sequences indexed by the q-ary sum-of-digits function and related functions and, furthermore, for sequences indexed by "moderately" monotonically increasing sequences, as for example $\lfloor n^\alpha \rfloor$ with $0 < \alpha < 1$. For an integer $q \geqslant 2$ and $n \in \mathbb{N}_0$ with base q expansion $n = r_0 + r_1 q + r_2 q^2 + \cdots$ the *q-ary sum-of-digits function* is defined by $s_q(n) := r_0 + r_1 + r_2 + \cdots$.

Previously, it has been shown in [18] that the sequence $(\boldsymbol{x}_{s_q(n)})_{n\geqslant 0}$, indexed by the q-ary sum-of-digits function, where $(\boldsymbol{x}_n)_{n\geqslant 0}$ denotes the Halton-sequence in co-prime bases $b_1, \ldots, b_s$ is uniformly distributed modulo one. The proof of this result is due to the fact that the sequence generated by the q-ary sum-of-digits function is uniformly distributed in $\mathbb{Z}$, see, for example, [12, 27]. In this paper we provide very tight lower and upper bounds on the discrepancy of index-transformed van der Corput-, Halton-, and (t,s)-sequences indexed by the sum-of-digits function.

This paper is structured as follows. In Section 2, we provide basic definitions and notation used throughout the subsequent sections. In Section 3, we prove a general theorem (Theorem 1) which will be of great importance in discussing sequences indexed by the sum-of-digits function. In Section 4 we present a concrete application of Theorem 1 which leads to the aforementioned tight bounds on the discrepancy of Halton- and (t, s)-sequences indexed by $s_q(n)$. Furthermore, we discuss a refinement of these results for van der Corput-sequences. Finally, in Section 5, we deal with discrepancy bounds for sequences which are obtained by certain moderately increasing index sequences, such as, e.g., $\lfloor n^\alpha \rfloor$ with $0 < \alpha < 1$.

2. Notation and basic definitions

We first outline the definitions of the sequences studied in this paper, namely van der Corput-, Halton-, and (t, s)-sequences.

Let $b \geqslant 2$ be an integer. A *van der Corput-sequence* $(x_n)_{n\geqslant 0}$ *in base* b is defined by $x_n = \varphi_b(n)$, where for $n \in \mathbb{N}_0$, with base b expansion $n = a_0 + a_1 b + a_2 b^2 + \cdots$, the so-called *radical inverse function* $\varphi_b : \mathbb{N}_0 \to [0, 1)$ is defined by

$$\varphi_b(n) := \frac{a_0}{b} + \frac{a_1}{b^2} + \frac{a_2}{b^3} + \cdots .$$

It is well known that for any base $b \geqslant 2$ the corresponding van der Corput-sequence is uniformly distributed modulo one and that $ND_N((x_n)_{n\geqslant 0}) = O(\log N)$, see, for example, [3, 6, 20].

If we choose co-prime integers $b_1, \dots, b_s \geqslant 2$, then s one-dimensional van der Corput-sequences can be combined to an s-dimensional uniformly distributed sequence with points $\boldsymbol{x}_n := (\varphi_{b_1}(n), \dots, \varphi_{b_s}(n))$ for $n \in \mathbb{N}_0$. This sequence is called a *Halton-sequence* and it is known that its discrepancy is of order $(\log N)^s/N$, see [1, 6, 10, 11, 13, 19, 22, 24]. Note that Halton-sequences are a direct generalization of van der Corput-sequences, so van der Corput-sequences can be viewed as one-dimensional Halton-sequences, and indeed Halton-sequences are sometimes also referred to as van der Corput-Halton-sequences (see, e.g., [20]). However, as there will be results in this paper which only hold for the one-dimensional case, it will be useful to explicitly distinguish van der Corput-sequences (which we use for the one-dimensional variant) from Halton-sequences (which we use for the multi-dimensional variant).

Another type of sequences we will be concerned with in this paper are (t, s)-sequences, for the definition of which we need the definition of elementary intervals and (t, m, s)-nets in base b.

For an integer $b \geqslant 2$, an *elementary interval* in base b is an interval of the form $\prod_{i=1}^{s} [a_i b^{-d_i}, (a_i + 1)b^{-d_i}) \subseteq [0, 1)^s$, where a_i, d_i are non-negative integers with $0 \leqslant a_i < b^{d_i}$ for $1 \leqslant i \leqslant s$.

Let t, m, with $0 \leqslant t \leqslant m$, be integers. Then a *$(t, m, s)$-net in base b* is a point set $(\boldsymbol{y}_n)_{n=0}^{b^m-1}$ in $[0, 1)^s$ such that any elementary interval in base b of volume b^{t-m} contains exactly b^t of the $\boldsymbol{y}_n$.

Furthermore, we call an infinite sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ a (t,s)-*sequence in base* b if the subsequence $(\boldsymbol{x}_n)_{n=kb^m}^{(k+1)b^m-1}$ is a (t,m,s)-net in base b for all integers $k\geqslant 0$ and $m\geqslant t$. It is known (see, e.g., [6, 23, 24]) that a (t,s)-sequence is particularly evenly distributed if the value of t is small. In particular, it can be shown that the discrepancy of a (t,s)-sequence in base b is of order $b^t(\log N)^s/N$, see, e.g., [6, 23, 24].

A very important sub-class of (t,s)-sequences is that of digital (t,s)-sequences, which are defined over algebraic structures like finite fields or rings. For the sake of simplicity, we restrict ourselves to digital sequences over finite fields $\mathbb{F}_p$ of prime order p. Again for the sake of simplicity we do not distinguish, here and later on, between elements in $\mathbb{F}_p$ and the set of integers $\{0,1,\dots,p-1\}$ (equipped with arithmetic operations modulo p).

For a vector $\boldsymbol{c}=(c_1,c_2,\dots)\in\mathbb{F}_p^{\infty}$ and for $m\in\mathbb{N}$ we denote the vector in $\mathbb{F}_p^m$ consisting of the first m components of $\boldsymbol{c}$ by $\boldsymbol{c}(m)$, i.e., $\boldsymbol{c}(m)=(c_1,\dots,c_m)$. Moreover, for an $\mathbb{N}\times\mathbb{N}$ matrix C over $\mathbb{F}_p$ and for $m\in\mathbb{N}$ we denote by $C(m)$ the left upper $m\times m$ submatrix of C.

For $s\in\mathbb{N}$ and $t\in\mathbb{N}_0$, choose $\mathbb{N}\times\mathbb{N}$ matrices $C_1,\dots,C_s$ over $\mathbb{F}_p$ with the following property. For every $m\in\mathbb{N}$, $m\geqslant t$, and all $d_1,\dots,d_s\in\mathbb{N}_0$ with $d_1+\dots+d_s=m-t$, the vectors

$$\boldsymbol{c}_1^{(1)}(m),\dots,\boldsymbol{c}_{d_1}^{(1)}(m),\dots,\boldsymbol{c}_1^{(s)}(m),\dots,\boldsymbol{c}_{d_s}^{(s)}(m)$$

are linearly independent in $\mathbb{F}_p^m$. Here $\boldsymbol{c}_i^{(j)}$ is the i-th row vector of the matrix C_j.

For $n\in\mathbb{N}_0$ let $n=n_0+n_1p+n_2p^2+\cdots$ be the base p representation of n. For every index $1\leqslant j\leqslant s$ multiply the digit vector $\boldsymbol{n}=(n_0,n_1,\dots)^\top$ by the matrix C_j,

$$C_j\cdot\boldsymbol{n}=:(x_{n,j}(1),x_{n,j}(2),\dots)^\top$$

(note that the matrix-vector multiplication is performed over $\mathbb{F}_p$), and set

$$x_n^{(j)}:=\frac{x_{n,j}(1)}{p}+\frac{x_{n,j}(2)}{p^2}+\cdots.$$

Finally set $\boldsymbol{x}_n:=(x_n^{(1)},\dots,x_n^{(s)})$. A sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ constructed in this way is called a *digital* (t,s)-*sequence over* $\mathbb{F}_p$. The matrices $C_1,\dots,C_s$ are called the *generator matrices* of the sequence.

To guarantee that the points $\boldsymbol{x}_n$ lie in $[0,1)^s$ (and not just in $[0,1]^s$) we assume that for each $1\leqslant j\leqslant s$ and $w\geqslant 0$ we have $c_{v,w}^{(j)}=0$ for all sufficiently large v, where $c_{v,w}^{(j)}$ are the entries of the matrix C_j (see [24, p.72, condition (S6)] for more information).

Throughout the paper we use the following notation. For functions $f,g:\mathbb{N}\to\mathbb{R}$, where $f\geqslant 0$, we write $g(n)=O(f(n))$ or $g(n)\ll f(n)$, if there exists a $C>0$ such that $|g(n)|\leqslant Cf(n)$ for all sufficiently large $n\in\mathbb{N}$. If we would like to stress that the quantity C may also depend on other variables than n, say $\alpha_1,\dots,\alpha_w$, which will be indicated by writing $\ll_{\alpha_1,\dots,\alpha_w}$.

3. A general theorem

In this section we present a general result for the discrepancy of sequences of the form $(\boldsymbol{x}_{g(n)})_{n\geqslant 0}$, for a particular class of functions $g:\mathbb{N}_0\to\mathbb{N}_0$. Here and in the following, a sequence $(a_k)_{k\in\mathbb{N}_0}$ is called *unimodal* if the sequence $(a_{k+1}-a_k)_{k\in\mathbb{N}_0}$ has exactly one change of sign.

Furthermore, we need the concept of the so-called *uniform discrepancy* of a sequence. The uniform discrepancy of a sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ in $[0,1)^s$ is defined as

$$\widetilde{D}_N((\boldsymbol{x}_n)_{n\geqslant 0}) := \sup_{k\in\mathbb{N}_0} D_N((\boldsymbol{x}_{n+k})_{n\geqslant 0}).$$

Theorem 1. *Let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be an s-dimensional sequence with uniform discrepancy $\widetilde{D}_N=\widetilde{D}_N((\boldsymbol{x}_n)_{n\geqslant 0})$, and let $f:\mathbb{N}_0\to\mathbb{R}$ be a non-decreasing function such that $N\widetilde{D}_N\leqslant f(N)$ for $N\in\mathbb{N}_0$.*

Let $g:\mathbb{N}_0\to\mathbb{N}_0$. Furthermore, let $(N_j)_{j\geqslant 0}$ be a strictly increasing sequence in $\mathbb{N}$ with $1=N_0$, and assume that $(N_j)_{j\geqslant 0}$ is a divisibility chain, i.e., $N_0|N_1$, $N_1|N_2$, $N_2|N_3$, etc. Define, for $k\in\mathbb{N}_0$,

$$G_{A,j}(k) := \#\{n \,:\, AN_j\leqslant n<(A+1)N_j, g(n)=k\}.$$

Then the following two assertions hold.

1. *For $N\in\mathbb{N}$ with $N_d\leqslant N<N_{d+1}$ we have*

$$ND_N((\boldsymbol{x}_{g(n)})_{n\geqslant 0}) \geqslant \max_{k\in\mathbb{N}_0} G_{0,d}(k).$$

2. *Assume that $G_{A,j}(k)$ is unimodal in k for all $j\in\mathbb{N}_0$ and all $A\in\mathbb{N}_0$, and put*

$$G_j := \max_{k,A\in\mathbb{N}_0} G_{A,j}(k) \qquad \text{for} \;\; j\in\mathbb{N}_0.$$

For $j\in\mathbb{N}_0$ and $A\in\mathbb{N}_0$ let

$$v_{A,j} := \#\{k\in\mathbb{N}_0 \,:\, g(n)=k \text{ for } AN_j\leqslant n<(A+1)N_j\}$$

and put

$$v_j := \max_{A\in\mathbb{N}_0} v_{A,j}.$$

Then for $N\in\mathbb{N}$ with $N_d\leqslant N<N_{d+1}$ we have

$$ND_N((\boldsymbol{x}_{g(n)})_{n\geqslant 0}) \leqslant \sum_{j=0}^{d}\frac{N_{j+1}}{N_j}G_j f(v_j).$$

Proof.

1. To show the lower bound choose a non-negative integer κ such that $\widetilde{G}_d = G_{0,d}(\kappa)=\max_{k\in\mathbb{N}_0}G_{0,d}(k)$. Then the number of $n\in\{0,\dots,N-1\}$ such that $\boldsymbol{x}_{g(n)}=\boldsymbol{x}_\kappa$ is at least $\widetilde{G}_d$ and hence, with an arbitrarily small interval containing $\boldsymbol{x}_\kappa$ we obtain

$$D_N((\boldsymbol{x}_{g(n)})_{n\geqslant 0}) \geqslant \frac{\widetilde{G}_d}{N}.$$

2. To prove the upper bound let
$$N = a_d N_d + a_{d-1} N_{d-1} + \cdots + a_0 N_0,$$
with $a_j \in \mathbb{N}_0$ and
$$a_j \leqslant \frac{N_{j+1}}{N_j}; \qquad \text{for } \ j \in \{0, \dots, d\}.$$
For $j \in \{0, \dots, d\}$ and $\ell \in \{0, \dots, a_j - 1\}$ we consider the sequence
$$X_{j,\ell} := (\boldsymbol{x}_{g(AN_j+k)})_{k=0}^{N_j-1}$$
where $AN_j := a_d N_d + \cdots + a_{j+1} N_{j+1} + \ell N_j$ (strictly speaking, $A = A(j, \ell)$). Since $G_{A,j}$ is unimodal we may assume that for $AN_j \leqslant n < (A+1)N_j$ the function $g(n)$ attains the values
$$w, w+1, \dots, w+v,$$
for some $w \in \mathbb{N}_0$ and some integer $v = v_{A,j} \leqslant v(j)$.
Assume that the value $w + u_1$ with $0 \leqslant u_1 \leqslant v$ is attained most often, the value $w + u_2$ with $0 \leqslant u_2 \leqslant v$ is attained second most often, etc. $\dots$, and $w + u_v$ with $0 \leqslant u_v \leqslant v$ (indeed, $u_v \in \{0, v\}$) is attained least often. If $w + u_r$ and $w + u_{r+1}$ are both attained the same number of times, then the order of them is of no relevance.
If we consider the sequence $X_{j,\ell}$ as a multi-set (i.e., multiplicity of the elements is relevant, but their order is not), then we can decompose $X_{j,\ell}$ into
$$\begin{array}{lll}
G_{A,j}(w+u_1) - G_{A,j}(w+u_2) & \text{times} & \{\boldsymbol{x}_{w+u_1}\} \\
G_{A,j}(w+u_2) - G_{A,j}(w+u_3) & \text{times} & \{\boldsymbol{x}_{w+u_1}, \boldsymbol{x}_{w+u_2}\} \\
G_{A,j}(w+u_3) - G_{A,j}(w+u_4) & \text{times} & \{\boldsymbol{x}_{w+u_1}, \boldsymbol{x}_{w+u_2}, \boldsymbol{x}_{w+u_3}\} \\
\qquad\qquad \dots & & \\
G_{A,j}(w+u_{v-1}) - G_{A,j}(w+u_v) & \text{times} & \{\boldsymbol{x}_{w+u_1}, \boldsymbol{x}_{w+u_2}, \dots, \boldsymbol{x}_{w+u_{v-1}}\} \\
G_{A,j}(w+u_v) - G_{A,j}(w+u_{v+1}) & \text{times} & \{\boldsymbol{x}_{w+u_1}, \boldsymbol{x}_{w+u_2}, \dots, \boldsymbol{x}_{w+u_v}\},
\end{array}$$
where we formally set $G_{A,j}(w + u_{v+1}) := 0$. Note that because of the unimodality of $G_{A,j}(k)$, for $r \in \{1, \dots, v\}$, the sequence $\boldsymbol{x}_{w+u_1}, \boldsymbol{x}_{w+u_2}, \dots, \boldsymbol{x}_{w+u_r}$ is a sequence of the form $\boldsymbol{x}_B, \dots, \boldsymbol{x}_{B+r-1}$ for some B.
Then, using the assumptions of the theorem and the triangle inequality for the discrepancy (see [20, p. 115, Theorem 2.6]), we obtain
$$\begin{aligned}
&N_j D_{N_j}(X_{j,\ell}) \\
&\leqslant \sum_{r=1}^{v} (G_{A,j}(w+u_r) - G_{A,j}(w+u_{r+1})) r D_r(\{\boldsymbol{x}_{w+u_1}, \boldsymbol{x}_{w+u_2}, \dots, \boldsymbol{x}_{w+u_r}\}) \\
&\leqslant G_{A,j}(w+u_1) f(v_{A,j}) \leqslant G_j f(v_j).
\end{aligned}$$
Using the triangle inequality for the discrepancy a second time, we finally obtain
$$N D_N((\boldsymbol{x}_{g(n)})_{n\geqslant 0}) \leqslant \sum_{j=0}^{d} a_j G_j f(v_j) \leqslant \sum_{j=0}^{d} \frac{N_{j+1}}{N_j} G_j f(v_j). \qquad \blacksquare$$

4. Indexing by the q-ary sum-of-digits function

We would now like to show results regarding index-transformed uniformly distributed sequences indexed by the q-ary sum-of-digits function. We first discuss an application of the general result in Theorem 1 (Section 4.1) to Halton- and (t,s)-sequences, and then show a refined result that applies to the particular case of van der Corput-sequences (Section 4.2).

4.1. Results for Halton- and (t,s)-sequences

Let $q \geqslant 2$ be an integer and $g(n) = s_q(n)$ the q-ary sum-of-digits function. For $j \in \mathbb{N}_0$ choose $N_j = q^j$. Then we have

$$G_{0,j}(k) = \#\{n : 0 \leqslant n < q^j, s_q(n) = k\}$$

and

$$(1 + x + x^2 + \cdots + x^{q-1})^j = \sum_{k \in \mathbb{N}_0} G_{0,j}(k) x^k,$$

by expanding the polynomial on the left hand side of the latter equation. Hence the sequence $(G_{0,j}(k))_{k\in\mathbb{N}_0}$ is the j-fold convolution of the sequence $(\underbrace{1,1,\ldots,1}_{q-\text{times}},0,0,\ldots)$, which implies by [25, Theorem 1] that $G_{0,j}(k)$ is unimodal for sufficiently large j. Since any $n \in \mathbb{N}_0$ with $Aq^j \leqslant n < (A+1)q^j$ can be written as $n = n' + Aq^j$, where $0 \leqslant n' < q^j$, it follows that $s_q(n) = s_q(n') + s_q(A)$ and hence $G_{A,j}(k) = G_{0,j}(k - s_q(A))$, where we set $G_{0,j}(k - s_q(A)) := 0$ if $k < s_q(A)$. Consequently, $G_{A,j}(k)$ is unimodal for any $A \in \mathbb{N}_0$ and for sufficiently large j.

We recall the following lemma from [8].

Lemma 1 (Drmota and Larcher, [8, Lemma 1]). *For integers $q \geqslant 2$, $j \geqslant 1$, and $0 \leqslant k \leqslant j(q-1)$ we have*

$$G_{0,j}(k) = \frac{q^j}{\sqrt{2\pi j}\sigma_q} \exp\left(-\frac{x_{j,k}^2}{2}\right)\left(1 + \frac{P_1(x_{j,k})}{\sqrt{j}} + \frac{P_2(x_{j,k})}{j}\right) + O\left(\frac{q^j}{j^2}\right),$$

where $P_1(x)$ and $P_2(x)$ are polynomials, $P_1(x)$ is odd, where $x_{j,k} := \frac{k - \frac{j(q-1)}{2}}{\sigma_q\sqrt{j}}$, and where $\sigma_q := \sqrt{\frac{q^2-1}{12}}$. The implied constant in the O-notation is uniform for all k and only depends on q.

Due to Lemma 1, there exists some $c_q > 0$ such that for sufficiently large j we have $G_{A,j}(k) \leqslant c_q q^j/\sqrt{j}$, uniformly in k and A. Thus we obtain

$$G_j \leqslant c_q \frac{q^j}{\sqrt{j}} \tag{2}$$

for sufficiently large j. On the other hand, for $\widetilde{k} = \lfloor j\frac{q-1}{2} \rfloor$ it follows that

$$\max_{k\in\mathbb{N}_0} G_{0,j}(k) \geqslant G_{0,j}(\widetilde{k}) \geqslant c_q' \frac{q^j}{\sqrt{j}}. \tag{3}$$

Furthermore it is clear that $v_0 = 1$ and $v_j \leqslant qj$ for all $j \in \mathbb{N}$. As an application of Theorem 1, we obtain the following result.

Theorem 2. *Let $X := (\boldsymbol{x}_n)_{n\geqslant 0}$ be an s-dimensional sequence such that $m\widetilde{D}_m((\boldsymbol{x}_n)_{n\geqslant 0}) \leqslant C(\log m)^s$ for all $m \in \mathbb{N}$, where C may depend on s or on the sequence X, but not on m. Let $q \geqslant 2$ be an integer. Then there exist $c_q^{(2)}, c_q^{(3)} > 0$, where $c_q^{(3)}$ may also depend on s and X, such that*

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leqslant D_N((\boldsymbol{x}_{s_q(n)})_{n\geqslant 0}) \leqslant c_q^{(3)} \frac{(\log\log N)^s}{\sqrt{\log N}}.$$

Proof. Assume that $q^d \leqslant N < q^{d+1}$. Then we obtain from Theorem 1 and Equation (3) that

$$D_N((\boldsymbol{x}_{s_q(n)})_{n\geqslant 0}) \geqslant \frac{c_q'}{N}\frac{q^d}{\sqrt{d}} \geqslant \frac{c_q^{(2)}}{\sqrt{\log N}}.$$

On the other hand, from Theorem 1 and Equation (2) ,

$$\begin{aligned} D_N((\boldsymbol{x}_{s_q(n)})_{n\geqslant 0}) &\leqslant \frac{1}{N}\sum_{j=1}^{d} qc_q \frac{q^j}{\sqrt{j}} C(\log(qj))^s \\ &\ll_q (\log d)^s \left(\frac{1}{N} \sum_{1\leqslant j<d/2} \frac{q^j}{\sqrt{j}} + \frac{1}{N} \sum_{d/2\leqslant j\leqslant d} \frac{q^j}{\sqrt{j}} \right) \\ &\ll_q (\log d)^s \left(\frac{\sqrt{\log N}}{\sqrt{N}} + \frac{1}{\sqrt{d}} \right) \ll_q \frac{(\log\log N)^s}{\sqrt{\log N}}, \end{aligned}$$

and the result follows. ■

The general lower bound in Theorem 2 is best possible with respect to the order of magnitude in N. This will follow from Theorem 3 below which deals with van der Corput-sequences.

There are several examples of sequences X which satisfy the conditions in Theorem 2 such as Halton- or (t, s)-sequences (for a proof of this fact, we refer to Section 6 of this paper). We thus obtain the following corollary.

Corollary 1. *Let $q \geqslant 2$ be an integer.*

1. *Let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be an s-dimensional Halton-sequence in pairwise co-prime bases $b_1, \ldots, b_s$. Then there exist $c_q^{(2)}, c_{q,s,b_1,\ldots,b_s}^{(4)} > 0$ such that*

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leqslant D_N((\boldsymbol{x}_{s_q(n)})_{n=0}^{N-1}) \leqslant c_{q,s,b_1,\ldots,b_s}^{(4)} \frac{(\log\log N)^s}{\sqrt{\log N}}.$$

2. *Let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be a (t,s)-sequence in base b. Then there exist $c_q^{(2)}, c_{q,b,s,t}^{(5)} > 0$ such that*

$$\frac{c_q^{(2)}}{\sqrt{\log N}} \leqslant D_N((\boldsymbol{x}_{s_q(n)})_{n\geqslant 0}) \leqslant c_{q,b,s,t}^{(5)} \frac{(\log\log N)^s}{\sqrt{\log N}}.$$

The result of the first part of Corollary 1 can be improved for the special instance of van der Corput-sequences, as we will show next.

4.2. The van der Corput-sequence indexed by the sum-of-digits function

The following results are based on a general discrepancy estimate which was first presented by Hellekalek [14]. The following definitions stem from [14, 15, 17]. We refer to these references for further information.

For an integer $b \geqslant 2$ let $\mathbb{Z}_b = \{z = \sum_{r=0}^{\infty} z_r b^r \,:\, z_r \in \{0,\dots,b-1\}\}$ be the set of b-adic numbers. $\mathbb{Z}_b$ forms an abelian group under addition. The set $\mathbb{N}_0$ is a subset of $\mathbb{Z}_b$. The *Monna map* $\phi_b : \mathbb{Z}_b \to [0,1)$ is defined by

$$\phi_b(z) := \sum_{r=0}^{\infty} \frac{z_r}{b^{r+1}}.$$

Note that the radical inverse function φ_b is nothing but ϕ_b restricted to $\mathbb{N}_0$. We also define the inverse $\phi_b^+ : [0,1) \to \mathbb{Z}_b$ by

$$\phi_b^+\left(\sum_{r=0}^{\infty} \frac{x_r}{b^{r+1}}\right) := \sum_{r=0}^{\infty} x_r b^r,$$

where we always use the finite b-adic representation for b-adic rationals in $[0,1)$.

For $k \in \mathbb{N}_0$ we can define characters $\chi_k : \mathbb{Z}_b \to \{c \in \mathbb{C} : |c| = 1\}$ of $\mathbb{Z}_b$ by

$$\chi_k(z) = \exp(2\pi \mathrm{i} \phi_b(k) z).$$

Finally, let $\gamma_k : [0,1) \to \{c \in \mathbb{C} : |c| = 1\}$ where $\gamma_k(x) = \chi_k(\phi_b^+(x))$.

For $b \geqslant 2$ we put $\rho_b(0) = 1$ and $\rho_b(k) = \frac{2}{b^{r+1}\sin(\pi\kappa_r/b)}$ for $k \in \mathbb{N}$ with base b expansion $k = \kappa_0 + \kappa_1 b + \cdots + \kappa_r b^r$, $\kappa_r \neq 0$.

We have the following general discrepancy bound which is based on the functions γ_k.

Lemma 2. *Let $g \in \mathbb{N}$. For any sequence $(y_n)_{n\geqslant 0}$ in $[0,1)$ we have*

$$D_N((y_n)_{n\geqslant 0}) \leqslant \frac{1}{b^g} + \sum_{k=1}^{b^g-1} \rho_b(k) \left| \frac{1}{N} \sum_{n=0}^{N-1} \gamma_k(y_n) \right|.$$

Proof. For the special case of a prime b, this result was shown by Hellekalek [14, Theorem 3.6]. Using [17, Lemma 2.10 and 2.11] it is easy to see that Hellekalek's result can be generalized to the one given in the lemma (cf. [16]). ■

We show a discrepancy bound for the van der Corput-sequence indexed by the q-ary sum-of-digits function for small values of q. This result improves on the first part of Corollary 1 for van der Corput-sequences. Moreover, it shows that the general lower bound from Theorem 2 is best possible in the order of magnitude in N.

Theorem 3. *Let $b, q \geqslant 2$ be integers with $q < 14$, let $(x_n)_{n\geqslant 0}$ be the van der Corput-sequence in base b and let $(s_q(n))_{n\geqslant 0}$ be the sequence of the q-adic sum-of-digits function. Then we have*

$$D_N((x_{s_q(n)})_{n\geqslant 0}) \ll_{b,q} \frac{1}{\sqrt{\log N}}.$$

Remark 1. In view of Theorem 2, the upper bound in Theorem 3 is best possible with respect to the order of magnitude in N.

Before we give the proof of Theorem 3, we need some preparations and auxiliary results. Writing $\mathrm{e}(x) := \exp(2\pi \mathrm{i} x)$ for short, we have

$$\frac{1}{N}\sum_{n=0}^{N-1} \gamma_k(x_{s_q(n)}) = \frac{1}{N}\sum_{n=0}^{N-1} \mathrm{e}\left(s_q(n)\phi_b(k)\right) =: T_k(N).$$

Lemma 3. *Let $b, q \geqslant 2$ be integers, let $k \in \mathbb{N}$ and let $(x_n)_{n\geqslant 0}$ be the van der Corput-sequence in base b. Then for any $m \in \mathbb{N}_0$ it is true that*

$$|T_k(q^m)| \leqslant \left(1 - \frac{16(q-1)}{q^2}\|\phi_b(k)\|^2\right)^{m/2},$$

where $\|x\|$ is the distance of a real x to the nearest integer.

Proof. First observe that

$$T_k(q^m) = \frac{1}{q^m}\sum_{n_0,\ldots,n_{m-1}=0}^{q-1} \mathrm{e}((n_0 + \ldots + n_{m-1})\phi_b(k)) = (T_k(q))^m.$$

We now proceed as in [27]. We use the identities $\exp(\mathtt{i}x) + \exp(-\mathtt{i}x) = 2\cos x$ and $\cos(2x) = 1 - 2\sin^2 x$ to obtain

$$\begin{aligned}
|T_k(q)|^2 &= \frac{1}{q^2} \sum_{n,n'=0}^{q-1} \mathrm{e}\left((n-n')\phi_b(k)\right) \\
&= \frac{1}{q^2} \left(q + \sum_{\substack{n,n'=0\\ n<n'}}^{q-1} \left(\mathrm{e}\left((n-n')\phi_b(k)\right) + \mathrm{e}\left(-(n-n')\phi_b(k)\right)\right) \right) \\
&= \frac{1}{q^2} \left(q + 2 \sum_{\substack{n,n'=0\\ n<n'}}^{q-1} \cos\left(2\pi(n-n')\phi_b(k)\right) \right) \\
&= \frac{1}{q^2} \left(q + 2 \sum_{\substack{n,n'=0\\ n<n'}}^{q-1} \left(1 - 2\sin^2\left(\pi(n-n')\phi_b(k)\right)\right) \right) \\
&= 1 - \frac{4}{q^2} \sum_{\substack{n,n'=0\\ n<n'}}^{q-1} \sin^2\left(\pi(n-n')\phi_b(k)\right) \leqslant 1 - \frac{4(q-1)}{q^2} \sin^2(\pi\phi_b(k)) \\
&\leqslant 1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2,
\end{aligned}$$

Therefore,

$$|T_k(q^m)| \leqslant \left(1 - \frac{16(q-1)}{q^2} \|\phi_b(k)\|^2\right)^{m/2}. \qquad \blacksquare$$

We also need the following lemma.

Lemma 4. *For $k \in \mathbb{N}$ and any $N \in \mathbb{N}$ with q-adic expansion $N = \sum_{r=0}^{R} a_r q^r$ we have*

$$|T_k(N)| \leqslant \frac{1}{N} \sum_{r=0}^{R} a_r q^r |T_k(q^r)|.$$

Proof. For $N = \sum_{r=0}^{R} a_r q^r$,

$$\{0, \ldots, N-1\} = \bigcup_{r=0}^{R} \{a_R q^R + \cdots + a_{r+1} q^{r+1}, \ldots, a_R q^R + \cdots + a_r q^r - 1\},$$

and hence

$$\begin{aligned} N|T_k(N)| &= \left|\sum_{n=0}^{N-1} \mathrm{e}\left(s_q(n)\phi_b(k)\right)\right| \\ &= \left|\sum_{r=0}^{R} \mathrm{e}\left((a_R+\cdots+a_{r+1})\phi_b(k)\right) \sum_{n=0}^{a_r q^r-1} \mathrm{e}\left(s_q(n)\phi_b(k)\right)\right| \\ &\leqslant \sum_{r=0}^{R}\left|\sum_{n=0}^{a_r q^r-1} \mathrm{e}\left(s_q(n)\phi_b(k)\right)\right| = \sum_{r=0}^{R}\left|\sum_{u=0}^{a_r-1} \mathrm{e}\left(u\phi_b(k)\right) \sum_{n=0}^{q^r-1} \mathrm{e}\left(s_q(n)\phi_b(k)\right)\right| \\ &\leqslant \sum_{r=0}^{R} a_r \left|\sum_{n=0}^{q^r-1} \mathrm{e}\left(s_q(n)\phi_b(k)\right)\right| = \sum_{r=0}^{R} a_r q^r |T_k(q^r)|. \qquad \blacksquare \end{aligned}$$

We are now ready to give the proof of Theorem 3.

Proof. For $k \in \{b^r, \ldots, b^{r+1}-1\}$ we have $\varphi_b(k) = \frac{A_k}{b^{r+1}}$ with $A_k \in \{1, \ldots, b^{r+1}-1\}$, where $A_{k_1} \neq A_{k_2}$ for $k_1 \neq k_2$. Hence we obtain from Lemma 3

$$\begin{aligned} \sum_{k=1}^{b^g-1} \rho_b(k)|T_k(q^m)| &\leqslant \sum_{r=0}^{g-1} \frac{2}{b^{r+1}\sin(\pi/b)} \sum_{k=b^r}^{b^{r+1}-1} \left(1-\frac{16(q-1)}{q^2}\left\|\frac{A_k}{b^{r+1}}\right\|^2\right)^{m/2} \\ &\leqslant \sum_{r=0}^{g-1} \frac{2}{b^{r+1}\sin(\pi/b)} \sum_{a=1}^{b^{r+1}-1} \left(1-\frac{16(q-1)}{q^2}\left\|\frac{a}{b^{r+1}}\right\|^2\right)^{m/2}. \end{aligned}$$

For the inner sum we have

$$\begin{aligned} &\sum_{a=1}^{b^{r+1}-1} \left(1-\frac{16(q-1)}{q^2}\left\|\frac{a}{b^{r+1}}\right\|^2\right)^{m/2} \\ &\quad= \sum_{1\leqslant a<b^{r+1}/2} \left(1-\frac{16(q-1)}{q^2}\frac{a^2}{b^{2r+2}}\right)^{m/2} \\ &\qquad+ \sum_{b^{r+1}/2\leqslant a<b^{r+1}} \left(1-\frac{16(q-1)}{q^2}\left(1-\frac{a}{b^{r+1}}\right)^2\right)^{m/2} \\ &\quad= \frac{1}{b^{m(r+1)}} \sum_{1\leqslant a<b^{r+1}/2} \left(b^{2r+2}-\frac{16(q-1)}{q^2}a^2\right)^{m/2} \\ &\qquad+ \frac{1}{b^{m(r+1)}} \sum_{b^{r+1}/2\leqslant a<b^{r+1}} \left(b^{2r+2}-\frac{16(q-1)}{q^2}(b^{r+1}-a)^2\right)^{m/2} \\ &\quad= \frac{2}{b^{m(r+1)}} \sum_{1\leqslant a<b^{r+1}/2} \left(b^{2r+2}-\frac{16(q-1)}{q^2}a^2\right)^{m/2} + \delta(b)\left(1-\frac{4(q-1)}{q^2}\right)^{m/2}, \end{aligned}$$

where $\delta(b) = 0$ when b is odd and $\delta(b) = 1$ when b is even.

The assumption $q < 14$ yields $\frac{16(q-1)}{q^2} \geqslant 1$, and hence

$$\begin{aligned}\sum_{a=1}^{b^{r+1}-1} &\left(1 - \frac{16(q-1)}{q^2}\left\|\frac{a}{b^{r+1}}\right\|^2\right)^{m/2} \\ &\leqslant \frac{2}{b^{m(r+1)}} \sum_{1\leqslant a<b^{r+1}/2} \left(b^{2r+2}-a^2\right)^{m/2} + \left(\frac{3}{4}\right)^{m/2} \\ &\leqslant \frac{2}{b^{m(r+1)}} \sum_{u=1}^{b^{2r+2}-1} u^{m/2} + \left(\frac{3}{4}\right)^{m/2} \\ &\leqslant \frac{2}{b^{m(r+1)}} \int_1^{b^{2r+2}} u^{m/2}\,\mathrm{d}u + \left(\frac{3}{4}\right)^{m/2} \\ &\ll_{b,q} \frac{b^{2r+2}}{m+1} + \left(\frac{3}{4}\right)^{m/2}\end{aligned}$$

with an implied constant depending only on b and q. Therefore

$$\sum_{k=1}^{b^g-1} \rho_b(k)|T_k(q^m)| \ll_{b,q} \sum_{r=0}^{g-1} \frac{1}{b^{r+1}}\left(\frac{b^{2(r+1)}}{m+1} + \left(\frac{3}{4}\right)^{m/2}\right) \ll_{b,q} \frac{b^g}{m+1}, \tag{4}$$

again with implied constants depending only on b and q.

Assume that $N = \sum_{r=0}^{R} a_r q^r$. Then, using Lemma 4 and (4), we obtain

$$\begin{aligned}\sum_{k=1}^{b^g-1} \rho_b(k)|T_k(N)| &\leqslant \frac{1}{N}\sum_{m=0}^{R} a_m q^m \sum_{k=1}^{b^g-1} \rho_b(k)|T_k(q^m)| \\ &\ll_{b,q} b^g \frac{1}{N}\sum_{m=0}^{R} a_m \frac{q^m}{m+1}.\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{N}\sum_{m=0}^{R} a_m \frac{q^m}{m+1} &\leqslant \frac{1}{N}\sum_{m=0}^{\lfloor R/2\rfloor} a_m q^m + \frac{1}{N}\sum_{m=\lfloor R/2\rfloor+1}^{R} a_m \frac{q^m}{m+1} \\ &\ll_q \frac{q^{R/2}}{N} + \frac{1}{R} \ll_q \frac{1}{\log N}\end{aligned}$$

we obtain

$$\sum_{k=1}^{b^g-1} \rho_b(k)|T_k(N)| \ll_{b,q} \frac{b^g}{\log N}.$$

From Lemma 2 it follows that

$$D_N((x_{s_q(n)})_{n\geqslant 0}) \ll_{b,q} \frac{1}{b^g} + \frac{b^g}{\log N}.$$

Choosing $g = \lfloor \log_b \sqrt{\log N} \rfloor$ yields

$$D_N((x_{s_q(n)})_{n\geqslant 0}) \ll_{b,q} \frac{1}{\sqrt{\log N}}. \qquad \blacksquare$$

Remark 2. We remark that, in principle, the method of proof based on Lemma 2 can not only be used for van der Corput-sequences, but also for Halton-sequences in higher dimensions. However, this leads to a discrepancy bound of order $(\log N)^{-\frac{1}{s+1}}$, which is considerably weaker than the one presented in Theorem 2.

5. Other index-transformations

In this section, we would now like to discuss index-transformed Halton- and digital (t,s)-sequences indexed by a different kind of sequence than the sum-of-digits function, as, e.g., $(\lfloor n^\alpha \rfloor)_{n\geqslant 0}$ with $0 < \alpha < 1$. The following theorem provides another general result, namely lower and upper bounds on the discrepancy of sequences indexed by functions which in some sense are “moderately“ monotonically increasing.

Theorem 4. *Let $A \in \mathbb{N}_0$ and write $\mathbb{N}_A := \{A, A+1, A+2, \ldots\}$. Let $f : \mathbb{N}_0 \to \mathbb{N}_A$ be surjective and monotonically increasing. Moreover, define, for $k \in \mathbb{N}_A$,*

$$F(k) := \#\{n \,:\, n \in \mathbb{N}_0, f(n) = k\}.$$

Under the assumption that $F(k)$ is monotonically increasing in k for sufficiently large k, the following three assertions hold.

1. *For an arbitrary sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ in $[0,1)^s$ it is true that*

$$\frac{F(f(N)-1)}{N} \leqslant D_N((\boldsymbol{x}_{f(n)})_{n\geqslant 0}).$$

2. *For a Halton-sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ in co-prime bases $b_1, \ldots, b_s$,*

$$D_N((\boldsymbol{x}_{f(n)})_{n\geqslant 0}) \leqslant C \frac{2F(f(N-1)+1)(\log N)^s}{N},$$

where C is a constant independent of N.

3. *For a digital (t,s)-sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ over $\mathbb{F}_p$ for prime p,*

$$D_N((\boldsymbol{x}_{f(n)})_{n\geqslant 0}) \leqslant \widetilde{C} p^t \frac{2F(f(N-1)+1)(\log N)^s}{N},$$

where $\widetilde{C}$ is a constant independent of N.

Proof.

1. Let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be an arbitrary sequence in $[0,1)^s$, and let f and F be as in the theorem. If $f(N)=A$, then, due to the properties of f, we obtain $F(f(N)-1)=0$, so the lower bound on the discrepancy is trivially fulfilled. If, on the other hand, $f(N)>A$, then it follows by the surjectivity of f that there exist $n\in\mathbb{N}_0$ such that $f(n)=f(N)-1$. Furthermore, whenever n is such that $f(n)=f(N)-1<f(N)$, it follows by the monotonicity of f that $n<N$. Hence, the value $f(N)-1$ occurs $F(f(N)-1)$ times among $f(0),\ldots,f(N-1)$, and the point $\boldsymbol{x}_{f(N)-1}$ is attained $F(f(N)-1)$ times in the sequence $\boldsymbol{x}_{f(0)},\ldots,\boldsymbol{x}_{f(N-1)}$. The lower bound follows by considering an arbitrarily small interval containing $\boldsymbol{x}_{f(N)-1}$.
2. Without loss of generality, assume $f(0)=0$, i.e., $A=0$.
   Furthermore, it is no loss of generality to assume that $f(1)=1$ and that $F(k)$ is monotonically increasing in k for $k\geqslant 0$. Indeed, if this is not the case, we can disregard a suitable number of initial elements $\boldsymbol{x}_{f(0)},\ldots,\boldsymbol{x}_{f(N_0)}$, without changing the discrepancy of the first N points of the sequence $(\boldsymbol{x}_{f(n)})_{n\geqslant 0}$ by more than $\frac{N_0}{N}$.
   Let $b_1,\ldots,b_s\geqslant 2$ be co-prime integers and let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be the corresponding Halton-sequence. For estimating the discrepancy, we consider an arbitrary interval
$$I:=\prod_{i=1}^{s}[0,\alpha^{(i)})\subseteq[0,1)^s,$$
   for some $\alpha^{(1)},\ldots,\alpha^{(s)}\in(0,1]$. For each $i\in\{1,\ldots,s\}$, choose m_i as the minimal integer such that $N\leqslant b_i^{m_i}$. Since $f(N-1)\leqslant N-1$, the i-th component $x_{f(n)}^{(i)}$ of a point $\boldsymbol{x}_{f(n)}$, $1\leqslant i\leqslant s$, $0\leqslant n\leqslant N-1$, has at most m_i non-zero digits in its base b_i representation. From this, it is easily derived that we can restrict ourselves to considering only $\alpha^{(i)}$ with at most m_i non-zero digits in their base b_i expansion, $1\leqslant i\leqslant s$, as this assumption changes $D_N((\boldsymbol{x}_{f(n)})_{n\geqslant 0})$ by a term of order of at most N^{-1}. We can therefore write I as the disjoint union of intervals
$$I(j_1,\ldots,j_s):=\prod_{i=1}^{s}\left[\sum_{r=1}^{j_i-1}\frac{\alpha_r^{(i)}}{b_i^r},\sum_{r=1}^{j_i}\frac{\alpha_r^{(i)}}{b_i^r}\right),$$
   where $1\leqslant j_i\leqslant m_i$ for $1\leqslant i\leqslant s$ and the $\alpha_r^{(i)}$ represent the base b_i digits of $\alpha^{(i)}$. Each of the $I(j_1,\ldots,j_s)$ can in turn be written as the disjoint union of intervals
$$\prod_{i=1}^{s}J(j_i,k_i):=\prod_{i=1}^{s}\left[\sum_{r=1}^{j_i-1}\frac{\alpha_r^{(i)}}{b_i^r}+\frac{k_i}{b_i^{j_i}},\sum_{r=1}^{j_i-1}\frac{\alpha_r^{(i)}}{b_i^r}+\frac{k_i+1}{b_i^{j_i}}\right),$$
   with $1\leqslant j_i\leqslant m_i$ and $0\leqslant k_i\leqslant\alpha_{j_i}^{(i)}-1$. If $\alpha_{j_i}^{(i)}=0$, then $J(j_i,k_i)$ is of zero volume containing no points. Hence we can restrict ourselves to considering only those $J(j_i,k_i)$ with $\alpha_{j_i}^{(i)}\geqslant 1$.

Let now $i \in \{1, \ldots, s\}$ and $v \geqslant 0$ be fixed. By the construction principle of the points of the Halton-sequence, we see that $x_v^{(i)}$ is contained in $J(j_i, k_i)$ if and only if

$$\begin{pmatrix} v_0^{(i)} \\ \vdots \\ v_{j_i-2}^{(i)} \\ v_{j_i-1}^{(i)} \end{pmatrix} = \begin{pmatrix} \alpha_i^{(1)} \\ \vdots \\ \alpha_i^{(j_i-1)} \\ k_i \end{pmatrix}, \tag{5}$$

where the $v_r^{(i)}$, $0 \leqslant r \leqslant j_i - 1$ are the digits of v in base b_i. Note that (5) has exactly one solution $(v_0^{(i)}, \ldots, v_{j_i-1}^{(i)})$ modulo b_i. Hence we can identify exactly one remainder $R^{(i)}$ modulo $b_i^{j_i}$, such that $x_v^{(i)} \in J(j_i, k_i)$ if and only if $v \equiv R^{(i)} \pmod{b_i^{j_i}}$. By the Chinese Remainder Theorem, there exists exactly one remainder R modulo $Q := \prod_{i=1}^s b_i^{j_i}$ such that

$$\boldsymbol{x}_v \in \prod_{i=1}^s J(j_i, k_i) \qquad \text{if and only if} \qquad v \equiv R \pmod{Q}.$$

We now deduce an estimate for the number of points among $\boldsymbol{x}_{f(0)}, \ldots, \boldsymbol{x}_{f(N-1)}$ that are contained in an interval of the type $\prod_{i=1}^s J(j_i, k_i)$. For short, we denote this number by $A\left(\prod_{i=1}^s J(j_i, k_i)\right)$.

Note that there exists a number $\theta = \theta(R, Q, f(N-1)) \in \{0, 1\}$ such that $0 = f(0) \leqslant R + wQ \leqslant f(N-1)$ if and only if $w \in \{0, \ldots, \lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta\}$, so

$$A\left(\prod_{i=1}^s J(j_i, k_i)\right) \geqslant \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(R + wQ) \geqslant \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ), \tag{6}$$

where we used the monotonicity of F. On the other hand, with the same argument,

$$A\left(\prod_{i=1}^s J(j_i, k_i)\right) \leqslant \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 1 + \theta} F(R + wQ) \leqslant \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor + \theta} F(wQ). \tag{7}$$

For the following, let $K = \left\lfloor \frac{f(N-1)}{Q} \right\rfloor + \theta$. Let

$$\Sigma_A := \sum_{r=0}^{(K-1)Q-1} F(r),$$

and note that we can write

$$\Sigma_A = \sum_{w=0}^{K-2} \sum_{r=0}^{Q-1} F(wQ + r) \geqslant Q \sum_{w=0}^{K-2} F(wQ) = Q \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor - 2 + \theta} F(wQ).$$

On the other hand, by the definition of θ,

$$\Sigma_A = \sum_{r=0}^{(\lfloor \frac{f(N-1)}{Q} \rfloor -1+\theta)Q-1} F(r) \leqslant \sum_{r=0}^{f(N-1)-1} F(r) \leqslant N-1,$$

from which we conclude that

$$\sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor -2+\theta} F(wQ) \leqslant \frac{N-1}{Q}. \tag{8}$$

Moreover, let

$$\Sigma_B := \sum_{r=1}^{KQ} F(r),$$

for which we can derive, in the same way as the corresponding estimate for Σ_A,

$$\Sigma_B \leqslant Q \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor +\theta} F(wQ).$$

Again by the definition of θ,

$$\begin{aligned}\Sigma_B = \sum_{r=1}^{(\lfloor \frac{f(N-1)}{Q} \rfloor +\theta)Q} F(r) &\geqslant \sum_{r=1}^{f(N-1)} F(r) \\ &= \#\{n \in \mathbb{N}_0 : 0 < f(n) \leqslant f(N-1)\} \geqslant N-1,\end{aligned}$$

where we used that $f(1) = 1$ and that f is monotonically increasing. Consequently,

$$\sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor +\theta} F(wQ) \geqslant \frac{N-1}{Q}. \tag{9}$$

Note, furthermore, that

$$\begin{aligned}0 \leqslant \sum_{w=1}^{\lfloor \frac{f(N-1)}{Q} \rfloor +\theta} F(wQ) - \sum_{w=0}^{\lfloor \frac{f(N-1)}{Q} \rfloor -2+\theta} F(wQ)& \\ \leqslant F\left(\left(\left\lfloor \frac{f(N-1)}{Q} \right\rfloor -1+\theta\right) Q\right)& \\ + F\left(\left(\left\lfloor \frac{f(N-1)}{Q} \right\rfloor +\theta\right) Q\right)& \\ \leqslant 2F(f(N-1)+1).& \end{aligned} \tag{10}$$

Combining Equations (6), (9), and (10), and noting that $\lambda\left(\prod_{i=1}^{s} J(j_i,k_i)\right) = \frac{1}{Q}$, gives

$$\begin{aligned}\frac{1}{N}A\left(\prod_{i=1}^{s} J(j_i,k_i)\right) - \frac{1}{Q} &\geqslant \frac{1}{N}\sum_{w=0}^{\lfloor \frac{f(N-1)}{Q}\rfloor -2+\theta} F(wQ) - \frac{1}{Q}\\ &\geqslant \frac{\sum_{w=1}^{\lfloor \frac{f(N-1)}{Q}\rfloor +\theta} F(wQ) - 2F(f(N-1)+1)}{N} - \frac{1}{Q}\\ &\geqslant \frac{-2F(f(N-1)+1)}{N} + \frac{N-1}{QN} - \frac{1}{Q}\\ &\geqslant \frac{-2F(f(N-1)+1)}{N} - \frac{1}{NQ}.\end{aligned}$$

In exactly the same way, using (7), (8), and (10), we get

$$\frac{1}{N}A\left(\prod_{i=1}^{s} J(j_i,k_i)\right) - \frac{1}{Q} \leqslant \frac{2F(f(N-1)+1)}{N} + \frac{1}{NQ},$$

from which we derive

$$\left|\frac{1}{N}A\left(\prod_{i=1}^{s} J(j_i,k_i)\right) - \frac{1}{Q}\right| \leqslant \frac{2F(f(N-1)+1)}{N} + \frac{1}{NQ}.$$

Finally, note that, by writing $A(I)$ for the number of points of $(\boldsymbol{x}_{f(n)})_{n=0}^{N-1}$ in I,

$$\begin{aligned}&\left|\frac{A(I)}{N} - \lambda(I)\right|\\ &\leqslant \sum_{j_1=1}^{m_1}\cdots\sum_{j_s=1}^{m_s}\sum_{k_1=0}^{\alpha_{j_1}^{(1)}-1}\cdots\sum_{k_s=0}^{\alpha_{j_s}^{(s)}-1}\left|\frac{1}{N}A\left(\prod_{i=1}^{s} J(j_i,k_i)\right) - \lambda\left(\prod_{i=1}^{s} J(j_i,k_i)\right)\right|\\ &\leqslant C\frac{(\log N)^s F(f(N-1)+1)}{N},\end{aligned}$$

for a suitably chosen constant C, and the result follows.

3. As in Item 2, assume without loss of generality that $f(0)=0$, $f(1)=1$, and that $F(k)$ is monotonically increasing in k for $k \geqslant 1$.
   Let p be a prime and let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be a digital (t,s)-sequence over $\mathbb{F}_p$. For estimating the discrepancy, we consider an arbitrary interval

$$I := \prod_{i=1}^{s}[0,\alpha^{(i)}) \subseteq [0,1)^s,$$

   for some $\alpha^{(1)},\ldots,\alpha^{(s)} \in (0,1]$. Choose m as the minimal integer such that $N \leqslant p^m$. By a similar argument as for the case of Halton sequences, we can restrict ourselves to considering only $\alpha^{(i)}$ with at most m non-zero digits $\alpha_1^{(i)}, \ldots, \alpha_m^{(i)}$ in their base p expansion. Moreover, with the same reasoning as in the Halton case, we see that we essentially only need to deal with intervals of the form

$$\prod_{i=1}^{s} J(j_i, k_i) := \prod_{i=1}^{s} \left[\sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{p^r} + \frac{k_i}{p^{j_i}}, \sum_{r=1}^{j_i-1} \frac{\alpha_r^{(i)}}{p^r} + \frac{k_i+1}{p^{j_i}} \right),$$

with $1 \leqslant j_i \leqslant m$ and $0 \leqslant k_i \leqslant \alpha_{j_i}^{(i)} - 1$. Again, if $\alpha_{j_i}^{(i)} = 0$, then $J(j_i, k_i)$ is of zero volume containing no points, so we can restrict ourselves to considering only those $J(j_i, k_i)$ with $\alpha_{j_i}^{(i)} \geqslant 1$.

As for the case of Halton sequences, we would like to derive an upper and a lower bound on the number $A\left(\prod_{i=1}^{s} J(j_i, k_i)\right)$ of points contained in $\prod_{i=1}^{s} J(j_i, k_i)$. To this end, denote the r-th row of a generator matrix C_j, $1 \leqslant j \leqslant s$ of $(\boldsymbol{x}_n)_{n \geqslant 0}$ by $\boldsymbol{c}_r^{(j)}$.

For an integer $v \geqslant 0$, the point $\boldsymbol{x}_v$ is contained in $\prod_{i=1}^{s} J(j_i, k_i)$ if and only if

$$\mathcal{C} \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \end{pmatrix} = A^\top, \tag{11}$$

where $v_0, v_1, v_2, \ldots$ are the base p digits of v, where

$$A := (\alpha_1^{(1)}, \ldots, \alpha_{j_1-1}^{(1)}, k_1, \alpha_1^{(2)}, \ldots, \alpha_{j_2-1}^{(2)}, k_2, \ldots\ldots, \alpha_1^{(s)}, \ldots, \alpha_{j_s-1}^{(s)}, k_s) \\ \in \mathbb{F}_p^{j_1+\cdots+j_s},$$

and

$$\mathcal{C} := \left(\boldsymbol{c}_1^{(1)}, \ldots, \boldsymbol{c}_{j_1}^{(1)}, \boldsymbol{c}_1^{(2)}, \ldots, \boldsymbol{c}_{j_2}^{(2)}, \ldots\ldots, \boldsymbol{c}_1^{(s)}, \ldots, \boldsymbol{c}_{j_s}^{(s)}\right)^\top \in \mathbb{F}_p^{(j_1+\cdots+j_s)\times\mathbb{N}}.$$

Let now $Q := p^{j_1+\cdots+j_s+t}$, let $w \in \mathbb{N}_0$ and consider those $v \geqslant 0$ with $wQ \leqslant v \leqslant (w+1)Q - 1$. For these v, the first $j_1 + j_2 + \cdots + j_s + t$ digits in their base p expansion vary, while all the other digits are fixed. Hence we can write (11) as

$$D_1 \cdot \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{j_1+\cdots j_s+t} \end{pmatrix} + D_2 \cdot \begin{pmatrix} v_{j_1+\cdots+j_s+t+1} \\ v_{j_1+\cdots+j_s+t+2} \\ \vdots \end{pmatrix} = A^\top,$$

where $\mathcal{C} = (D_1|D_2)$ and where D_1 is an $(j_1+\cdots+j_s)\times(j_1+\cdots+j_s+t)$-matrix and D_2 is an $(j_1 + \cdots + j_s) \times \mathbb{N}$-matrix over $\mathbb{F}_p$.

Due to the fact that $(\boldsymbol{x}_n)_{n\geqslant 0}$ is a digital (t,s)-sequence, it follows that D_1 has full rank, and hence there are exactly p^t values v in $\{wQ, wQ+1,\ldots,(w+1)Q-1\}$ such that $\boldsymbol{x}_v$ is contained in $\prod_{i=1}^s J(j_i,k_i)$.
Now note again that there exists a number $\theta=\theta(Q,f(N-1))\in\{0,1\}$ such that $0=f(0)\leqslant wQ\leqslant f(N-1)$ if and only if $w\in\{0,\ldots,\lfloor\frac{f(N-1)}{Q}\rfloor-1+\theta\}$. By our observations above, for each of these $w\in\{0,\ldots,\lfloor\frac{f(N-1)}{Q}\rfloor-1+\theta\}$ there exist p^t integers $R_{w,1},\ldots,R_{w,p^t}\in\{0,\ldots,Q-1\}$ such that exactly the points $\boldsymbol{x}_{R_{w,1}+wQ},\ldots,\boldsymbol{x}_{R_{w,p^t}+wQ}$ among $\boldsymbol{x}_{wQ},\boldsymbol{x}_{wQ+1},\ldots,\boldsymbol{x}_{(w+1)Q-1}$ are contained in $\prod_{i=1}^s J(j_i,k_i)$. Therefore, we can estimate

$$A\left(\prod_{i=1}^s J(j_i,k_i)\right)\geqslant\sum_{w=0}^{\lfloor\frac{f(N-1)}{Q}\rfloor-2+\theta}\sum_{z=1}^{p^t}F(R_{w,z}+wQ)\geqslant p^t\sum_{w=0}^{\lfloor\frac{f(N-1)}{Q}\rfloor-2+\theta}F(wQ),\tag{12}$$

and

$$A\left(\prod_{i=1}^s J(j_i,k_i)\right)\leqslant\sum_{w=0}^{\lfloor\frac{f(N-1)}{Q}\rfloor-1+\theta}\sum_{z=1}^{p^t}F(R_{w,z}+wQ)\leqslant p^t\sum_{w=1}^{\lfloor\frac{f(N-1)}{Q}\rfloor+\theta}F(wQ).\tag{13}$$

In exactly the same way as for a Halton sequence, we obtain, by noting that $\lambda\left(\prod_{i=1}^s J(j_i,k_i)\right)=\frac{1}{p^{l_1+\cdots+l_s}}=\frac{p^t}{Q}$,

$$\left|\frac{1}{N}A\left(\prod_{i=1}^s J(j_i,k_i)\right)-\frac{1}{Q}\right|\leqslant\frac{p^t 2F(f(N-1)+1)}{N}+\frac{p^t}{NQ},$$

and the result follows. ∎

Examples of functions f and F satisfying the assumptions of Theorem 4 are obtained as follows. Let $g:\mathbb{R}_0^+\to\mathbb{R}_0^+$ be a function that is twice differentiable on $(0,\infty)$, with $g'(x)>0$ and $g''(x)<0$ for $x\in(0,\infty)$. Moreover, define $f(n):=\lfloor g(n)\rfloor$ for $n\in\mathbb{N}$. It then easily follows that f and F indeed fulfill the assumptions of the theorem and we obtain

$$F(k+1)=\left\lceil g^{-1}(k+1)\right\rceil-\left\lceil g^{-1}(k)\right\rceil.\tag{14}$$

We thus obtain the following exemplary corollary to Theorem 4.

Corollary 2. *Let $\alpha\in(0,1)$. Then the following assertions hold.*

1. *For a Halton-sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ in co-prime bases $b_1,\ldots,b_s$,*

$$\overline{C}_1\frac{1}{N^\alpha}\leqslant D_N((\boldsymbol{x}_{\lfloor n^\alpha\rfloor})_{n\geqslant 0})\leqslant\overline{C}_2\frac{(\log N)^s}{N^\alpha},$$

where $\overline{C}_1$, $\overline{C}_2$ are constants that depend on the sequence and on α, but are independent of N.

2. *For a digital (t,s)-sequence $(\boldsymbol{x}_n)_{n\geqslant 0}$ over $\mathbb{Z}_p$ for prime p,*

$$\overline{\overline{C}}_1 \frac{1}{N^\alpha} \leqslant D_N((\boldsymbol{x}_{\lfloor n^\alpha \rfloor})_{n\geqslant 0}) \leqslant \overline{\overline{C}}_2 \frac{(\log N)^s}{N^\alpha},$$

where $\overline{\overline{C}}_1$, $\overline{\overline{C}}_2$ are constants that depend on the sequence and on α, but are independent of N.

Proof. The result follows by combining Theorem 2 with the observation that

$$c'_\alpha k^{\frac{1}{\alpha}-1} \leqslant F(k) \leqslant c_\alpha k^{\frac{1}{\alpha}-1},$$

with constants $c'_\alpha, c_\alpha > 0$ that depend on α, but not on k. ■

6. Appendix: Uniform discrepancy

In Corollary 1 we implicitly used the fact that (t,s)-sequences in base b as well as Halton-sequences in pairwise co-prime bases $b_1,\ldots,b_s$ have uniform discrepancy of order $(\log N)^s/N$. Since we are not aware of a proof of these facts in the existing literature, we provide one here.

6.1. Uniform discrepancy of (t,s)-sequences in base b

Assume that $\Delta_b(t,m,s)$ is a number for which

$$b^m D_{b^m}(\mathscr{P}) \leqslant \Delta_b(t,m,s)$$

holds for the discrepancy of any (t,m,s)-net $\mathscr{P}$ in base b.

Theorem 5. *Let $(\boldsymbol{x}_n)_{n\geqslant 0}$ be a (t,s)-sequence in base b. Then we have*

$$N\widetilde{D}_N((\boldsymbol{x}_n)_{n\geqslant 0}) \leqslant (2b-1)\left(tb^t + \sum_{m=t}^{\lfloor \log_b N \rfloor} \Delta_b(t,m,s)\right).$$

Proof. Let $k \in \mathbb{N}_0$. We show that

$$ND_N((\boldsymbol{x}_{n+k})_{n\geqslant 0}) \leqslant (2b-1)\left(tb^t + \sum_{m=t}^{\lfloor \log_b N \rfloor} \Delta_b(t,m,s)\right)$$

uniformly in $k \in \mathbb{N}_0$.

For $N < b^t$, the assertion follows trivially by $ND_N((\boldsymbol{x}_{n+k})_{n\geqslant 0}) \leqslant N$.

Let now $N \in \mathbb{N}$, $N \geqslant b^t$ with b-adic expansion $N = a_r b^r + a_{r-1}b^{r-1} + \cdots + a_1 b + a_0$ where $a_j \in \{0,\ldots,b-1\}$ for $0 \leqslant j \leqslant r$ and $a_r \neq 0$ (note that $r \geqslant t$). For given $k \in \mathbb{N}_0$, choose $\ell \in \mathbb{N}$ such that $(\ell-1)b^r \leqslant k < \ell b^r$. Then we can write

$$k = \ell b^r - (d_{r-1}b^{r-1} + \cdots + d_1 b + d_0) - 1$$

with some $d_j \in \{0, \dots, b-1\}$ for $0 \leqslant j \leqslant r-1$, and

$$k = (\ell - 1)b^r + \kappa_{r-1}b^{r-1} + \cdots + \kappa_1 b + \kappa_0$$

with some $\kappa_j \in \{0, \dots, b-1\}$ for $0 \leqslant j \leqslant r-1$. Note that therefore $d_j + \kappa_j = (b-1)$ for $0 \leqslant j < r$.

We split up the point set $\mathscr{P}_{k,N} := \{\boldsymbol{x}_n : k \leqslant n \leqslant k+N-1\}$ in the following way:

$$\mathscr{P}_{k,N} = \bigcup_{1 \leqslant d \leqslant d_0+1} \mathscr{P}'_{0,d} \cup \bigcup_{\substack{1 \leqslant m \leqslant t-1 \\ 1 \leqslant d \leqslant d_m}} \mathscr{P}'_{m,d} \cup \bigcup_{\substack{t \leqslant m \leqslant r-1 \\ 1 \leqslant d \leqslant d_m}} \mathscr{P}'_{m,d}$$
$$\cup \bigcup_{0 \leqslant a \leqslant a_r - 2} \mathscr{P}''_a \cup \bigcup_{\substack{0 \leqslant m \leqslant t-1 \\ 0 \leqslant x \leqslant a_m + \kappa_m - 1}} \mathscr{P}'''_{m,x} \cup \bigcup_{\substack{t \leqslant m \leqslant r-1 \\ 0 \leqslant x \leqslant a_m + \kappa_m - 1}} \mathscr{P}'''_{m,x},$$

where

$$\mathscr{P}'_{m,d} := \{\boldsymbol{x}_{\ell b^r - d_{r-1}b^{r-1} - \cdots - d_{m+1}b^{m+1} - d b^m + j} : 0 \leqslant j < b^m\},$$
$$\mathscr{P}''_a := \{\boldsymbol{x}_{\ell b^r + a b^r + j} : 0 \leqslant j < b^r\},$$
$$\mathscr{P}'''_{m,x} := \{\boldsymbol{x}_{(\ell + a_r - 1)b^r + (\kappa_{r-1} + a_{r-1})b^{r-1} + \cdots + (\kappa_{m+1} + a_{m+1})b^{m+1} + x b^m + j} : 0 \leqslant j < b^m\}.$$

For $m \leqslant t-1$, we can bound the discrepancy of $\mathscr{P}'_{m,d}$ and $\mathscr{P}'''_{m,x}$, respectively, by the trivial bound 1. For $m \geqslant t$, the point sets $\mathscr{P}'_{m,d}$ and $\mathscr{P}'''_{m,x}$ are (t, m, s)-nets in base b, and the $\mathscr{P}''_a$ are (t, r, s)-nets in base b. From the triangle inequality for the discrepancy we obtain

$$\begin{aligned} N D_N(\mathscr{P}_{k,N}) &\leqslant (d_0 + a_0 + \kappa_0 + 1)b^0 + \sum_{m=1}^{t-1} (d_m + a_m + \kappa_m) b^m \\ &\quad + \sum_{m=t}^{r-1} (d_m + a_m + \kappa_m)\Delta_b(t, m, s) + \max(a_r - 2, 0)\Delta_b(t, r, s) \\ &\leqslant (2b-1) + (2b-2)\left((t-1)b^t + \sum_{m=t}^{r-1} \Delta_b(t, m, s)\right) \\ &\quad + \max(b-3, 0)\Delta_b(t, r, s) \\ &\leqslant (2b-1)\left(t b^t + \sum_{m=t}^{r} \Delta_b(t, m, s)\right) \end{aligned}$$

and the result follows, since $r = \lfloor \log_b N \rfloor$. ■

Corollary 3. *Let $(\boldsymbol{x}_n)_{n \geqslant 0}$ be a (t, s)-sequence in base b. Then we have*

$$N \widetilde{D}_N((\boldsymbol{x}_n)_{n \geqslant 0}) \ll_{s,b} b^t (\log N)^s.$$

Proof. The result follows from Theorem 5 together with the fact that

$$\Delta_b(t, m, s) \ll_{s,b} b^t m^{s-1}$$

for $m \geqslant t$ (see, for example, [6, 24]). ■

6.2. Uniform discrepancy of Halton-sequences

Theorem 6. *Let* $(\boldsymbol{x})_{n\geqslant 0}$ *be a Halton-sequence in pairwise co-prime bases* $b_1,\dots,b_s$. *Then we have*

$$N\widetilde{D}_N((\boldsymbol{x}_n)_{n\geqslant 0}) = \frac{1}{s!}\prod_{j=1}^{s}\left(\frac{\lfloor b_j/2\rfloor \log N}{\log b_j} + s\right) + O((\log N)^{s-1}),$$

where the implied constant depends on $b_1,\dots,b_s$ *and* s.

Proof. The result follows from an adaption of the proof of [6, Theorem 3.36]. Note that [6, Lemma 3.37] also holds true for $A(J,k,N,\mathcal{S}) := \#\{n\in\mathbb{N} : k\leqslant n < k+N \text{ and } \boldsymbol{x}_n\in J\}$ instead of $A(J,N,\mathcal{S}) := A(J,0,N,\mathcal{S})$. The rest of the proof of [6, Theorem 3.36] remains unchanged. ∎

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