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SELF-APPROXIMATION OF HURWITZ ZETA-FUNCTIONS

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Abstract: We are looking for real numbers α and d for which there exist "many" real numbers τ such that the shifts of the Hurwitz-zeta function $\zeta(s+i\tau,\alpha)$ and $\zeta(s+id\tau,\alpha)$ are 'near' each other.

Keywords: Hurwitz zeta-function, strong recurrence, universality theorem.

1. Introduction

Let $s=\sigma+it$ denote a complex variable. For $\sigma>1$, the Hurwitz zeta-function is given by

$$\zeta(s,\alpha)=\sum_{n=0}^{\infty}\frac{1}{(n+\alpha)^s},$$

where α is a parameter from the interval (0,1]. The Hurwitz zeta-function can be continued analytically to the entire complex plane except for a simple pole at $s=1$. For $\alpha=1$ we get $\zeta(s,1)=\zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function.

In this paper we consider the following problem. Find all real numbers $0<\alpha\leqslant 1$ and d such that, for any compact subset $\mathcal{K}$ of the strip $1/2<\sigma<1$ and any $\varepsilon>0$,

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]:\max_{s\in\mathcal{K}}|\zeta(s+i\tau,\alpha)-\zeta(s+id\tau,\alpha)|<\varepsilon\right\}>0, \tag{1}$$

where meas A stands for the Lebesgue measure of a measurable set A. This problem is motivated by Bagchi [1, 2, 3] result that the Riemann hypothesis for the Riemann zeta-function is valid if and only if the inequality (1) is valid for $\alpha=1$ and $d=0$. In the case of the Riemann zeta-function ($\alpha=1$) the inequality (1) was proved by Nakamura [11] for all algebraic irrational d, afterwards by Pańkowski [14]

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for all irrational d, and recently by Nakamura and Pańkowski [13] for $0 \neq d = a/b$ with $|a-b| \neq 1$, $\gcd(a,b)=1$ (the papers [5, 12], where non-zero rational d were considered, contain a gap in the proof of the main theorem, see [13]). The case, when $\alpha \neq 1/2, 1$ is a rational or transcendental number and $d=0$, is a partial case of the universality theorem for the Hurwitz zeta-function which is proved independently by Bagchi [1] and Gonek [7], see also [9]. More on the universality theorems see books of Laurinčikas [8], Steuding [15], and the survey of Matsumoto [10]. Here we will prove the case then α is a transcendental number and d is a rational number, we will also show that for any transcendental number α the inequality (1) is true for almost all numbers d and that for any irrational number d the inequality (1) is true for almost all numbers α. Next we state our results more precisely.

Let $d_1, d_2, \dots, d_k, \alpha$ be real numbers and let α be a transcendental number from the interval (0,1] .
Let

$$A(d_1, d_2, \dots, d_k; \alpha) = \{d_j \log(n_j + \alpha) : j = 1, \dots, k;\ n_j \in \mathbb{N}_0\}$$

be a multiset, where $\mathbb{N}_0$ denotes the set of all non-negative integers. Note that in a multiset the elements can appear more than once. For example $\{1,2\}$ and $\{1,1,2\}$ are different multisets, but $\{1,2\}$ and $\{2,1\}$ are equal multisets. If a multiset $A(d_1, d_2, \dots, d_k; \alpha)$ is linearly independent over rational numbers then $A(d_1, d_2, \dots, d_k; \alpha)$ is a set and the numbers $d_1, \dots, d_k$ are linearly independent over $\mathbb{Q}$. We prove the following theorem.

Theorem 1. *Let $l \leqslant m$ be positive integers and let α be a transcendental number from the interval (0,1]. Let $d_1, \dots, d_l \in \mathbb{R}$ be such that $A(d_1, d_2, \dots, d_l; \alpha)$ is linearly independent over $\mathbb{Q}$. For $m > l$, let $d_{l+1}, \dots, d_m \in \mathbb{R}$ be such that each d_k, $k = l+1, \dots, m$ is a linear combination of $d_1, \dots, d_l$ over $\mathbb{Q}$. Then*

$$\liminf_{T\to\infty} \frac{1}{T} meas \Big\{\tau \in [0,T] : \max_{1 \leqslant j,k \leqslant m} \max_{s \in \mathcal{K}} |\zeta(s + id_j\tau, \alpha) - \zeta(s + id_k\tau, \alpha)| < \varepsilon \Big\} > 0. \tag{2}$$

In the inequality (2), for almost all ε, 'lim inf' can be replaced by 'lim' similarly as in Theorem 2 of [5]. Note that for any transcendental number α, $0 < \alpha \leqslant 1$, and for any real number d_1, the set $A(d_1; \alpha)$ is linearly independent over $\mathbb{Q}$. The following propositions show that for any positive integer l 'most' collections of real numbers $d_1, d_2, \dots, d_l, \alpha$, where $0 < \alpha \leqslant 1$, are such that $A(d_1, d_2, \dots, d_l; \alpha)$ is linearly independent over $\mathbb{Q}$.

Proposition 2. *Let α be a transcendental number and $l \geqslant 2$. If the set $A(d_1, d_2, \dots, d_{l-1}; \alpha)$ is linearly independent over $\mathbb{Q}$, then the set*

$$D = \{d_l \in \mathbb{R} : A(d_1, d_2, \dots, d_l; \alpha) \text{ is linearly dependent over } \mathbb{Q}\}$$

is countable.

Proposition 3. *Let $d_1, d_2, \dots, d_l$ be real numbers linearly independent over $\mathbb{Q}$. Then the set*

$$B = \{\alpha \in (0,1] : A(d_1, d_2, \dots, d_l; \alpha) \text{ is linearly dependent over } \mathbb{Q}\}$$

is countable.

In the next section we prove Theorem 1. Section 3 is devoted to proofs of Propositions 2 and 3.

2. Proof of Theorem 1

We follow the proof of Theorem 1 in [5]. Also lemmas from [5] will be used. As it was already mentioned the proof of Theorem 1 in [5] contains a gap, however here we avoid this gap because we work directly with $\zeta(s,\alpha)$ instead of $\log \zeta(s,\alpha)$.

Let us start with a truncated Hurwitz zeta-function

$$\zeta_v(s,\alpha) = \sum_{q \leqslant v} \frac{1}{(q+\alpha)^s}.$$

By conditions of the theorem there are integers $a \neq 0$ and $a_{k,1}, a_{k,2}, \dots, a_{k,l}$ such that

$$d_k = \frac{1}{a}(a_{k,1}d_1 + a_{k,2}d_2 + \dots + a_{k,l}d_l) \qquad \text{for } \ l < k \leqslant m. \tag{3}$$

Let

$$A = \max_{l<k\leqslant m} \{|a_{k,1}| + |a_{k,2}| + \dots + |a_{k,l}|\}.$$

Denote by $\|x\|$ the minimal distance of $x \in \mathbb{R}$ to an integer. If

$$\left\| \tau \frac{d_n \log(q+\alpha)}{2\pi a} \right\| < \delta \qquad \text{for } \ q \leqslant v \ \text{ and } \ 1 \leqslant n \leqslant l \tag{4}$$

then, by the relation (3),

$$\left\| \tau \frac{d_k \log(q+\alpha)}{2\pi} \right\| < A\delta \qquad \text{for } \ q \leqslant v \ \text{ and } \ l < k \leqslant m.$$

By this and by the continuity in s of the function $\zeta_v(s,\alpha)$ we have that for any $\varepsilon > 0$ there is $\delta > 0$ such that for τ satisfying (4)

$$\max_{1\leqslant k,n\leqslant m} \max_{s\in\mathcal{K}} |\zeta_v(s+id_k\tau,\alpha) - \zeta_v(s+id_n\tau,\alpha)| < \varepsilon. \tag{5}$$

For positive numbers δ, v, and T we define the set

$$S_T = S_T(\delta, v) = \left\{ \tau : \tau \in [0,T], \left\| \tau \frac{d_n \log(q+\alpha)}{2\pi a} \right\| < \delta, \ q \leqslant v, \ 1 \leqslant n \leqslant l \right\}. \tag{6}$$

Let U be an open bounded rectangle with vertices on the lines $\sigma = \sigma_1$ and $\sigma = \sigma_2$, where $1/2 < \sigma_1 < \sigma_2 < 1$, such that the set $\mathcal{K}$ is in U. Let $p > v$ be a positive integers. We have

$$\frac{1}{T}\int\limits_{S_T}\int\limits_{U}\sum_{k=1}^{m}\left|\zeta_p(s+id_k\tau,\alpha)-\zeta_v(s+id_k\tau,\alpha)\right|^2 d\sigma dt d\tau$$
$$=\sum_{k=1}^{m}\int\limits_{U}\frac{1}{T}\int\limits_{S_T}\left|\zeta_p(s+id_k\tau,\alpha)-\zeta_v(s+id_k\tau,\alpha)\right|^2 d\tau d\sigma dt.$$

To evaluate the inner integrals of the right-hand side of the last equality we will apply Lemma 6 from [5]. By generalized Kronecker's theorem (see Lemma 5 in [5]) and by linear independence of $A(d_1,d_2,\dots,d_l;\alpha)$ the curve

$$\omega(\tau)=\left(\tau\frac{d_k\log(q+\alpha)}{2\pi a}\right)_{0\leqslant q\leqslant p}^{1\leqslant k\leqslant l}$$

is uniformly distributed mod 1 in $\mathbb{R}^{l(p+1)}$. Let R' be a subregion of the $l(p+1)$-dimensional unit cube defined by inequalities

$$\|y_{k,q}\|\leqslant\delta \qquad \text{for } 1\leqslant k\leqslant l \text{ and } 0\leqslant q\leqslant v$$

and

$$\left|y_{k,q}-\frac{1}{2}\right|\leqslant\frac{1}{2} \qquad \text{for } 1\leqslant k\leqslant l \text{ and } v+1\leqslant q\leqslant p.$$

Let R be a subregion of the $l(v+1)$-dimensional unit cube defined by inequalities

$$\|y_{k,q}\|\leqslant\delta \qquad \text{for } 1\leqslant k\leqslant l \text{ and } 0\leqslant q\leqslant v$$

Clearly $\operatorname{meas} R' = \operatorname{meas} R = (2\delta)^{l(v+1)}$. Let

$$\zeta_{p,v}(s+id_k\tau,\alpha)=\zeta_p(s+id_k\tau,\alpha)-\zeta_v(s+id_k\tau,\alpha). \tag{7}$$

Then in view of the linear dependence (3) we get

$$\lim_{T\to\infty}\frac{1}{T}\int_{S_T}\sum_{k=1}^{m}\left|\zeta_{p,v}(s+id_k\tau,\alpha)\right|^2 d\tau$$
$$=\lim_{T\to\infty}\frac{1}{T}\int_{S_T}\left(\sum_{k=1}^{l}\left|\zeta_{p,v}(s+id_k\tau,\alpha)\right|^2\right.$$
$$\left.+\sum_{k=l+1}^{m}\left|\zeta_{p,v}\left(s+\frac{i}{a}(a_{k,1}d_1+a_{k,2}d_2+\dots+a_{k,l}d_l)\tau,\alpha\right)\right|^2\right)d\tau.$$

By Lemma 6 in [5] and equality (7) we obtain that the last limit is equal to

$$\int_{R'} \left(\sum_{k=1}^{l} \left| \sum_{v<q\leqslant p} \frac{e^{-2\pi i a y_{k,q}}}{(q+\alpha)^s} \right|^2 \right.$$

$$\left. + \sum_{k=l+1}^{m} \left| \sum_{v<q\leqslant p} \frac{e^{-2\pi i (a_{k,1}y_{1,q}+a_{k,2}y_{2,q}+\cdots+a_{k,l}y_{l,q})}}{(q+\alpha)^s} \right|^2 \right) dy_{1,1}\dots dy_{l,p}$$

$$= \operatorname{meas} R \int_0^1 \dots \int_0^1 \left(\sum_{k=1}^{l} \left| \sum_{v<q\leqslant p} \frac{e^{-2\pi i y_{k,q}}}{(q+\alpha)^s} \right|^2 \right.$$

$$\left. + \sum_{k=l+1}^{m} \left| \sum_{v<q\leqslant p} \frac{e^{-2\pi i (a_{k,1}y_{1,q}+a_{k,2}y_{2,q}+\cdots+a_{k,l}y_{l,q})}}{(q+\alpha)^s} \right|^2 \right) dy_{1,v+1}\dots dy_{l,p}$$

$$= m \operatorname{meas} R \sum_{v<q\leqslant p} \frac{1}{(q+\alpha)^{2\sigma}} \ll \operatorname{meas} R \sum_{q>v} \frac{1}{(q+\alpha)^{2\sigma}}.$$

Consequently

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{S_T} \int\limits_{U} \sum_{k=1}^{m} \left|\zeta_p(s+id_k\tau,\alpha) - \zeta_v(s+id_k\tau,\alpha)\right|^2 d\sigma dt d\tau \tag{8}$$

$$\ll \operatorname{meas} R \sum_{q>v} \frac{1}{(q+\alpha)^{2\sigma_1}}.$$

Again by Lemma 5 in [5],

$$\lim_{T\to\infty} \frac{1}{T} \operatorname{meas} S_T = \operatorname{meas} R. \tag{9}$$

By (8) and (9), for large v, as $T\to\infty$, we have

$$\operatorname{meas} \left\{ \tau : \tau \in S_T, \int_U \sum_{k=1}^{m} \left|\zeta_{p,v}(s+id_k\tau,\alpha)\right|^2 d\sigma dt < \sqrt{\sum_{q>v} \frac{1}{(q+\alpha)^{2\sigma_1}}} \right\}$$

$$> \frac{1}{2} T \operatorname{meas} R.$$

Then Lemma 4 in [5] gives

$$\operatorname{meas} \left\{ \tau : \tau \in S_T, \max_{s\in\mathcal{K}} \sum_{k=1}^{m} \left|\zeta_{p,v}(s+id_k\tau,\alpha)\right| \leqslant \frac{m}{d\sqrt{\pi}} \left(\sum_{q>v} \frac{1}{(q+\alpha)^{2\sigma_1}} \right)^{\frac{1}{4}} \right\}$$

$$> \frac{1}{2} T \operatorname{meas} R,$$

where $d = \min_{z\in\partial U} \min_{s\in\mathcal{K}} |s-z|$. Therefore we obtain that for any $\varepsilon > 0$ there is $v = v(\varepsilon)$ such that for any $p > v$

$$\operatorname{meas}\left\{\tau : \tau \in S_T, \max_{s\in\mathcal{K}} \sum_{k=1}^{m} |\zeta_p(s+id_k\tau,\alpha) - \zeta_v(s+id_k\tau,\alpha)| < \varepsilon\right\} \qquad (10)$$
$$> \frac{1}{2} T \operatorname{meas} R.$$

Now we will prove that for any $\delta > 0$ there is $p = p(\delta)$ such that

$$\operatorname{meas}\left\{\tau : \max_{s\in\mathcal{K}} \sum_{k=1}^{m} |\zeta(s+id_k\tau,\alpha) - \zeta_p(s+id_k\tau,\alpha)| < \delta\right\} \qquad (11)$$
$$> (1-\delta)T.$$

The last formula together with (5), (6) and (10) yields Theorem 1. We return to the proof of (11). By the mean value theorem of the Hurwitz zeta-function (see Garunkštis, Laurinčikas, and Steuding [6]) and by Carlson's Theorem (see Carlson [4]) we obtain

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |\zeta(s+ix\tau,\alpha) - \zeta_p(s+ix\tau,\alpha)|^2 \, d\tau = \sum_{q>p} \frac{1}{(q+\alpha)^{2\sigma}},$$

where x is fixed. Thus (11) follows in view of

$$\int_0^T \int_U \sum_{k=1}^{m} |\zeta(s+id_k\tau,\alpha) - \zeta_p(s+ix\tau,\alpha)|^2 \, d\sigma dt d\tau \ll T \sum_{q>p} \frac{1}{(q+\alpha)^{2\sigma_1}}.$$

Theorem 1 is proved.

3. Proofs of Propositions 2 and 3

Proof of Proposition 2. Let Ω be a set of all rational numbers sequences, where each sequence has only finitely many nonzero elements. Then Ω is a countable set. By $\mathbf{0}$ we denote the sequence all elements of which are zeros. Let $d_1 = 1$. Recall that the set $A(1;\alpha)$ is linearly independent. Then in view of the linear independence of $A(d_1, d_2, \ldots, d_{l-1}; \alpha)$ we obtain that

$$D = \Bigg\{ -\frac{d_1 \sum_{n=0}^{\infty} a_{1n} \log(n+\alpha) + \cdots + d_{l-1} \sum_{n=0}^{\infty} a_{l-1n} \log(n+\alpha)}{\sum_{n=0}^{\infty} a_{ln} \log(n+\alpha)} :$$
$$(a_{10}, a_{11}, \ldots, a_{(l-1)0}, a_{(l-1)1}, \ldots, a_{l0}, a_{l1}, \ldots) \in \Omega \setminus \mathbf{0},$$
$$(a_{l0}, a_{l1}, \ldots) \neq \mathbf{0} \Bigg\}.$$

Thus D is a countable set. This proves the proposition. ■

Proof of Proposition 3. We use the same notations as in the proof of Proposition 2. Similarly as before we have that

$$B = \left\{ \alpha \in I : d_1 \sum_{n=0}^{\infty} a_{1n} \log(n+\alpha) + \cdots + d_l \sum_{n=0}^{\infty} a_{ln} \log(n+\alpha) = 0, \right.$$
$$\left. (a_{10}, a_{11}, \ldots, a_{20}, a_{21}, \ldots, \ldots, a_{l0}, a_{l1}, \ldots) \in \Omega \setminus \mathbf{0} \right\}.$$

Recall that Ω is a countable set. If, for fixed

$$(a_{10}, a_{11}, \ldots, a_{20}, a_{21}, \ldots, \ldots, a_{l0}, a_{l1}, \ldots) \in \Omega \setminus \mathbf{0},$$

the function

$$f(\alpha) = d_1 \sum_{n=0}^{\infty} a_{1n} \log(n+\alpha) + \cdots + d_l \sum_{n=0}^{\infty} a_{ln} \log(n+\alpha)$$

has only finite number of zeros in $(0,1]$, then the set B is countable. Thus to prove the proposition it remains to show that $f(\alpha)$ has finitely many zeros in the interval $(0,1]$. In view of the condition that $d_1, d_2, \ldots, d_k$ are linearly independent and by the definition of Ω we have that there is a finite collection of real numbers $b_0, b_1, \ldots, b_m$, such that $b_m \neq 0$ and

$$f(\alpha) = b_0 \log(\alpha) + b_1 \log(1+\alpha) + \cdots + b_m \log(m+\alpha).$$

Let b_n, $n \leqslant m$ be the first coefficient not equal to zero. Then we see that $f(\alpha)$ is unbounded in $(-n, 1/2)$ and is bounded in $(1/2, 1]$. Thus $f(\alpha)$ is not a constant in $(-n, 1]$. Moreover there is a small positive number α_0 such that $f(\alpha) \neq 0$ if $\alpha \in (-n, -n+\alpha_0)$. We consider $f(\alpha)$ as an analytic function in the half-plane $\Re\alpha > -n$ of the complex plane. A set of zeros of a non-constant analytic function is discrete. Thus there are finitely many zeros in the disc $|1-\alpha| \leqslant 1+n-\alpha_0$. We obtained that the function $f(\alpha)$ has finitely many zeros in $(0,1]$. This proves the proposition. ■

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