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ANDREWS' SINGULAR OVERPARTITIONS WITH ODD PARTS

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Abstract: Recently singular overpartitions was defined and studied by G. E. Andrews. He showed that such partitions can be enumerated by $\overline{C}_{\delta,i}(n)$, the number of overpartitions of n such that no part is divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. In this paper, we establish several infinite families of congruences $\overline{CO}_{\delta,i}(n)$, the number of singular overpartitions of n into odd parts such that no part is divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. For example, for all $n \geqslant 0$ and $\alpha \geqslant 0$, $\overline{CO}_{3,1}(4\cdot 3^{\alpha+3}n+7\cdot 3^{\alpha+2}) \equiv 0 \pmod{8}$.

Keywords: partitions, singular overpartitions, congruences.

1. Introduction

G.E. Andrews [2] defined combinatorial objects which he called singular overpartitions and proved that these singular overpartitions which depends on two parameters δ and i can be enumerated by the function $\overline{C}_{\delta,i}(n)$ which gives the number of overpartitions of n in which no part is divisible by δ and parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{C}_{\delta,i}(n)$ is

$$\sum_{n=0}^{\infty} \overline{C}_{\delta,i}(n) q^n = \frac{(q^\delta;q^\delta)_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q)_\infty}. \tag{1.1}$$

Throughout the paper, we use the standard q-series notation, and f_k is defined as

$$f_k := (q^k;q^k)_\infty = \lim_{n\to\infty} \prod_{m=1}^{n} (1-q^{mk}).$$

For $|ab|<1$, Ramanujan's general theta function $f(a,b)$ is defined as

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}. \tag{1.2}$$

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Using Jacobi's triple product identity [4, Entry 19, p. 35], the equation (1.2) becomes

$$f(a,b) = (-a,ab)_\infty (-b,ab)_\infty (ab,ab)_\infty .$$

The most important special cases of $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.3}$$

$$\psi(q) := f\left(q,q^3\right) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{1.4}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{n(3n-1)}{2}} = (q;q)_\infty = f_1. \tag{1.5}$$

Andrews [2] has found the following congruence results, for each integer $n \geqslant 0$,

$$\overline{C}_{3,1}(9n+3) \equiv 0 \pmod 3, \tag{1.6}$$

$$\overline{C}_{3,1}(9n+6) \equiv 0 \pmod 3. \tag{1.7}$$

Recently S-C. Chen, M.D. Hirschhorn and J.A. Sellers [5] have found some infinite families of congruences modulo 3 for $\overline{C}_{3,1}(n)$, $\overline{C}_{6,1}(n)$, $\overline{C}_{6,2}(n)$ and modulo powers of 2 for $\overline{C}_{4,1}(n)$. For example, for all $k, m \geqslant 0$,

$$\overline{C}_{3,1}(2^k(4m+3)) \equiv 0 \pmod 3, \tag{1.8}$$

$$\overline{C}_{3,1}(4^k(16m+6)) \equiv 0 \pmod 3. \tag{1.9}$$

The authors Z. Ahmed and N.D. Baruah [1] have found some new congruences for $\overline{C}_{3,1}(n)$ modulo 18 and 36 and $\overline{C}_{8,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$ modulo 2. For example, for all $n \geqslant 0$,

$$\overline{C}_{3,1}(48n+12) \equiv 0 \pmod{18}, \tag{1.10}$$

$$\overline{C}_{3,1}(24n+22) \equiv 0 \pmod{36}. \tag{1.11}$$

Chen [6] has also found some new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{4,1}(n)$ modulo powers of 2. For example, for all $m \geqslant 0$,

$$\overline{C}_{3,1}(6m+5) \equiv 0 \pmod{16}. \tag{1.12}$$

O.X.M. Yao [11] has proved some congruences modulo 16, 32, 14 for $\overline{C}_{3,1}(n)$. For example, for all $n \geqslant 0$,

$$\overline{C}_{3,1}(18n+15) \equiv 0 \pmod{32}. \tag{1.13}$$

M.S. Mahadeva Naika and D.S. Gireesh [9] have found some modulo 6, 12, 16, 18, 24, 48, 72 for $\overline{C}_{3,1}(n)$. For example, for all $n \geqslant 0$,

$$\overline{C}_{3,1}(24n+14) \equiv 0 \pmod{32}. \tag{1.14}$$

Motivated by the above works, in this paper, we defined the function $\overline{CO}_{\delta,i}(n)$, the number of singular overpartitions of n into odd parts such that no part is divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. The generating function of $\overline{CO}_{\delta,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{CO}_{\delta,i}(n)q^n = \frac{(q^\delta;q^{2\delta})_\infty(-q^i;q^\delta)_\infty(-q^{\delta-i};q^\delta)_\infty}{(q;q^2)_\infty(-q^{2i};q^{2\delta})_\infty(-q^{2(\delta-i)};q^{2\delta})_\infty}, \tag{1.15}$$

where $0 < i < \delta$.

2. Preliminaries

We list a few dissection formulas to prove our main results.

Lemma 2.1 ([4, Entry 25 p. 40]). *The following* 2*-dissection formulas hold:*

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{2.1}$$

and

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{2.2}$$

Lemma 2.2 ([10]). *The following* 2*-dissection formula holds:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}. \tag{2.3}$$

Lemma 2.3 ([3, Lemma 2.6]). *The following* 3*-dissection formula holds:*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{2.4}$$

Lemma 2.4. *The following* 2*-dissection formulas hold:*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}} \tag{2.5}$$

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}. \tag{2.6}$$

Equation (2.5) was proved by Baruah and K.K. Ojah [3]. Replace q by $-q$ in (2.5) and using the fact that $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$, we get (2.6).

Lemma 2.5 ([8]). *The following* 3*-dissection formula holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2\frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{2.7}$$

Lemma 2.6 ([7, Theorem 2.1]). *For any odd prime p,*

$$\psi(q)=\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right)+q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{2.8}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod p$ *for* $0 \leqslant m \leqslant \frac{p-3}{2}$.

Lemma 2.7 ([7, Theorem 2.2]). *For any prime $p \geqslant 5$,*

$$\begin{aligned} f_1 = & \sum_{\substack{k=-\frac{p-1}{2} \\ k\neq(\pm p-1)/6}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) \\ & + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}. \end{aligned} \tag{2.9}$$

Furthermore, for $-(p-1)/2 \leqslant k \leqslant (p-1)/2$ and $k \neq (\pm p-1)/6$,

$$\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p.$$

3. Congruences for $\overline{CO}_{3,1}(n)$

Theorem 3.1. *For each integer $n \geqslant 0$,*

$$\overline{CO}_{3,1}(12n+7) \equiv 0 \pmod 8, \tag{3.1}$$

$$\overline{CO}_{3,1}(24n+19) \equiv 0 \pmod{16}, \tag{3.2}$$

$$\overline{CO}_{3,1}(24n+7) \equiv \psi(q)f_4 \pmod{16}. \tag{3.3}$$

Proof. Setting $\delta=3$ and $i=1$ in (1.15), we find that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(n)q^n = \frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^2 (q^{12};q^{12})}{(q^6;q^6)_\infty^3 (q^4;q^4)_\infty (q;q)_\infty^2}. \tag{3.4}$$

Substituting (2.3) into (3.4), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(n)q^n = \frac{f_4^3 f_{12}^3}{f_2^2 f_6^2 f_8 f_{24}} + 2q\frac{f_8 f_{24}}{f_2 f_6}, \tag{3.5}$$

which yields, for each $n \geqslant 0$,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n = 2\frac{f_4 f_{12}}{f_1 f_3}. \tag{3.6}$$

Employing (2.4) into (3.6), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n = 2\frac{f_{12}^2 f_{18}^4}{f_3^4 f_{36}^2} + 2q\frac{f_6^2 f_9^3 f_{12} f_{36}}{f_3^5 f_{18}^2} + 4q^2\frac{f_6 f_{12} f_{18} f_{36}}{f_3^4}. \tag{3.7}$$

Extracting the terms involving q^{3n} in the above equation and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+1)q^n = 2\frac{f_4^2 f_6^4}{f_1^4 f_{12}^2}. \tag{3.8}$$

By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{2k}^m \equiv f_k^{2m} \pmod 2 \tag{3.9}$$

and

$$f_{2k}^{2m} \equiv f_k^{4m} \pmod 4. \tag{3.10}$$

Using (3.10) in (3.8), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+1)q^n \equiv 2f_2^2 \pmod 8. \tag{3.11}$$

Congruences (3.1) follows by extracting the terms involving q^{2n+1} from (3.11).

Collecting the terms involving q^{2n} from (3.11) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+1) \equiv 2f_1^2 \pmod 8. \tag{3.12}$$

Substituting (2.2) into (3.8), we find that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+1)q^n = 2\frac{f_4^{16} f_6^4}{f_2^{14} f_8^4 f_{12}^2} + 8q\frac{f_4^4 f_6^4 f_8^4}{f_2^{10} f_{12}^2}, \tag{3.13}$$

which implies that,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+7)q^n = 8\frac{f_2^4 f_3^4 f_4^4}{f_1^{10} f_6^2}. \tag{3.14}$$

Using (3.9) in (3.14), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+7)q^n \equiv 8f_2^7 \pmod{16}. \tag{3.15}$$

Extracting the terms involving q^{2n+1} from (3.15) we get (3.2).

Collecting the terms involving q^{2n} from (3.15) and replacing q^2 by q, reduces to

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7)q^n \equiv 8f_1^7 \pmod{16}, \tag{3.16}$$

Using (3.9) in (3.16), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7)q^n \equiv 8\left(\frac{f_2^2}{f_1}\right) f_4 \pmod{16}. \tag{3.17}$$

Using (1.4) in (3.17), we arrive at (3.3). ■

Theorem 3.2. *For any prime $p \equiv 5 \pmod 6$, $\alpha \geqslant 1$, and $n \geqslant 0$, we have*

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2p^{2\alpha}n + p^{2\alpha})q^n \equiv 2\psi(q)\psi(q^3) \pmod 4. \tag{3.18}$$

Proof. Using (3.9) in (3.6), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n \equiv 2\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 4. \tag{3.19}$$

Using (1.4) in (3.19), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n \equiv 2\psi(q)\psi(q^3) \pmod 4. \tag{3.20}$$

Define

$$\sum_{n=0}^{\infty} g(n)q^n = \psi(q)\psi(q^3). \tag{3.21}$$

Combining (3.20) and (3.21), we find that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n \equiv 2\sum_{n=0}^{\infty} g(n)q^n \pmod 4. \tag{3.22}$$

Now, we consider the congruence equation

$$\frac{k^2+k}{2} + 3\cdot\frac{m^2+m}{2} \equiv \frac{4p^2-4}{8} \pmod p, \tag{3.23}$$

which is equivalent to

$$(2k+1)^2 + 3\cdot(2m+1)^2 \equiv 0 \pmod p,$$

where $0 \leqslant k, m \leqslant \frac{p-1}{2}$ and p is a prime such that $(\frac{-3}{p}) = -1$. Since $(\frac{-3}{p}) = -1$ for $p \equiv 5 \pmod 6$, the congruence relation (3.23) holds if and only if both $k = m = \frac{p-1}{2}$. Therefore, if we substitute (2.8) into (3.21) and then extracting the terms in which the powers of q are congruent to $\frac{p^2-1}{2}$ modulo p and then divide by $q^{\frac{p^2-1}{2}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn + \frac{p^2-1}{2}\right) q^{pn} = \psi(q^{p^2})\psi(q^{3p^2}),$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2 n + \frac{p^2-1}{2}\right) q^{n} = \psi(q)\psi(q^3) \tag{3.24}$$

and for $n \geqslant 0$,

$$g\left(p^2n + pi + \frac{p^2-1}{2}\right) = 0, \tag{3.25}$$

where i is an integer and $1 \leqslant i \leqslant p-1$. By induction, we see that for $n \geqslant 0$ and $\alpha \geqslant 0$,

$$g\left(p^{2\alpha}n + \frac{p^{2\alpha}-1}{2}\right) = g(n). \tag{3.26}$$

Replacing n by $p^{2\alpha}n + \frac{p^{2\alpha}-1}{2}$ in (3.22), we arrive at (3.18). ■

Theorem 3.3. *For any prime $p \equiv 5 \pmod 6$, $\alpha \geqslant 1$, and $n \geqslant 0$, we have*

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24p^{2\alpha}n + 7p^{2\alpha}) \equiv (-1)^{\alpha.\frac{\pm p-1}{6}} \psi(q) f_4 \pmod{16}. \tag{3.27}$$

Proof. Define

$$\sum_{n=0}^{\infty} a(n)q^n = \psi(q) f_4. \tag{3.28}$$

Combining (3.3) and (3.28), we see that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7)q^n \equiv \sum_{n=0}^{\infty} a(n)q^n \pmod{16}. \tag{3.29}$$

Now, we consider the congruence equation

$$\frac{k^2+k}{2} + 4\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p}, \tag{3.30}$$

which is equivalent to

$$3\cdot(2k+1)^2 + (12m+2)^2 \equiv 0 \pmod{p},$$

where $\frac{-(p-1)}{2} \leqslant m \leqslant \frac{p-1}{2}$, $0 \leqslant k \leqslant \frac{p-1}{2}$ and p is a prime such that $(\frac{-3}{p}) = -1$. Since $(\frac{-3}{p}) = -1$ for $p \equiv 5 \pmod 6$, the congruence relation (3.30) holds if and only if $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$. Therefore, if we substitute (2.8) and (2.9) into (3.28) and then extracting the terms in which the powers of q are $pn + \frac{7p^2-7}{24}$, we arrive at

$$\sum_{n=0}^{\infty} a\left(pn + \frac{7p^2-7}{24}\right) q^{pn+\frac{7p^2-7}{24}} = (-1)^{\frac{\pm p-1}{6}} q^{\frac{7p^2-7}{24}} \psi(q^{p^2}) f_{4p^2}. \tag{3.31}$$

Dividing by $q^{\frac{7p^2-7}{24}}$ on both sides of (3.31) and on simplification, we find that

$$\sum_{n=0}^{\infty} a\left(pn + \frac{7p^2-7}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} \psi(q^p) f_{4p},$$

which implies that

$$\sum_{n=0}^{\infty} a\left(p^2n+\frac{7p^2-7}{24}\right)q^n=(-1)^{\frac{\pm p-1}{6}}\psi(q)f_4 \tag{3.32}$$

and for $n\geqslant 0$,

$$a\left(p^2n+pi+\frac{7p^2-7}{24}\right)=0, \tag{3.33}$$

where i is an integer and $1\leqslant i\leqslant p-1$. Combining (3.28) and (3.32), we see that for $n\geqslant 0$,

$$a\left(p^2n+\frac{7p^2-7}{24}\right)=(-1)^{\frac{\pm p-1}{6}}a(n). \tag{3.34}$$

By (3.34) and mathematical induction, we deduce that for $n\geqslant 0$ and $\alpha\geqslant 0$,

$$a\left(p^{2\alpha}n+\frac{7p^{2\alpha}-7}{24}\right)=(-1)^{\alpha\cdot\frac{\pm p-1}{6}}a(n). \tag{3.35}$$

Replacing n by $p^{2\alpha}n+\frac{7p^{2\alpha}-7}{24}$ in (3.29), we arrive at (3.27). ■

Theorem 3.4. *For all $n\geqslant 0$ and $\alpha\geqslant 0$,*

$$\overline{CO}_{3,1}(36n+21)\equiv 0 \pmod 8, \tag{3.36}$$

$$\overline{CO}_{3,1}(36n+3)\equiv \overline{CO}_{3,1}(12n+1) \pmod 8, \tag{3.37}$$

$$\overline{CO}_{3,1}(4\cdot 3^{\alpha+3}n+7\cdot 3^{\alpha+2})\equiv 0 \pmod 8, \tag{3.38}$$

$$\overline{CO}_{3,1}(36n+33)\equiv 0 \pmod 8, \tag{3.39}$$

$$\overline{CO}_{3,1}(18n+15)\equiv \overline{CO}_{3,1}(6n+5) \pmod 8. \tag{3.40}$$

Proof. Equating the coefficients of q^{3n+1} from both sides of (3.7), dividing by q and then replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(6n+3)q^n=2\frac{f_2^2f_3^3f_4f_{12}}{f_1^5f_6^2}. \tag{3.41}$$

Using (3.10) in (3.41), we obtain

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(6n+3)q^n\equiv 2\frac{f_4f_{12}}{f_1f_3} \pmod 8. \tag{3.42}$$

Substituting (2.4) into (3.42), we get

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(6n+3)q^n\equiv 2\frac{f_{12}^2f_{18}^4}{f_3^4f_{36}^2}+2q\frac{f_6^2f_9^3f_{12}f_{36}}{f_3^5f_{18}^2}+4q^2\frac{f_6f_{12}f_{18}f_{36}}{f_3^4} \pmod 8, \tag{3.43}$$

which implies that for all $n \geqslant 0$,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+3)q^n \equiv 2\frac{f_4^2 f_6^4}{f_1^4 f_{12}^2} \pmod 8. \tag{3.44}$$

Using (3.10) in (3.44), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+3)q^n \equiv 2f_2^2 \pmod 8. \tag{3.45}$$

Equating the coefficients of q^{2n+1} from both sides of (3.45), dividing by q and then replacing q^2 by q, we arrive at (3.36).

Extracting the terms involving q^{2n} from (3.45) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+3)q^n \equiv 2f_1^2 \pmod 8. \tag{3.46}$$

In view of congruences (3.46) and (3.12), we obtain (3.37).

Extracting the terms involving q^{3n+1} from (3.43), dividing by q and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+9)q^n \equiv 2\frac{f_2^2 f_3^3 f_4 f_{12}}{f_1^5 f_6^2} \pmod 8. \tag{3.47}$$

Using (3.10) in (3.47), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+9)q^n \equiv 2\frac{f_4 f_{12}}{f_1 f_3} \pmod 8. \tag{3.48}$$

In view of congruences (3.48) and (3.42), we get

$$\overline{CO}_{3,1}(18n+9) \equiv \overline{CO}_{3,1}(6n+3) \pmod 8. \tag{3.49}$$

Utilizing (3.49) and by mathematical induction on α, we arrive at

$$\overline{CO}_{3,1}(2\cdot 3^{\alpha+2}n + 3^{\alpha+2}) \equiv \overline{CO}_{3,1}(6n+3) \pmod 8. \tag{3.50}$$

Using (3.36) in (3.50), we obtain (3.38).

From (3.43), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+15)q^n \equiv 4\frac{f_2 f_4 f_6 f_{12}}{f_1^4} \pmod 8. \tag{3.51}$$

Using (3.9) in (3.51), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+15)q^n \equiv 4f_2 f_6 f_{12} \pmod 8. \tag{3.52}$$

Congruences (3.39) follows extracting the terms involving q^{2n+1} from (3.52).

Extracting the terms involving q^{3n+2} from (3.7), dividing by q^2 and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(6n+5)q^n = 4\frac{f_2f_4f_6f_{12}}{f_1^4}. \tag{3.53}$$

Using (3.9) in (3.53), we have

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(6n+5)q^n \equiv 4f_2f_6f_{12} \pmod 8. \tag{3.54}$$

Combining (3.52) and (3.54), we arrive at (3.40). ∎

Theorem 3.5. *For all $n \geqslant 0$ and $\alpha \geqslant 0$,*

$$\overline{CO}_{3,1}(12n+7) \equiv 0 \pmod 8, \tag{3.55}$$
$$\overline{CO}_{3,1}(12n+11) \equiv 0 \pmod 8, \tag{3.56}$$
$$\overline{CO}_{3,1}(108n+63) \equiv 0 \pmod 8, \tag{3.57}$$
$$\overline{CO}_{3,1}(108n+99) \equiv 0 \pmod 8, \tag{3.58}$$
$$\overline{CO}_{3,1}(972n+567) \equiv 0 \pmod 8, \tag{3.59}$$
$$\overline{CO}_{3,1}(972n+891) \equiv 0 \pmod 8, \tag{3.60}$$
$$\overline{CO}_{3,1}(12\cdot 9^{\alpha+2}n+3\cdot 9^{\alpha+2}) \equiv \overline{CO}_{3,1}(108n+27) \pmod 8. \tag{3.61}$$

Proof. Substituting (2.5) into (3.6), we obtain

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(2n+1)q^n = 2\frac{f_8^2f_{12}^6}{f_2^2f_6^4f_{24}^2} + 2q\frac{f_4^6f_{24}^2}{f_2^4f_6^2f_8^2}, \tag{3.62}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(4n+3)q^n = 2\frac{f_2^6f_{12}^2}{f_1^4f_3^2f_4^2}. \tag{3.63}$$

Using (3.10) in (3.63), we get

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(4n+3)q^n \equiv 2\frac{f_{12}^2}{f_3^2} \pmod 8. \tag{3.64}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} from (3.64) we get (3.55) and (3.56).

Extracting the terms involving q^{3n} from (3.64) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+3)q^n \equiv 2\frac{f_4^2}{f_1^2} \pmod{8}. \tag{3.65}$$

Substituting (2.4) into (3.65) and equating the terms q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+27)q^n \equiv 2\frac{f_2^4 f_3^6 f_{12}^2}{f_1^8 f_6^4} \pmod{8}. \tag{3.66}$$

Using (3.10) in (3.66), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+27)q^n \equiv 2f_3^6 \pmod{8}. \tag{3.67}$$

Congruences (3.57) and (3.58) follows extracting the terms involving q^{3n+1} and q^{3n+2} from (3.66).

Extracting the terms involving q^{3n} from (3.67) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(108n+27)q^n \equiv 2f_1^6 \pmod{8}, \tag{3.68}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(108n+27)q^n \equiv 2f_1^2 f_2^2 \pmod{8}. \tag{3.69}$$

Employing (2.7) into (3.69) and equating the terms involving q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(324n+243)q^n \equiv 2f_3^2 f_6^2 \pmod{8}. \tag{3.70}$$

Using (3.10) in (3.70), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(324n+243)q^n \equiv 2f_3^6 \pmod{8}. \tag{3.71}$$

Extracting the terms involving q^{3n+1} and $q^{3n+2}$from (3.71), we arrive at (3.59) and (3.60).

Extracting the terms involving q^{3n} from (3.71) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(972n+243)q^n \equiv 2f_1^6 \pmod{8}. \tag{3.72}$$

In view of congruences (3.72) and (3.68), we get

$$\overline{CO}_{3,1}(972n+243) \equiv \overline{CO}_{3,1}(108n+27) \pmod{8}. \tag{3.73}$$

Utilizing (3.73) and by mathematical induction on α, we arrive at (3.61). ∎

Theorem 3.6. *For all $n \geqslant 0$ and $\alpha \geqslant 0$,*

$$\overline{CO}_{3,1}(24n+14) \equiv 0 \pmod 8, \tag{3.74}$$

$$\overline{CO}_{3,1}(4\cdot 3^{\alpha+2}n + 2\cdot 3^{\alpha+2}) \equiv 3^{\alpha+1}\overline{CO}_{3,1}(12n+6) \pmod 8, \tag{3.75}$$

$$\overline{CO}_{3,1}(108n+27) \equiv 3\overline{CO}_{3,1}(24n+6) \pmod 8, \tag{3.76}$$

$$\overline{CO}_{3,1}(72n+6) \equiv 3\overline{CO}_{3,1}(24n+2) \pmod 8, \tag{3.77}$$

$$\overline{CO}_{3,1}(72n+42) \equiv 0 \pmod 8, \tag{3.78}$$

$$\overline{CO}_{3,1}(72n+66) \equiv 0 \pmod 8, \tag{3.79}$$

$$\overline{CO}_{3,1}(24n+22) \equiv 0 \pmod 8, \tag{3.80}$$

$$\overline{CO}_{3,1}(36n+30) \equiv \overline{CO}_{3,1}(12n+10) \pmod 8. \tag{3.81}$$

Proof. From (3.5), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n)q^n = \frac{f_2^3 f_6^3}{f_1^2 f_3^2 f_4 f_{12}}. \tag{3.82}$$

Substituting (2.5) into (3.82) and equating the terms q^{2n+1},

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+2)q^n = 2\frac{f_2^3 f_6^3}{f_1^3 f_3^3}. \tag{3.83}$$

Using (3.10) in (3.83), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+2)q^n \equiv 2\frac{f_6^3}{f_3^3}(f_1 f_2) \pmod 8. \tag{3.84}$$

Employing (2.7) into (3.84), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+2)q^n \equiv 2\frac{f_6^4 f_9^4}{f_3^4 f_{18}^2} - 2q\frac{f_6^3 f_9 f_{18}}{f_3^3} - 4q^2\frac{f_6^2 f_{18}^4}{f_3^2 f_9^2} \pmod 8, \tag{3.85}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+2)q^n \equiv 2\frac{f_2^4 f_3^4}{f_1^4 f_6^2} \pmod 8. \tag{3.86}$$

Using (3.10) in (3.86), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+2)q^n \equiv 2f_2^2 \pmod 8. \tag{3.87}$$

Congruence (3.74) follows extracting the terms involving q^{2n+1} from (3.87).

Extracting the terms involving q^{2n} from (3.87), we arrive at

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+2) \equiv 2f_1^2 \pmod 8. \tag{3.88}$$

Extracting the terms involving q^{3n+1} from (3.85), dividing by q and then replacing q^{3n} by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6\frac{f_2^3 f_3 f_6}{f_1^3} \pmod 8. \tag{3.89}$$

Using (3.10) in (3.89), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6(f_1 f_2) f_3 f_6 \pmod 8. \tag{3.90}$$

Substituting (2.7) into (3.90), we arrive at

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6\frac{f_6^2 f_9^4}{f_{18}^2} - 6q f_3 f_6 f_9 f_{18} - 12q^2 \frac{f_3^2 f_{18}^4}{f_9^2} \pmod 8, \tag{3.91}$$

which implies that for all $n \geqslant 0$

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+18)q^n \equiv 2 f_1 f_2 f_3 f_6 \pmod 8. \tag{3.92}$$

In the view of congruence (3.92) and (3.90), we have

$$\overline{CO}_{3,1}(36n+18) \equiv 3\overline{CO}_{3,1}(12n+6) \pmod 8. \tag{3.93}$$

Utilizing (3.93) and by mathematical induction on α, we arrive at (3.75).

Employing (2.6) into (3.90), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6\frac{f_2^2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 6q\frac{f_4^4 f_6^2 f_{24}^2}{f_8^2 f_{12}^2} \pmod 8. \tag{3.94}$$

Extracting the terms involving q^{2n} from (3.94) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+6)q^n \equiv 6\frac{f_1^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod 8. \tag{3.95}$$

Using (3.10) in (3.95), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+6)q^n \equiv 6 f_1^2 f_2^2 \pmod 8. \tag{3.96}$$

Combining (3.96) and (3.69), we obtain (3.76).

Extracting the terms involving q^{3n} from (3.91) and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+6)q^n \equiv 6\frac{f_2^2 f_3^4}{f_6^2} \pmod 8. \tag{3.97}$$

Using (3.10) in (3.97), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+6)q^n \equiv 6f_2^2 \pmod{8}. \tag{3.98}$$

Congruences (3.78) follows by extracting the terms involving q^{2n+1} from (3.98).

Extracting the terms involving q^{2n} from (3.98) and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(72n+6)q^n \equiv 6f_1^2 \pmod{8}. \tag{3.99}$$

Combining the equations (3.99) and (3.88), we arrive at (3.77).

Equating the coefficients of q^{3n+2} from both sides of (3.91), dividing by q^2 and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+30)q^n \equiv 4\frac{f_1^2 f_6^4}{f_3^2} \pmod{8}. \tag{3.100}$$

Using (3.9) in (3.100), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+30)q^n \equiv 4f_2 f_6^3 \pmod{8}. \tag{3.101}$$

Extracting the terms involving q^{2n+1} from (3.101), we arrive at (3.79).

Equating the coefficients of q^{3n+2} from both sides of (3.85), dividing by q^2 and then replacing q^3 by q,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+10)q^n \equiv 4\frac{f_2^2 f_6^4}{f_1^2 f_3^2} \pmod{8}. \tag{3.102}$$

Using (3.9) in (3.102), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+22)q^n \equiv 4f_2 f_6^3 \pmod{8}. \tag{3.103}$$

Congruences (3.80) follows by extracting the terms involving q^{2n+1} from (3.103).

In the view of congruences (3.103) and (3.101), we get (3.81). ■

Theorem 3.7. *For all integers $n \geqslant 0$,*

$$\overline{CO}_{3,1}(12n+6) \equiv 0 \pmod{6}, \tag{3.104}$$

$$\overline{CO}_{3,1}(12n+10) \equiv 0 \pmod{6}. \tag{3.105}$$

Proof. By the binomial theorem, it is easy to see that for positive integers k and m,

$$f_{3k}^m \equiv f_k^{3m} \pmod{3}. \tag{3.106}$$

Using (3.106) in (3.83), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+2)q^n \equiv 2\frac{f_6^4}{f_3^4}. \tag{3.107}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} from (3.107), we arrive at (3.104) and (3.105). ∎

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