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To Professor Włodzimierz Staś on his 75th birthday

ESSENTIAL NORMS OF TOEPLITZ OPERATORS ON BERGMAN-HARDY SPACES ON THE UNIT DISK

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Abstract: For a Borel measure μ on $[0,1]$ with $1 \in \operatorname{supp}(\mu)$ we consider the Toeplitz operators T_f with $f \in L^\infty(\mu \otimes \lambda)$ on the space $H^2(\mu)$ consisting of all holomorphic on $\mathbb{D}$ functions in the Lebegue class $L^2(\mu \otimes \lambda)$ on $\overline{\mathbb{D}}$. We show that for every Bergman-Hardy space $H^2(\mu)$ the following estimations hold: $\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu))) \geqslant |f(t_0)|$ if f is continuous at point $t_0 \in \mathbb{T}$, and $\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu))) = \sup\{|f(z)| : z \in \mathbb{D}\}$ if f is a bounded harmonic function on $\mathbb{D}$.
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The Bergman-Hardy spaces on the disk are generalization of the weighted Bergman spaces on the disk. Informally we can say that a weighted Bergman space on the disk is generated by the measure which in polar coordinates has the form $(1-r^2)^\alpha r dr \times \lambda$ for some $\alpha > -1$, where dr and λ are the Lebesgue measures on $[0,1]$ and $\mathbb{T}$, respectively. A Bergman-Hardy space $H^p(\mu)$ on the disk is generated by the measure of the form $\mu \times \lambda$ in polar coordinates, where μ is an arbitrary positive finite Borel measure on $[0,1]$ which do not vanish near 1. We do not have explicit formula for the Bergman kernel as well as for a reproducing kernel for all Bergman-Hardy spaces $H^2(\mu)$, and we can not apply the Berezin transform techniques (see [9], [8]) for these spaces. Our approach is more directly, we apply sequences of functions in $H^2(\mu)$ which are closely related to the Cauchy and Poisson kernels in $\mathbb{D}$. We show that the classical results (see [1], [9]) for Bergman and weighted Bergman spaces on the disk concerning essential norms of Toeplitz operators T_f, when f has a point of continuity on the boundary or f is harmonic, hold for every Bergman-Hardy space $H^2(\mu)$.

The paper is divided into two sections. The terminology and basic facts are explained in the first. We gather in this section basic properties of Bergman-Hardy spaces $H^p(\mu)$ for $0 < p < \infty$. In the second section we estimate the

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essential norms of Toeplitz operators T_f if f has a point of continuity on the boundary $\mathbb{T}$ or is harmonic.

1. Preliminaries

We start by explaining basic notation used in this paper. As usual, $\mathbb{D}$ will stand for the open unit disk and $\mathbb{T}$ for the unit circle in the complex plain $\mathbb{C}$. The Lebesgue measures on $[0,1]$ and $\mathbb{T}$ will be denoted by dr and λ, respectively. Throughout the paper μ will be a positive Borel measure on $[0,1]$ with $1 \in \operatorname{supp}(\mu)$ (the support of μ is the smallest closed set $C \subset [0,1]$ such that $\mu(C) = \mu([0,1])$). By $\mu \otimes \lambda$ we will denote the Borel measure on $\overline{\mathbb{D}}$ given by $\mu \otimes \lambda(A) = \mu \times \lambda(\Phi^{-1}(A))$ where $\Phi : [0,1] \times \mathbb{T} \to \overline{\mathbb{D}}$ is given by $\Phi((r,t)) = rt$. The norm of an element $f \in L^\infty(\mu \otimes \lambda)$ will be denoted by $\|f\|_\infty$. The space of all continuous linear operators on a given Hilbert space H taking values in H equipped with the operator norm will be denoted by $\mathcal{L}(H)$ and its subspace consisting of all compact operators by $\mathcal{K}(H)$. The essential norm of an operator $T \in \mathcal{L}(H)$ is its distance to compact operators,

$$\operatorname{dist}(T, \mathcal{K}(H)) = \inf\{\|T - S\| : S \in \mathcal{K}(H)\}.$$

For a positive finite Borel measure μ on $[0,1]$ with $1 \in \operatorname{supp}(\mu)$ and $0 < p < \infty$, we denote by $H^p(\mu)$ the space of all holomorphic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\| = \left(\int_{[0,1)} \int_{\mathbb{T}} |f(rt)|^p \, d\lambda(t) d\mu(r) + \mu(\{1\}) \sup_{0<r<1} \int_{\mathbb{T}} |f(rt)|^p \, d\lambda(t)\right)^{1/p} < \infty.$$

equipped with the norm (when $1 \leqslant p < \infty$) and quasi-norm (when $0 < p < 1$) defined above. There are many classical examples of spaces of holomorphic functions on $\mathbb{D}$ that are Bergman-Hardy spaces. For example: $H^p(\delta_1)$ is the classical Hardy space $H^p(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ (δ_1 is the Dirac measure on $[0,1]$ concentrated at 1), the space $H^p(r\,dr)$ is the classical Bergman space of holomorphic functions on $\mathbb{D}$, the space $H^p((\alpha+1)(1-r^2)^\alpha r\,dr)$ for some $\alpha > -1$ is the classical weighted Bergman space of holomorphic functions on $\mathbb{D}$. The definition above seems to be artificial, but it generates spaces consisting of holomorphic functions on $\mathbb{D}$ which are closed subspaces of $L^p(\mu \otimes \lambda)$ with a dense subset of polynomials. In order to check these facts we will need

Proposition 1. *Let μ be a positive finite Borel measure on $[0,1]$ such that $1 \in \operatorname{supp}(\mu)$ and $0 < p < \infty$. For every $f \in H^p(\mu)$ and $z \in \mathbb{D}$*

$$|f(z)| \leqslant \begin{cases} \left(\dfrac{4}{(1-|z|)\mu([(1+|z|)/2, 1])}\right)^{1/p} \|f\| & \text{if } \mu(\{1\}) = 0 \\ \left(\dfrac{4}{(1-|z|)\mu(\{1\})}\right)^{1/p} \|f\| & \text{if } \mu(\{1\}) > 0 \end{cases}$$

Proof. Applying the fact that $|f|^p$ is a subharmonic function we get

$$|f(z)|^p \leqslant \int_{\mathbb{T}} \frac{R^2 - |z|^2}{|Rt - z|^2} |f(Rt)|^p \, d\lambda(t) \leqslant \frac{R + |z|}{R - |z|} \int_{\mathbb{T}} |f(Rt)|^p \, d\lambda(t)$$

for every $|z| < R < 1$. Therefore, if $\mu(\{1\}) = 0$,

$$\begin{aligned} |f(z)|^p &\leqslant \frac{4}{(1 - |z|)\mu([(1 + |z|)/2, 1])} \int_{[(1+|z|)/2,1]} \int_{\mathbb{T}} |f(rt)|^p d\lambda(t) \, d\mu(r) \\ &\leqslant \frac{4 \, \|f\|^p}{(1 - |z|)\mu([(1 + |z|)/2, 1])}. \end{aligned}$$

If $\mu(\{1\}) > 0$,

$$|f(z)|^p \leqslant \frac{4 \, \|f\|^p}{(1 - |z|)\mu(\{1\})}.$$

■

From the proposition follows that the topology of $H^p(\mu)$ is stronger then the topology of uniform convergence on compact subsets of $\mathbb{D}$. Then spaces $H^p(\mu)$ are Banach for $1 \leqslant p < \infty$ and p-Banach for $0 < p < 1$. If $\mu(\{1\}) = 0$, then $H^p(\mu)$ is a closed subspace of $L^p(\mu \otimes \lambda)$. If $\mu(\{1\}) > 0$, then $H^p(\mu)$ coincides with $H^p(\mathbb{D})$, moreover the norms $\|\cdot\|_{H^p(\mu)}$ and $\|\cdot\|_{H^p(\mathbb{D})}$ are equivalent. For every f from the Hardy class $H^p(\mathbb{D})$

$$\sup_{0<r<1} \int_{\mathbb{T}} |f(rt)|^p \, d\lambda(t) = \int_{\mathbb{T}} |f^*(t)|^p \, d\lambda(t)$$

where f^* is the radial limit function of f. If $\mu(\{1\}) > 0$, we will identify $f \in H^p(\mu)$ with the function $\tilde{f}$ on $\overline{\mathbb{D}}$ given by $\tilde{f}|_{\mathbb{D}} = f$ and $\tilde{f}|_{\mathbb{T}} = f^*$. Then the norm ($p$-norm) in $H^p(\mu)$ can be evaluate applying the following formula

$$\|f\| = \left(\int_{\overline{\mathbb{D}}} |f|^p \, d\mu \otimes \lambda \right)^{\frac{1}{p}}.$$

Hence, $H^p(\mu)$ is a closed subspace of $L^p(\mu \otimes \lambda)$ also if $\mu(\{1\}) > 0$. The space $L^p(\mu \otimes \lambda)$ is separable for every $0 < p < \infty$, it is a straightforward consequence of the Lusin theorem and the fact that the space $C(\overline{\mathbb{D}})$ of all complex continuous functions on $\overline{\mathbb{D}}$ is separable. Therefore the space $H^p(\mu)$ is separable for every μ. For every $t \in \mathbb{T}$ and $f \in H^p(\mu)$, $\|f\| = \|f_t\|$ where $f_t(z) = f(zt)$. By standard arguments concerning properties of translations (see [3], [7]) polynomials are dense in $H^p(\mu)$ for every μ. Furthermore, the closed unit ball $B_{H^p(\mu)}$ of $H^p(\mu)$ is compact in the topology of uniform convergence on compact subsets of $\mathbb{D}$. Consequently, the weak topology and the topology of uniform convergence on compact subsets of $\mathbb{D}$ coincides on $B_{H^p(\mu)}$ for every $1 < p < \infty$.

2. The main results

The space $H^2(\mu)$ is a closed subspace of the Hilbert space $L^2(\mu\otimes\lambda)$. Let P be the orthogonal projection of $L^2(\mu\otimes\lambda)$ onto $H^2(\mu)$. For every $f\in L^\infty(\mu\otimes\lambda)$ the Toeplitz operator $T_f : H^2(\mu)\to H^2(\mu)$ is given by

$$T_f(h) = P(fh).$$

Since P is orthogonal, for every $g,h\in H^2(\mu)$

$$\langle g\,,\,T_f(h)\rangle = \langle P^*(g)\,,\,fh\rangle = \langle g\,,\,fh\rangle.$$

In the sequel we will also need the following elementary

Fact 2. *For every $\pi\geqslant\delta>0$ and $R>r>0$*

$$\int_{-\delta}^{\delta}\frac{dt}{|R-re^{it}|^2} = \frac{4}{R^2-r^2}\operatorname{arctg}\left(\frac{R+r}{R-r}\operatorname{tg}\left(\frac{\delta}{2}\right)\right).$$

Now we are ready to state our main result.

Theorem 3. *Let μ be a positive finite Borel measure on $[0,1]$ such that $1\in\operatorname{supp}(\mu)$. If $f\in L^\infty(\mu\otimes\lambda)$ is continuous at point $t_0\in\mathbb{T}$, then*

$$\operatorname{dist}(T_f,\mathcal{K}(H^2(\mu)))\geqslant|f(t_0)|.$$

Proof. We can assume that $t_0=1$ and $f(1)>0$. Let us take any $\varepsilon>0$. Since f is continuous at 1, there exists an open neighborhood U of 1 such that $|f(rt)-f(1)|<\varepsilon$ for every $rt\in U$. Hence, there exists $\delta\in(0,1)$ such that $re^{it}\in U$ for every $t\in[-\delta,\delta]$ and $r\in[1-\delta,1]$. Let

$$g_n(z) = \frac{z^n}{1+\delta/n-z}.$$

Then

$$\|g_n\|^2 = \int_0^1\int_{\mathbb{T}}\frac{r^{2n}}{|1+\delta/n-rt|^2}\,d\lambda(t)\,d\mu(r) = \int_0^1\frac{r^{2n}}{(1+\delta/n)^2-r^2}\,d\mu(r).$$

Since $r\leqslant1$ and $(1+\delta/n)^2-r^2\geqslant2\delta/n$ for $0\leqslant r\leqslant1$,

$$\|g_n\|^2\leqslant n\,\frac{\mu([0,1])}{2\,\delta}$$

On the other hand, for every $0<r_0<1$

$$\|g_n\|^2\geqslant\frac{r_0^{2n}}{4}\,\mu([r_0,1]).$$

It follows that $(g_n/\|g_n\|)$ is a null sequence in the topology of uniform convergence on compact subsets of $\mathbb{D}$. Consequently, it converges to zero in the weak topology of $H^2(\mu)$. Let us introduce the following numbers

$$\alpha_{n,l} = \int_{1-\delta/l}^{1} \frac{1}{2\pi} \int_{-\delta}^{\delta} \frac{r^{2n}}{|1+\delta/n - re^{it}|^2}\, dt\, d\mu(r)$$
$$= \frac{2}{\pi} \int_{1-\delta/l}^{1} \frac{r^{2n}}{(1+\delta/n)^2 - r^2} \operatorname{arctg}\left(\frac{1+\delta/n+r}{1+\delta/n-r} \operatorname{tg}\left(\frac{\delta}{2}\right)\right) d\mu(r),$$

and

$$\beta_{n,l} = \frac{2}{\pi} \operatorname{arctg}\left(\frac{1+\delta/n+1-\delta/l}{1+\delta/n-(1-\delta/l)} \operatorname{tg}\left(\frac{\delta}{2}\right)\right).$$

Then

$$\left|\left\langle \frac{g_n}{\|g_n\|}, T_f\left(\frac{g_n}{\|g_n\|}\right)\right\rangle\right|$$
$$= \left| \frac{1}{\|g_n\|^2} \int_0^1 \int_{\mathbb{T}} f\,|g_n|^2(rt)\, d\lambda(t)\, d\mu(r)\right|$$
$$= \frac{1}{2\pi\,\|g_n\|^2} \left| \int_{1-\delta/l}^{1} \int_{-\delta}^{\delta} + \left(\int_0^1 \int_{-\pi}^{\pi} - \int_{1-\delta/l}^{1} \int_{-\delta}^{\delta} \right) f\,|g_n|^2(re^{it})\, dt\, d\mu(r) \right|$$
$$\geqslant \frac{1}{\|g_n\|^2} \big((f(1)-\varepsilon)\alpha_{n,l} - \|f\|_\infty (\|g_n\|^2 - \alpha_{n,l}) \big)$$
$$= (f(1) + \|f\|_\infty - \varepsilon) \frac{\alpha_{n,l}}{\|g_n\|^2} - \|f\|_\infty .$$

Applying the fact that functions $r \to r^{2n}/((1+\delta/n)^2 - r^2)$ and $r \to \operatorname{arctg}((1+\delta/n+r)/(1+\delta/n-r)\operatorname{tg}(\delta/2))$ are increasing on $[0,1]$ we get

$$\|g_n\|^2 = \frac{1}{2\pi} \int_{1-\delta/l}^{1} \int_{-\delta}^{\delta} + \int_{1-\delta/l}^{1} \int_{\delta\leqslant|t|\leqslant\pi} + \int_0^{1-\delta/l} \int_{-\pi}^{\pi} |g_n(re^{it})|^2\, dt\, d\mu(r)$$
$$\leqslant \alpha_{n,l} + \int_{1-\delta/l}^{1} \frac{r^{2n}(1-\beta_{n,l})}{(1+\delta/n)^2 - r^2}\, d\mu(r) \ + \frac{(1-\delta/l)^{2n} \mu([0,1])}{((1+\delta/n)^2 - (1-\delta/l)^2)}$$
$$\leqslant \alpha_{n,l} + (1-\beta_{n,l}) \frac{\alpha_{n,l}}{\beta_{n,l}}$$
$$+ \frac{(1-\delta/l)^{2n} \mu([0,1])}{((1+\delta/n)^2 - (1-\delta/l)^2)} \frac{\alpha_{n,l}}{\int_{1-\delta/2l}^{1} r^{2n} \beta_{n,l}/((1+\delta/n)^2 - r^2)\, d\mu(r)}$$
$$\leqslant \frac{\alpha_{n,l}}{\beta_{n,l}} \left(1 + \left(\frac{1-\delta/l}{1-\delta/2l}\right)^{2n} \frac{\mu([0,1])}{\mu([1-\delta/2l, 1])}\right).$$

Hence for every $l \in \mathbb{N}$

$$\lim_n \frac{\alpha_{n,l}}{\|g_n\|^2} \geqslant \lim_n \frac{(2/\pi)\operatorname{arctg}\big(((2+\delta/n-\delta/l)/(\delta/n+\delta/l))\operatorname{tg}(\delta/2)\big)}{1 + ((1-\delta/l)/(1-\delta/2l))^{2n} \mu([0,1])/\mu([1-\delta/2l,1])}$$
$$= \frac{2}{\pi} \operatorname{arctg}\left(\frac{2l-\delta}{\delta} \operatorname{tg}\left(\frac{\delta}{2}\right)\right).$$

Since the sequence $(g_n/\|g_n\|)$ is weakly null in $H^2(\mu)$,

$$\lim_n K\left(\frac{g_n}{\|g_n\|}\right) = 0 \quad \text{for every compact operator } K \in \mathcal{K}(H^2(\mu)).$$

Hence

$$\begin{aligned}\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu)) &\geqslant \limsup_n \left\|T_f\left(\frac{g_n}{\|g_n\|}\right)\right\| \geqslant \limsup_n \left|\left\langle \frac{g_n}{\|g_n\|}, T_f\left(\frac{g_n}{\|g_n\|}\right)\right\rangle\right| \\ &\geqslant (f(1) + \|f\|_\infty - \varepsilon)\,\frac{2}{\pi}\operatorname{arctg}\left(\frac{2l-\delta}{\delta}\operatorname{tg}\left(\frac{\delta}{2}\right)\right) - \|f\|_\infty.\end{aligned}$$

Since the estimation above holds for every $l \in \mathbb{N}$,

$$\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu))) \geqslant f(1) - \varepsilon$$

To finish the proof is enough to note that $\varepsilon > 0$ we took arbitrarily. ∎

In the sequel we will need the following well known

Fact 4. *For every* $T \in \mathcal{L}(H^2(\mu))$

$$\begin{aligned}\operatorname{dist}(T, \mathcal{K}(H^2(\mu))) &= \sup\{\limsup_n \|T(h_n)\| : (h_n) \subset B_{H^2(\mu)} \textit{ is weakly null}\} \\ &= \lim_n \|T - T \circ P_n\|,\end{aligned}$$

where P_n *is the orthogonal projection of* $H^2(\mu)$ *onto the linear span of* $e_1, \dots, e_n$ *for some orthonormal Schauder basis* (e_n) *in* $H^2(\mu)$.

Proof. Since

$$\|T - T \circ P_{n+1}\| = \|T \circ (Id - P_n) \circ (Id - P_{n+1})\| \leqslant \|T - T \circ P_n\|,$$

the sequence $(\|T - T \circ P_n\|)$ is decreasing. The inequality $\operatorname{dist}(T, \mathcal{K}(H^2(\mu))) \leqslant \lim_n \|T - T \circ P_n\|$ is clear. If $(h_n) \subset B_{H^2(\mu)}$ is a weakly null sequence in $H^2(\mu)$, then

$$\limsup_n \|T(h_n)\| = \limsup_n \|(T - K)(h_n)\| \leqslant \|T - K\|$$

for every $K \in \mathcal{K}(H^2(\mu))$. Hence

$$\sup\{\limsup_n \|T(h_n)\| : (h_n) \subset B_{H^2(\mu)} \text{ is weakly null}\} \leqslant \operatorname{dist}(T, \mathcal{K}(H^2(\mu))).$$

Let us select $h_n \in P_n^{-1}(\{0\}) \cap B_{H^2(\mu)} = (Id - P_n)(B_{H^2(\mu)})$ such that

$$\|(T - T \circ P_n)(h_n)\| \geqslant \|T - T \circ P_n\| - \frac{1}{n}.$$

Then (h_n) is a weakly null sequence in $H^2(\mu)$ and

$$\limsup_n \|T(h_n)\| = \limsup_n \|(T - T \circ P_n)(h_n)\| \geqslant \lim_n \|T - T \circ P_n\|.$$

∎

Straightforward consequence of the theorem and the fact above is the following

Corollary 5. *Let μ be a positive finite Borel measure on $[0,1]$ such that $1 \in \operatorname{supp}(\mu)$. If $f \in L^\infty(\mu \otimes \lambda)$ is continuous at each point of $\mathbb{T}$, then*

$$\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu))) = \sup_{t\in\mathbb{T}} |f(t)|.$$

Proof. In view of Theorem 3

$$\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu)) \geqslant \sup_{t\in\mathbb{T}} |f(t)|.$$

On the other hand, for every $\varepsilon > 0$ there exists $R \in (0,1)$ such that $\sup\{|f(z)| : z \in \mathbb{D} \setminus R\,\mathbb{D}\} < \varepsilon + \sup\{|f(t)| : t \in \mathbb{T}\}$. Let $(h_n) \subset B_{H^2(\mu)}$ be a weakly null sequence. Since (h_n) converges uniformly to zero on $R\,\mathbb{D}$, for every $(g_n) \subset B_{H^2(\mu)}$

$$\begin{aligned}\limsup_n |\langle g_n, f\,h_n\rangle| &\leqslant \limsup_n \Big(\int_{R\,\mathbb{D}} |f\,h_n\,g_n|\,d\mu\otimes\lambda \\ &\qquad + (\varepsilon + \sup_{t\in\mathbb{T}}|f(t)|) \int_{\mathbb{D}\setminus R\,\mathbb{D}} |h_n g_n|\,d\mu\otimes\lambda \Big) \\ &\leqslant \varepsilon + \sup_{t\in\mathbb{T}} |f(t)|.\end{aligned}$$

Hence

$$\operatorname{dist}(T_f, \mathcal{K}(H^2(\mu))) \leqslant \sup_{t\in\mathbb{T}} |f(t)|.$$

■

The corollary show that every bounded Borel function f on $\overline{\mathbb{D}}$ with compact support in $\mathbb{D}$ generates the compact Toeplitz operator T_f in each Bergman-Hardy space $H^2(\mu)$. The next result generalizes the following well known fact for Toeplitz operators on the Hardy space H^2 on $\mathbb{T}$: for every $f \in L^\infty(\lambda)$

$$\operatorname{dist}(T_f, \mathcal{K}(H^2)) = \|f\|_\infty.$$

Recall that for $f \in L^1(\lambda)$ its Poisson integral $\mathbb{P}(f) : \mathbb{D} \to \mathbb{C}$ is given by

$$\mathbb{P}(f)(z) = \int_{\mathbb{T}} \frac{1-|z|^2}{|t-z|^2} f(t)\,d\lambda(t).$$

Every bounded harmonic function F on $\mathbb{D}$ is the Poisson integral of its radial limit function F^*, i.e. $F = \mathbb{P}(F^*)$, and $\sup\{|F(z)| : z \in \mathbb{D}\} = \sup\operatorname{ess}\{|F^*(t)| : t \in \mathbb{T}\}$ (see [2]).

Theorem 6. *Let μ be a positive Borel measure on $[0,1]$ such that $1 \in \operatorname{supp}(\mu)$. If $f \in L^\infty(\lambda)$, then*

$$\operatorname{dist}(T_{\mathbb{P}(f)}, \mathcal{K}(H^2(\mu))) = \|f\|_\infty.$$

Proof. Since $|\mathbb{P}(f)| \leqslant \|f\|_\infty$ $\mu \otimes \lambda$-a.e. on $\overline{\mathbb{D}}$, $\operatorname{dist}(T_{\mathbb{P}(f)}, \mathcal{K}(H^2(\mu))) \leqslant \|f\|_\infty$. Then we only have to show the inequality in the other direction. Since $\mathbb{P}(f)$ is a bounded and harmonic function on $\mathbb{D}$, the limit

$$\lim_{r\to 1} \mathbb{P}(f)(rt) = \mathbb{P}(f)^* \quad \text{exists for } \lambda\text{-a.e. } t \in \mathbb{T}.$$

This limit exists for every Lebesgue point of f (see [2]). Let us take any $\|f\|_\infty > \varepsilon > 0$ and any point t_0 for which the limit above exists and $|\mathbb{P}(f)^*(t_0)| > \|f\|_\infty - \varepsilon$. We can assume that $\mathbb{P}(f)^*(t_0) > 0$. Then there exists $\delta > 0$ such that $|\mathbb{P}(f)(rt_0) - \mathbb{P}(f)^*(t_0)| < \varepsilon$ for every $r \in [(1-\delta)^3, 1)$. Let

$$g_n(z) = \frac{z^n}{1+\delta/n - z\bar{t_0}}.$$

Then

$$\|g_n\|^2 = \int_0^1 \int_{\mathbb{T}} \frac{r^{2n}}{|(1+\delta/n) - rt\bar{t_0}|^2}\, d\lambda(t)\, d\mu(r) = \int_0^1 \frac{r^{2n}}{(1+\delta/n)^2 - r^2}\, d\mu(r).$$

We know from the proof of Theorem 3 that $(g_n/\|g_n\|)$ is a weakly null sequence in $H^2(\mu)$. Let

$$\alpha_n = \int_{1-\delta}^1 \int_{\mathbb{T}} \frac{r^{2n}}{|(1+\delta/n) - rt\bar{t_0}|^2}\, d\lambda(t)\, d\mu(r) = \int_{1-\delta}^1 \frac{r^{2n}}{(1+\delta/n)^2 - r^2}\, d\mu(r)$$

Then

$$\begin{aligned}
&\left|\left\langle \frac{g_n}{\|g_n\|}, T_{\mathbb{P}(f)}\left(\frac{g_n}{\|g_n\|}\right)\right\rangle\right| \\
&= \frac{1}{2\pi\|g_n\|^2}\left|\int_{1-\delta}^1 \int_{-\pi}^{\pi} + \int_0^{1-\delta} \int_{-\pi}^{\pi} (\mathbb{P}(f)\,|g_n|^2)(re^{it})\ dt\, d\mu(r)\right| \\
&\geqslant \frac{1}{\|g_n\|^2}\Bigg(\left|\int_{1-\delta}^1 \frac{r^{2n}}{(1+\delta/n)^2 - r^2}\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1+\delta/n)^2 - r^2}{|(1+\delta/n) - rt\bar{t_0}|^2}\mathbb{P}(f)(re^{it})\, dt\ d\mu(r)\right| \\
&\qquad\qquad - \|f\|_\infty(\|g_n\|^2 - \alpha_n)\Bigg) \\
&\geqslant \frac{1}{\|g_n\|^2}\left(\left|\int_{1-\delta}^1 \frac{r^{2n}}{(1+\delta/n)^2 - r^2}\mathbb{P}(f)\left(\frac{r^2}{1+\delta/n}t_0\right) d\mu(r)\right| - \|f\|_\infty(\|g_n\|^2 - \alpha_n)\right) \\
&\geqslant \frac{1}{\|g_n\|^2}\big((\mathbb{P}(f)^*(t_0) - \varepsilon)\alpha_n - \|f\|_\infty(\|g_n\|^2 - \alpha_n)\big) \\
&= (\mathbb{P}(f)^*(t_0) + \|f\|_\infty - \varepsilon)\frac{\alpha_n}{\|g_n\|^2} - \|f\|_\infty.
\end{aligned}$$

Furthermore

$$\begin{aligned}\|g_n\|^2 &= \frac{1}{2\pi}\int_{1-\delta}^{1}\int_{-\pi}^{\pi} + \int_{0}^{1-\delta}\int_{-\pi}^{\pi} |g_n(re^{it})|^2\,dt\,d\mu(r)\\ &\leqslant \alpha_n + \frac{(1-\delta)^{2n}\mu([0,1])}{((1+\delta/n)^2-(1-\delta)^2)}\,\frac{\alpha_n}{\int_{1-\delta/2}^{1} r^{2n}/((1+\delta/n)^2-r^2)\,d\mu(r)}\\ &\leqslant \alpha_n\left(1+\left(\frac{1-\delta}{1-\delta/2}\right)^{2n}\frac{\mu([0,1])}{\mu([1-\delta/2,1])}\right)\end{aligned}$$

Hence

$$\lim_n \frac{\alpha_n}{\|g_n\|^2} \geqslant \lim_n \frac{1}{1+\left((1-\delta)/(1-\delta/2)\right)^{2n}\mu([0,1])/\mu([1-\delta/2,1])} = 1$$

Since $(g_n/\|g_n\|)$ converges weakly to zero in $H^2(\mu)$,

$$\begin{aligned}\operatorname{dist}(T_f,\mathcal{K}(H^2(\mu))) &\geqslant \limsup_n \left\|T_f\left(\frac{g_n}{\|g_n\|}\right)\right\| \geqslant \limsup_n \left|\left\langle \frac{g_n}{\|g_n\|}, T_{\mathbb{P}(f)}\left(\frac{g_n}{\|g_n\|}\right)\right\rangle\right|\\ &\geqslant \mathbb{P}(f)^*(t_0)-\varepsilon \geqslant \|f\|_\infty - 2\varepsilon.\end{aligned}$$

To finish the proof is enough to note that $\varepsilon > 0$ we took arbitrarily. ■

W. Lusky [4] showed that a similar fact holds for the angular extension $\tilde{f}$ of $f \in L^\infty(\lambda)$,

$$\operatorname{dist}(T_{\tilde{f}},\mathcal{K}(H^2(\mu))) = \|f\|_\infty.$$

where $\tilde{f}(rt) = f(t)$ for every $r \in [0,1)$ and $t \in \mathbb{T}$.

For a bounded holomorphic function f on $\mathbb{D}$ the multiplication operator $M_f : H^2(\mu) \to H^2(\mu)$ is given by

$$M_f(g) = f\,g = T_f(g).$$

Multiplication operators on $H^2(\mu)$ form a subclass of Toeplitz operators. Straightforward consequence of the theorem above is the following

Corollary 7. *Let μ be a positive finite Borel measure on $[0,1]$ such that $1 \in \operatorname{supp}(\mu)$. If f is a bounded holomorphic function on $\mathbb{D}$, then*

$$\operatorname{dist}(M_f,\mathcal{K}(H^2(\mu))) = \sup_{z\in\mathbb{D}} |f(z)|.$$

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