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ON THE MOMENTS OF HECKE SERIES AT CENTRAL POINTS II

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Abstract: We prove, in standard notation from spectral theory, the asymptotic formula ($B > 0$)

$$\sum_{\kappa_j \le T} \alpha_j H_j(\tfrac{1}{2}) = \left(\frac{T}{\pi}\right)^2 - BT\log T + O(T(\log T)^{1/2}),$$

by using an approximate functional equation for $H_j(\frac{1}{2})$ and the Bruggeman-Kuznetsov trace formula. We indicate how the error term may be improved to $O(T(\log T)^{\varepsilon})$.
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1. Introduction and statement of results

The purpose of this paper is to continue the work begun by the first author in [6]. Therein he obtained asymptotic formulas for sums of $H_j^3(\frac{1}{2})$ and $H_j^4(\frac{1}{2})$, where $H_j(s)$ is the Hecke series ($s = \sigma + it$ will denote a complex variable)

$$H_j(s) \;=\; \sum_{n=1}^{\infty} t_j(n)n^{-s} \qquad (\sigma > 1), \tag{1.1}$$

associated with the Maass wave form $\psi_j(z)$, where $\rho_j(1)t_j(n) = \rho_j(n)$ and $\rho_j(n)$ is the n-th Fourier coefficient of $\psi_j(z)$. The function $H_j(s)$ can be continued to an entire function. It satisfies the functional equation

$$\begin{aligned} H_j(s) = {}& 2^{2s-1}\pi^{2s-2}\Gamma(1-s+i\kappa_j)\Gamma(1-s-i\kappa_j) \\ & \times (\varepsilon_j\cosh(\pi\kappa_j) - \cos(\pi s))H_j(1-s), \end{aligned} \tag{1.2}$$

where ε_j $(=\pm 1)$ is the so-called parity sign of $\psi_j(z)$. By $\{\lambda_j = \kappa_j^2 + \frac{1}{4}\} \cup \{0\}$ we denote the eigenvalues (discrete spectrum) of the hyperbolic Laplacian

$$\Delta = -y^2\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right)$$

2000 Mathematics Subject Classification: 11F72, 11F66, 11M41, 11M06 acting over the Hilbert space composed of all Γ-automorphic functions which are square integrable with respect to the hyperbolic measure ($\Gamma = \mathrm{PSL}(2,\mathbb{Z})$). For other relevant notation involving spectral theory the reader is referred to [5], [6] or Y. Motohashi's comprehensive monograph [12]. The method used in [6] could not furnish the asymptotic formula for sums of $H_j(\frac{1}{2})$, but only the bounds

$$T^2(\log T)^{-7/2} \ll \sum_{\kappa_j \le T} \alpha_j H_j(\tfrac{1}{2}) \ll T^2(\log T)^{1/2} \tag{1.3}$$

were obtained, where as usual we set

$$\alpha_j = |\rho_j(1)|^2(\cosh \pi\kappa_j)^{-1}.$$

The aim of this paper is to improve (1.3) to a sharp asymptotic formula, given by

Theorem 1. *We have*

$$\sum_{\kappa_j \le T} \alpha_j H_j(\tfrac{1}{2}) + \frac{2}{\pi}\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^2}{|\zeta(1+2it)|^2}\,dt = \left(\frac{T}{\pi}\right)^2 + O(T(\log T)^{1/2}). \tag{1.4}$$

It remains yet to evaluate the weighted integral of the mean square of $|\zeta(\frac{1}{2}+it)|$ in (1.4). The evaluation of this integral is given by

Theorem 2. *There exist constants $A\,(>0)$ and B which are effectively computable such that*

$$\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^2}{|\zeta(1+2it)|^2}\,dt = T(A\log T + B) + O_\varepsilon(T^{\frac{33}{35}+\varepsilon}). \tag{1.5}$$

Corollary. *If A is the constant appearing in (1.5), then*

$$\sum_{\kappa_j \le T} \alpha_j H_j(\tfrac{1}{2}) = \left(\frac{T}{\pi}\right)^2 - \frac{2A}{\pi}T\log T + O(T(\log T)^{1/2}). \tag{1.6}$$

In (1.5) and later ε denotes positive, arbitrarily small constants, not necessarily the same ones at each occurrence. The formula (1.6) shows that there are actually two main terms in the asymptotic formula for the sum of $\alpha_j H_j(\frac{1}{2})$. Although the error term in (1.6) is probably too large by a factor of $\sqrt{\log T}$, the method of proof of Theorem 1 does not allow any further improvement, if we use the weight function (2.14). However, by a suitable choice of the weight function the error terms in (1.4), (1.6) (and (1.7)) may be improved to $O(T(\log T)^\varepsilon)$. We preferred to work directly with the Gaussian weight function (2.14) because of its classical flavour. This already leads to (1.6) with two main terms, which is the novelty of the paper.

It may be remarked that, with our method of proof, we can obtain the asymptotic formula

$$\sum_{\kappa_j \le T} \alpha_j = \left(\frac{T}{\pi}\right)^2 + O(T(\log T)^{1/2}). \tag{1.7}$$

This should be compared to a result of N.V. Kuznetsov (see [12, p. 92] with $m = 1$), who had (1.7) with the error term $O(T \log T)$, so that our result is somewhat sharper.

In what concerns the true order of sums of $\alpha_j H_j^k(\frac{1}{2})$, it was conjectured in [6] that, for $k \in \mathbb{N}$ fixed,

$$\sum_{\kappa_j \le T} \alpha_j H_j^k(\tfrac{1}{2}) + \frac{2}{\pi}\int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2k}}{|\zeta(1+2it)|^2}\,dt = T^2 P_{\frac{1}{2}(k^2-k)}(\log T) + O_{\varepsilon,k}(T^{1+c_k+\varepsilon}), \tag{1.8}$$

where $P_{\frac{1}{2}(k^2-k)}(z)$ is a suitable polynomial of degree $\frac{1}{2}(k^2-k)$ in z whose coefficients depend on k, and $0 \le c_k < 1$. We actually have $c_1 = c_2 = 0$, and even sharper results in these cases by (1.6) and Y. Motohashi's result [11], respectively. Namely he proved the asymptotic formula ($\gamma = 0.5772157\ldots$ is Euler's constant)

$$\sum_{\kappa_j \le T} \alpha_j H_j^2(\tfrac{1}{2}) = 2\pi^{-2}T^2(\log T + \gamma - \tfrac{1}{2} - \log(2\pi)) + O(T\log^6 T),$$

while the proofs in [6], in the cases $k = 3, 4$, show that (1.8) holds with $c_3 = 1/7, c_4 = 1/3$. We also note that the main term in Theorem 1, namely $(T/\pi)^2$, is exactly of the form predicted by Random matrix theory (see J.B. Conrey [1] and the work by J.B. Conrey et al. [2]). This theory also gives the correct value of the leading coefficient of the polynomial $P_{\frac{1}{2}(k^2-k)}(z)$ for the cases $k = 2, 3, 4$, when the asymptotic formulas for the sums in question are known.

Our method of proof consists of using the Bruggeman-Kuznetsov trace formula (cf. Lemma 1), coupled with a simple approximate functional equation for $H_j(\frac{1}{2})$ (of length $\asymp \kappa_j^2$) for Theorem 1 (cf. Lemma 2). This is proved in Section 2, which contains the necessary lemmas. The crucial lemma is Lemma 3, which shows that, in our case, the contribution of the Kloosterman sum part in the trace formula is negligible. Theorem 1 is proved in Section 3, and Theorem 2 in Section 4. Finally in Section 5 we discuss how the error terms in (1.4), (1.6) and (1.7) may be improved to $O(T(\log T)^{\varepsilon})$.

2. The necessary lemmas

Lemma 1. (The first Bruggeman-Kuznetsov trace formula). *Let $f(r)$ be an even, regular function for $|\Im\mathrm{m}\, r| \le \frac{1}{2}$ such that $f(r) \ll (1+|r|)^{-2-\delta}$ for some $\delta > 0$.*

Then

$$\sum_{j=1}^{\infty}\alpha_j t_j(m)t_j(n)f(\kappa_j)+\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sigma_{2ir}(m)\sigma_{2ir}(n)}{(mn)^{ir}|\zeta(1+2ir)|^2}f(r)\,\mathrm{d}r$$
$$=\frac{1}{\pi^2}\delta_{m,n}\int_{-\infty}^{\infty}r\tanh(\pi r)f(r)\,\mathrm{d}r+\sum_{\ell=1}^{\infty}\frac{1}{\ell}S(m,n;\ell)f_+\left(\frac{4\pi\sqrt{mn}}{\ell}\right), \tag{2.1}$$

where $\delta_{m,n}=1$ *if* $m=n$ *and zero otherwise* $(m,n>0)$, $\sigma_a(d)=\sum_{d|n}d^a$, $S(m,n;\ell)$ *is the Kloosterman sum and*

$$f_+(x)=\frac{2i}{\pi}\int_{-\infty}^{\infty}\frac{r}{\cosh(\pi r)}J_{2ir}(x)f(r)\,\mathrm{d}r. \tag{2.2}$$

The J-Bessel function is defined (see e.g., N.N. Lebedev [9]) as

$$J_\nu(z)=\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{\nu+2k}}{\Gamma(k+1)\Gamma(k+\nu+1)}\qquad(|\arg z|<\pi). \tag{2.3}$$

The proof of Lemma 1 is to be found e.g., in Y. Motohashi [12, Chapter 2].

Lemma 2. *Let* $\kappa_j=(1+o(1))K$, $r=(1+o(1))K\quad(r\in\mathbb{R})$ *as* $K\to\infty, Y=(1+\delta)\frac{K^2}{4\pi^2}$, *with* $\delta>0$ *a given constant. Then, for any fixed positive constant* $A>0$, *there exists a constant* $C=C(A,\delta)>0$ *such that, for* $h=C\log K$, *we have*

$$H_j(\tfrac12)=\sum_{n\le(1+\delta)Y}t_j(n)n^{-1/2}\mathrm{e}^{-(n/Y)^h}+O(K^{-A}), \tag{2.4}$$

and

$$\zeta(\tfrac12+ir)\zeta(\tfrac12-ir)=\sum_{n\le(1+\delta)Y}\sigma_{2ir}(n)n^{-\frac12-ir}\mathrm{e}^{-(n/Y)^h}+O(K^{-A}). \tag{2.5}$$

Proof. We start from the Mellin inversion integral (see e.g., [4, (A.7)])

$$\mathrm{e}^{-(n/Y)^h}=\frac{1}{2\pi i}\int_{(c)}\left(\frac{Y}{n}\right)^w\Gamma(1+\frac{w}{h})\frac{\mathrm{d}w}{w}\quad(c>0,\ Y\gg1), \tag{2.6}$$

where $\int_{(c)}$ denotes integration over the line $\Re e\,w=c$. We use (1.1) and (see [4, Chapter 1])

$$\zeta(s)\zeta(s-a)=\sum_{n=1}^{\infty}\sigma_a(n)n^{-s}\qquad(\sigma>\max(1,\,1+\Re e\,a)), \tag{2.7}$$

to obtain from (2.6)

$$\sum_{n=1}^{\infty} t_j(n)n^{-1/2}e^{-(n/Y)^h} = \frac{1}{2\pi i}\int_{(1)} H_j(\tfrac12 + w)\Gamma(1+\frac{w}{h})\frac{Y^w}{w}\,dw \tag{2.8}$$

and

$$\begin{aligned}&\sum_{n=1}^{\infty} \sigma_{2ir}(n)n^{-\frac12-ir}e^{-(n/Y)^h}\\ &= \frac{1}{2\pi i}\int_{(1)} \zeta(w+\tfrac12+ir)\zeta(w+\tfrac12-ir)\Gamma(1+\frac{w}{h})\frac{Y^w}{w}\,dw.\end{aligned} \tag{2.9}$$

We shall give only the detailed proof of the more complicated formula (2.4). The proof of (2.5) is analogous, being based on the use of (2.9). The series in (2.8) can be truncated at $n = (1+\delta)Y$ with the error $\ll K^{-A}$. On the right-hand side of (2.8) we replace the line of integration by $\mathcal{L} = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4 \cup \gamma_5$, where γ_1 is the line from $-1-i\infty$ to $-1-ih^2$, γ_2 is the line segment from $-1-ih^2$ to $-\frac12 h - ih^2$, γ_3 is the line segment from $-\frac12 h - ih^2$ to $-\frac12 h + ih^2$, γ_4 is the line segment from $-\frac12 h + ih^2$ to $-1+ih^2$, and γ_5 is the line from $-1+ih^2$ to $-1+i\infty$. In doing this we pass the pole $w=0$ which, by the residue theorem, gives us the desired contribution $H_j(\frac12)$. By the functional equation (1.2) we have

$$H_j(\tfrac12 + w) = X_j(\tfrac12 + w)H_j(\tfrac12 - w) \tag{2.10}$$

with

$$\begin{aligned}X_j(\tfrac12 + w) = (2\pi)^{2w}\pi^{-1}\Gamma(\tfrac12 - w + i\kappa_j)\Gamma(\tfrac12 - w - i\kappa_j)\\ \times(\varepsilon_j\cosh(\pi\kappa_j) + \sin(\pi w)).\end{aligned} \tag{2.11}$$

To bound the gamma factors on $\mathcal{L}$ we use Stirling's formula in the form

$$\Gamma(\sigma + it) \ll |t|^{\sigma-\frac12}e^{-\pi|t|/2} \qquad (|t| \ge t_0), \tag{2.12}$$

which is valid uniformly for $0 \le \sigma \le |t|^{2/3}$. To see this, note that

$$\begin{aligned}&\Re\mathrm{e}\,\{\log\Gamma(\sigma+it) - \log\Gamma(it)\}\\ &= \Re\mathrm{e}\left(\int_0^\sigma \frac{\Gamma'(x+it)}{\Gamma(x+it)}\,dx\right)\\ &= \Re\left\{\int_0^\sigma \left(\log(x+it) - \frac{1}{2(x+it)} + O(\frac{1}{(x+it)^2})\right)dx\right\}\\ &\le \tfrac12\sigma\log(t^2+\sigma^2) + O(\sigma t^{-2}) \le \sigma\log|t| + O((\sigma+\sigma^3)t^{-2}),\end{aligned}$$

hence (2.12) follows from Stirling's formula for $\Gamma(it)$, and can be used to bound the gamma-factors appearing in the expression for $X_j(\frac12 + w)$.

We have first

$$\int_{\gamma_1} H_j(\tfrac12 + w)\Gamma(1+\frac{w}{h})\frac{Y^w}{w}\,dw \ll \int_{h^2}^{\infty}\exp\Big(-\frac{\pi v}{2h}\Big)(K^2+v^2)\,dv \ll K^{-A},$$

if C in the formulation of the lemma is sufficiently large, and an analogous bound holds for the integral over γ_5.

Next, on γ_2 and on γ_4, the integrand is

$$\ll (\kappa_j^2 - h^4)^{-\sigma}(4\pi^2 Y)^{\sigma} \mathrm{e}^{-\pi h/2} \ll \mathrm{e}^{-\pi h/2} \ll K^{-A},$$

so that the corresponding integrals are of the desired order of magnitude.

Finally, on γ_3, the integrand is

$$\ll \kappa_j^h (4\pi^2 Y)^{-h/2} \le \left(\frac{(1+o(1))K^2}{4\pi^2 Y}\right)^{h/2} \le (1+\tfrac{1}{2}\delta)^{-h/2} \le K^{-A}$$

for any fixed $A > 0$. Combining the above bounds we obtain (2.4). ∎

Lemma 3. *For* $C\sqrt{\log K} \le G \le K$ *and a sufficiently large constant* $C > 0$ *we have*

$$\sum_{K \le \kappa_j \le K+G} \alpha_j H_j(\tfrac{1}{2}) \ll GK. \tag{2.13}$$

Proof. First we remark that the slightly weaker bound $GK\sqrt{\log K}$ for the sum in (2.13) follows by applying the Cauchy-Schwarz inequality and the bound for sums of α_j and $\alpha_j H_j^2(\frac{1}{2})$ in short intervals; such bounds are given by Y. Motohashi [12, pp. 121-122 and (3.5.13)].

Secondly, in the proof of Lemma 3 we may restrict G to $G = G_0 = C\sqrt{\log K}$. For larger G we divide $[K, K+G]$ into $\ll G/G_0$ subintervals of length G_0, to each of which we apply (2.13) with suitable K and $G = G_0$. Adding up all the results we arrive at (2.13).

The idea of proof of (2.13) is actually the same as the one that will be used in the proof of Theorem 1, and for the proof of Theorem 1 we need (2.13) only with $G = C\sqrt{\log K_0}, K_0 \le K \le 2K_0$. Lemma 3 is in fact a local version of Theorem 1. Thus let, for $G = C\sqrt{\log K}$,

$$f(r, K) := \frac{(r^2 + \frac{1}{4})}{(r^2 + 1000)} \left\{ \exp\left(-\left(\frac{r-K}{G}\right)^2\right) + \exp\left(-\left(\frac{r+K}{G}\right)^2\right)\right\}. \tag{2.14}$$

This function, which is a Gaussian weight function and a slightly modified function of the function used systematically by Y. Motohashi [11], [12], clearly satisfies the conditions of Lemma 1. To begin the proof, we apply Lemma 1 (taking $n = 1$), combined with Lemma 2, where $\delta > 0$ is a small constant. This yields, since

$H_j(\frac{1}{2}) \geq 0$ (see S. Katok–P. Sarnak [8] for a proof),

$$\begin{aligned}
\sum_{K \le \kappa_j \le K+G} \alpha_j H_j(\tfrac{1}{2}) &\le 2 \sum_{j=1}^{\infty} \alpha_j H_j(\tfrac{1}{2}) f(\kappa_j, K) \\
&= \frac{2}{\pi^2} \int_{-\infty}^{\infty} r \tanh(\pi r) f(r, K)\, \mathrm{d}r - \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+ir)|^2}{|\zeta(1+2ir)|^2} f(r,K)\, \mathrm{d}r \\
&\quad + 2 \sum_{m \le (1+\delta)^2 K^2/(4\pi^2)} m^{-1/2} \mathrm{e}^{-(m/Y)^h} \sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,1;\ell) f_+\left(\frac{4\pi}{\ell}\sqrt{m}\right) + o(1) \\
&\le \frac{2}{\pi^2} \int_{-\infty}^{\infty} r \tanh(\pi r) f(r, K)\, \mathrm{d}r \\
&\quad + 2 \sum_{m \le (1+\delta)^2 K^2/(4\pi^2)} m^{-1/2} \mathrm{e}^{-(m/Y)^h} \sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,1;\ell) f_+\left(\frac{4\pi}{\ell}\sqrt{m}\right) + o(1),
\end{aligned} \tag{2.15}$$

where f_+ is given by (2.2) with $f(r) = f(r, K)$.

We have first

$$\int_{-\infty}^{\infty} r \tanh(\pi r) f(r,K)\, \mathrm{d}r \ll K \int_{K-G\log^2 K}^{K+G\log^2 K} \mathrm{e}^{-(r-K)^2/G^2}\, \mathrm{d}r + 1 \ll GK. \tag{2.16}$$

The crucial step in the proof is to show that, for any fixed $A > 0$,

$$\sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,1;\ell) f_+\left(\frac{4\pi}{\ell}\sqrt{m}\right) \ll K^{-A}, \tag{2.17}$$

provided that we choose $G \geq C\sqrt{\log K}$.

To begin with, we may truncate the ℓ-sum in (2.17) to the range $1 \le \ell \le K^B$ for some constant $B > 1$. To see this, we move the line of integration in the integral defining f_+ (cf. (2.2)) to $\Im\mathrm{m}\, r = -1$. Since $f(-\frac{1}{2}i, K) = 0$, there is no pole of the integrand. Then we use the series representation (see (2.3))

$$J_{2+ix}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2+ix+2k}}{\Gamma(k+1)\Gamma(k+2+ix+1)} \quad (z = 4\pi\sqrt{m}/\ell \ll K^{1-B}),$$

which shows that the contribution of $\ell > K^B$ is $\ll K^{-A}$ for any fixed $A > 0$, provided that $B = B(A)$ is sufficiently large.

In the remaining sum, we substitute (see e.g., [9, p. 139])

$$J_{2ir}(x) - J_{-2ir}(x) = \frac{2i}{\pi} \sinh(\pi r) \int_{-\infty}^{\infty} \cos(x \cosh u) \cos(2ru)\, \mathrm{d}u.$$

Integration by parts shows that, for $x > 0$ and $r \ge 0$,

$$\begin{aligned} J_{2ir}(x) - J_{-2ir}(x) = \frac{2i}{\pi}\sinh(\pi r)\int_{-\log^2 K}^{\log^2 K} \cos(x\cosh u)\cos(2ru)\,du \\ + O\left(x^{-1}(r+1)\exp(\pi r - \tfrac12\log^2 K)\right). \end{aligned} \tag{2.18}$$

The error term in (2.18) clearly contributes $\ll K^{-A}$ to the sum in (2.17). The main term in (2.18) will contribute to f_+

$$-\frac{4}{\pi^2}\int_{-\log^2 K}^{\log^2 K} \cos(x\cosh u)\int_0^\infty rf(r,K)\tanh(\pi r)\cos(2ru)\,dr\,du. \tag{2.19}$$

In the inner integral we use

$$r\tanh(\pi r) = r\,\mathrm{sign}\,r + O(|r|\exp(-\pi|r|)), \tag{2.20}$$

and make the change of variable $r = K + Gx$. The x integral can be truncated at $|x| = \log^2 K$ with error $\ll K^{-A}$. The rational function in x in the integrand is expanded by Taylor's series, taking so many terms that the error will again make a contribution which will be $\ll K^{-A}$. Then (2.19) will become

$$= \Re\mathrm{e}\int_{-\log^2 K}^{\log^2 K} P(u,K,G)\cos(x\cosh u)\exp(-(G^2u^2 + 2iKu))\,du + O(K^{-A}), \tag{2.21}$$

where $P(u,K,G)$ is a polynomial in u, K and G. Here we used the familiar integral

$$\int_{-\infty}^{\infty}\exp(Ax - Bx^2)\,dx = \sqrt{\frac{\pi}{B}}\exp\left(\frac{A^2}{4B}\right) \qquad (\Re\mathrm{e}\,B > 0), \tag{2.22}$$

and $P(u,K,G)$ may be evaluated by successive differentiation of (2.22) as the function of A.

If $G \ge C\sqrt{\log K}$ with large $C > 0$, then the integration in (2.21) can be restricted to the interval $|u| \le u_0$, where u_0 is a small positive constant, and the error thus made will be $\ll K^{-A}$. Then the relevant exponential factor will be of the form

$$\exp(ig(u)),\; g(u) = \pm x\cosh u + 2Ku,\; g'(u) = \pm x\sinh u + 2K \gg K$$

for $|x| \le BK$ and any constant $B > 0$, and $|u| \le u_0$ with sufficiently small u_0, since $\sinh u = u + O(|u|^3)$ for small u. In our case $x = 4\pi\sqrt{m}/\ell \le 2(1+\delta)K$ by (2.4). Thus the corresponding integral will have no saddle points, and by a large number of successive integrations by parts it transpires that the integral in question will be $\ll K^{-A}$, and so will also be $f_+(4\pi\sqrt{m}/\ell)$. Therefore (2.17) holds, and Lemma 3 follows from (2.15)–(2.17). ■

Lemma 4. *If $A(s) = \sum_{m \le M} a(m) m^{-s}$ with $a(m) \ll_\varepsilon m^\varepsilon$, then we have*

$$\begin{aligned}&\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 |A(\tfrac{1}{2}+it)|^2 \,\mathrm{d}t \\ &= T \sum_{h,k \le M} \frac{a(h)\overline{a(k)}}{hk}(h,k)\left(\log \frac{T(h,k)}{2\pi hk} + 2\gamma - 1\right) + E(T,A),\end{aligned} \tag{2.23}$$

with $E(T,A) \ll_\varepsilon T^{1/3+\varepsilon} M^{4/3}$ if $M \ll T^C$ for some $C > 0$.

This mean value result was proved by Y. Motohashi [10].

3. The proof of Theorem 1

As in the proof of Lemma 2, we let $f(r,K)$ be defined by (2.14). We suppose additionally that $K_0 \le K \le 2K_0$, and that $G = G(K_0)$ is a function of K_0 (later we shall choose $G = C\sqrt{\log K_0}$). We apply Lemma 1 and Lemma 2, similarly as in (2.15). Then we divide by $\sqrt{\pi}G$ and integrate the resulting expression over K from K_0 to $2K_0$. It follows that

$$\begin{aligned}&\sum_{j=1}^{\infty} \alpha_j H_j(\tfrac{1}{2}) w(\kappa_j) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2}+ir)|^2}{|\zeta(1+2ir)|^2} w(r)\,\mathrm{d}r \\ &= \frac{1}{\pi^2}\int_{-\infty}^{\infty} r\tanh(\pi r) w(r)\,\mathrm{d}r + o(1) \\ &+ \frac{1}{\sqrt{\pi}G}\int_{K_0}^{2K_0} \sum_{m \le (1+\delta)^2 K^2/(4\pi^2)} m^{-1/2} \mathrm{e}^{-(m/Y)^h} \sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m,1;\ell) f_+\left(\frac{4\pi}{\ell}\sqrt{m}\right) \mathrm{d}K,\end{aligned} \tag{3.1}$$

where we set

$$w(r) := \frac{1}{\sqrt{\pi}G}\int_{K_0}^{2K_0} f(r,K)\,\mathrm{d}K. \tag{3.2}$$

Since $w(r)$ is even, it suffices to consider $r \ge 0$. From (2.14) we obtain, with the change of variable $K = r + Gx$,

$$w(r) = \frac{1}{\sqrt{\pi}}\int_{(K_0-r)/G}^{(2K_0-r)/G} \mathrm{e}^{-x^2}\,\mathrm{d}x + O(K_0^{-2}). \tag{3.3}$$

If $r \in [K_0 + CG\sqrt{\log K_0}, 2K_0 - CG\sqrt{\log K_0}]$ with large $C > 0$, then the integral in (3.3) equals $1 + O(K_0^{-2})$. If $r > 2K_0 + CG\sqrt{\log K_0}$ or $r < K_0 - CG\sqrt{\log K_0}$, the integral is $O(K_0^{-2})$. Otherwise note that, for $x \ge 0$, we have $2\mathrm{e}^x \ge 2 + 2x + x^2$, which implies that

$$\mathrm{e}^{-x} \le 2(x+1)^{-2} \qquad (x \ge 0). \tag{3.4}$$

Hence using (3.2)-(3.4) we obtain ($\chi_{\mathcal{I}}(x)$ is the characteristic function of the set $\mathcal{I}$), for $r \geq 0$,

$$w(r) = \chi_{[K_0, 2K_0]}(r) + O(K_0^{-2}) + O\Big\{G^3(G + \min(|r - K_0|, |r - 2K_0|))^{-3}\Big\}. \tag{3.5}$$

Using (3.5) and Lemma 2 we have, for $C > 0$ sufficiently large,

$$\begin{aligned}
\sum_{j=1}^{\infty} \alpha_j H_j(\tfrac{1}{2}) w(\kappa_j) &= \sum_{K_0 - CG\sqrt{\log K_0} \leq \kappa_j \leq 2K_0 + CG\sqrt{\log K_0}} \alpha_j H_j(\tfrac{1}{2}) w(\kappa_j) + O(1) \\
&= \sum_{K_0 \leq \kappa_j \leq 2K_0} \alpha_j H_j(\tfrac{1}{2}) + O(1) \\
&\quad + O\left(G^3 \sum_{K_0 - CG\sqrt{\log K_0} \leq \kappa_j \leq K_0} \alpha_j H_j (G + K_0 - \kappa_j)^{-3} \right) \\
&\quad + O\left(G^3 \sum_{2K_0 < \kappa_j \leq 2K_0 + CG\sqrt{\log K_0}} \alpha_j H_j (G + \kappa_j - 2K_0)^{-3} \right) \\
&= \sum_{K_0 \leq \kappa_j \leq 2K_0} \alpha_j H_j(\tfrac{1}{2}) + O(GK_0).
\end{aligned} \tag{3.6}$$

Similarly we obtain, since $w(r) = w(-r)$,

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|\zeta(\frac{1}{2} + ir)|^2}{|\zeta(1 + 2ir)|^2} w(r)\, \mathrm{d}r = \frac{2}{\pi} \int_{K_0}^{2K_0} \frac{|\zeta(\frac{1}{2} + ir)|^2}{|\zeta(1 + 2ir)|^2} w(r)\, \mathrm{d}r + O(GK_0), \tag{3.7}$$

on using $1/\zeta(1 + it) \ll \log t$ and $\zeta(\frac{1}{2} + it) \ll t^{1/6}$. Finally we have, since (2.20) holds,

$$\begin{aligned}
\frac{1}{\pi^2} \int_{-\infty}^{\infty} r \tanh(\pi r) w(r)\, \mathrm{d}r &= \frac{2}{\pi^2} \int_{K_0}^{2K_0} r\, \mathrm{d}r + O(K_0 G) \\
&= \frac{1}{\pi^2} \left\{ (2K_0)^2 - K_0^2 \right\} + O(GK_0).
\end{aligned} \tag{3.8}$$

We note that the contribution of the Kloosterman-sum part in (3.1), analogously to (2.17), is $\ll K_0^{-A}$ for any fixed $A > 0$. Therefore from (3.1) and (3.6)–(3.8) it follows that

$$\begin{aligned}
&\sum_{K_0 < \kappa_j \leq 2K_0} \alpha_j H_j(\tfrac{1}{2}) + \frac{2}{\pi} \int_{K_0}^{2K_0} \frac{|\zeta(\frac{1}{2} + ir)|^2}{|\zeta(1 + 2ir)|^2}\, \mathrm{d}r \\
&= \frac{1}{\pi^2} \left\{ (2K_0)^2 - K_0^2 \right\} + O(GK_0).
\end{aligned} \tag{3.9}$$

Theorem 1 follows now from (3.9) if we choose $G = C\sqrt{\log K_0}$ with a sufficiently large constant $C > 0$, replace K_0 by $T2^{-j}$ and then sum over $j = 1, 2, \ldots$. The asymptotic formula (1.7) follows similarly as the proof of Theorem 1, if one uses the technique of proof of Theorem 2. One simply takes $m = n = 1$ in Lemma 1 and proceeds as in the proof of Theorem 1, only the argument is simpler and the details are thus omitted. Namely the integral in (1.4) will appear without $|\zeta(\frac{1}{2}+it)|^2$, and will be asymptotic to CT.

4. The proof of Theorem 2

In the general problem of evaluating $\sum_{\kappa_j\le T}\alpha_j H_j^k(\frac{1}{2})$ one encounters the integrals (see (1.8))

$$I_k(T) := \int_0^T \frac{|\zeta(\frac{1}{2}+it)|^{2k}}{|\zeta(1+2it)|^2}\,dt \qquad (k\in\mathbb{N}), \tag{4.1}$$

where k is fixed. By general convexity results for Dirichlet series one has (see K. Ramachandra [13])

$$I_k(T) \gg_k T(\log T)^{k^2}. \tag{4.2}$$

Although one expects the lower bound in (4.2) to be of the correct order of magnitude this, like in the case of the integral without the zeta-factor in the denominator, seems at present impossible to prove for $k\ge 3$. In fact, even for $k=2$, when precise results on $\int_0^T|\zeta(\frac{1}{2}+it)|^4\,dt$ are known (see e.g., [5] and [12]), an upper bound for $I_2(T)$ corresponding to the lower bound in (4.2) seems difficult to obtain and represents an open problem. A slightly weaker bound, namely $I_2(T)\ll T(\log T)^4(\log\log T)^2$, follows from [14, eqs. (3.34)-(3.36)] by a method similar to the one used in the proof of Theorem 2.

What we can obtain, though, is the asymptotic formula (1.5) of Theorem 2, which will be proved now. We remark that the exponent of the error term is by no means best possible, and the use of optimal known zero-density estimates would certainly lead to small improvements.

We start from

$$J_1(T) := \int_T^{2T} \frac{|\zeta(\frac{1}{2}+it)|^2}{|\zeta(1+2it)|^2}\,dt = \int_{\mathcal{A}(T)} + \int_{\mathcal{B}(T)}. \tag{4.3}$$

Here $\mathcal{A}(T)$ is the subset of points $t\in[T,2T]$ such that there are no zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ satisfying $\frac{3}{4}\le\beta\le 1$, $2t-\log^4 T\le\gamma\le 2t+\log^4 T$, and $\mathcal{B}(T)=[T,2T]\setminus\mathcal{A}(T)$. From M.N. Huxley's zero-density estimate (see [4, Chapter 11])

$$N(\sigma,T) = \sum_{\beta\ge\sigma,|\gamma|\le T} 1 \ll T^{(3-3\sigma)/(3\sigma-1)}\log^C T \qquad (C>0,\ \tfrac{3}{4}\le\sigma\le 1)$$

it follows that

$$\mu(\mathcal{B}(T)) \ll T^{3/5}\log^C T, \tag{4.4}$$

where $\mu(\cdot)$ denotes measure. Thus, by the Cauchy-Schwarz inequality for integrals,

$$\int_{\mathcal{B}(T)} \frac{|\zeta(\frac12+it)|^2}{|\zeta(1+2it)|^2}\,dt \le \left\{\int_T^{2T} \frac{|\zeta(\frac12+it)|^4}{|\zeta(1+2it)|^4}\,dt\cdot\mu(\mathcal{B}(T))\right\}^{1/2}$$
$$\ll T^{4/5}\log^C T,$$

where C denotes generic positive constants, and where the integral with the fourth moment of $|\zeta(\frac12+it)|$ was estimated trivially as $\ll T\log^6 T$, using $1/\zeta(1+2it)\ll \log t$. If $t\in\mathcal{A}(T)$, then $1/\zeta(\sigma+2it+iv)\ll_\varepsilon t^\varepsilon$ for $\sigma>3/4$ and $|v|\le\frac12\log^4 T$ (e.g., by the technique of [15, Chapter 14]). Hence from (2.6) we obtain ($h=\log^2 T$, $T^\varepsilon\ll Y\ll T^{1/2}$)

$$\sum_{n=1}^\infty \mu(n)n^{-1-2it}e^{-(n/Y)^h} = \frac{1}{2\pi i}\int_{(1)} \frac{Y^w}{\zeta(1+2it+w)}\Gamma(1+\frac{w}{h})\frac{dw}{w}$$
$$= \frac{1}{2\pi i}\int_{\Re e\, w=1,|\Im m\, w|\le\frac12 h^2} \frac{Y^w}{\zeta(1+2it+w)}\Gamma(1+\frac{w}{h})\frac{dw}{w}+O(T^{-10}) \tag{4.5}$$
$$= \frac{1}{\zeta(1+2it)}+\frac{1}{2\pi i}\int_{\Re e\, w=\varepsilon-\frac14,|\Im m\, w|\le\frac12 h^2} \frac{Y^w}{\zeta(1+2it+w)}\Gamma(1+\frac{w}{h})\frac{dw}{w}+O(T^{-10})$$
$$= \frac{1}{\zeta(1+2it)}+O(Y^{-1/4}T^\varepsilon)+O(T^{-10}).$$

Set $a(m)=\mu(n)$ if $m=n^2$ and $a(m)=0$ otherwise. From (4.5) it follows that, for $t\in\mathcal{A}(T)$,

$$\frac{1}{\zeta(1+2it)} = \sum_{m\le 4Y^2} a(m)m^{-1/2-it}\exp(-(\sqrt{m}/Y)^h)+O(T^\varepsilon Y^{-1/4}). \tag{4.6}$$

We then obtain, using (4.4), (4.6) and the Cauchy-Schwarz inequality,

$$\int_{\mathcal{A}(T)}\dots\,dt = \int_T^{2T}|\zeta(\tfrac12+it)|^2\Big|\sum_{m\le 4Y^2} a(m)m^{-1/2-it}\exp(-(\sqrt{m}/Y)^h)\Big|^2 dt$$
$$+O_\varepsilon(T^{1+\varepsilon}Y^{-1/4})+O(T^{4/5}\log^C T).$$

To evaluate the last integral we use (2.23) of Lemma 4. We obtain

$$\int_0^T|\zeta(\tfrac12+it)|^2\Big|\sum_{m\le 4Y^2} a(m)m^{-1/2-it}\exp(-(\sqrt{m}/Y)^h)\Big|^2 dt$$
$$=T\sum_{\ell,k\le 2Y}\frac{\mu(\ell)\mu(k)}{\ell^2k^2}e^{-(\ell/Y)^h-(k/Y)^h}(\ell,k)^2\left(\log\frac{T(\ell,k)^2}{2\pi\ell^2k^2}+2\gamma-1\right)$$
$$+O_\varepsilon(T^{1/3+\varepsilon}Y^{8/3}).$$

Setting $d = (\ell, k), \ell = d\ell_1, k = dk_1, (\ell_1,\, k_1) = 1$, we see that the double sum above equals

$$\sum_{d \le 2Y} \frac{\mu^2(d)}{d^2} \sum_{k_1 \le \frac{2Y}{d}, \ell_1 \le \frac{2Y}{d}, (k_1,\ell_1)=(k_1,d)=(\ell_1,d)=1} \frac{\mu(k_1)\mu(\ell_1)}{k_1^2\ell_1^2} \times$$
$$\times \mathrm{e}^{-(d\ell_1/Y)^h - (dk_1/Y)^h} \left\{ \log\left(\frac{T}{2\pi k_1^2 \ell_1^2 d^2} \right) + 2\gamma - 1 \right\}.$$

The terms $k_1 > Y/(2d)$, and then $\ell_1 > Y/(2d)$ are estimated trivially, producing an error which is $O(TY^{-1}\log^2 T)$. In the remaining terms we get rid of the exponential factor by using $\mathrm{e}^{-x} = 1 + O(x)$ for $x > 0$. In the inner sum we extend the summation to all k_1, ℓ_1, obtaining again an error which is $O(TY^{-1}\log^2 T)$, and similarly we extend the summation over all d. Finally we obtain that the double sum above equals

$$A \log T + B + O\left(\frac{\log^2 T}{Y} \right) \qquad (A > 0),$$

where the constants A and B may be explicitly evaluated. Putting together all the expressions we wind up with

$$\int_0^T \frac{|\zeta(\frac{1}{2} + it)|^2}{|\zeta(1 + 2it)|^2}\, \mathrm{d}t = T(A \log T + B)$$
$$+ O_\varepsilon(T^{1/3+\varepsilon} Y^{8/3}) + O_\varepsilon(T^{1+\varepsilon} Y^{-1/4}) + O(T^{4/5} \log^C T).$$

The choice $Y = T^{8/35}$ completes the proof of (1.5) of Theorem 2.

5. The choice of the weight function

We shall discuss now how the error terms in (1.4) (and thus also in (1.6) and (1.7)) can be improved to $O(T(\log T)^\varepsilon)$. Let S_α^β be the class of smooth functions $f(x)\,(\in C^\infty)$ introduced by I.M. Gel'fand and G.E. Shilov [3]. The functions $f(x)$ satisfy for any real x the inequalities

$$|x^k f^{(q)}(x)| \le CA^k B^q k^{k\alpha} q^{q\beta} \qquad (k, q = 0, 1, 2, \ldots) \tag{5.1}$$

with suitable constants $A, B, C > 0$ depending on f alone. For $\alpha = 0$ it follows that $f(x)$ is of bounded support, namely it vanishes for $|x| \ge A$. For $\alpha > 0$ the condition (5.1) is equivalent (see [3]) to the condition

$$|f^{(q)}(x)| \le CB^q q^{q\beta} \exp(-a|x|^{1/\alpha}) \qquad (a = \alpha/(eA^{1/\alpha})) \tag{5.2}$$

for all x and $q \ge 0$. We shall denote by E_α^β the subclass of S_α^β with $\alpha > 0$ consisting of even functions $f(x)$ such that $f(x)$ is not the zero-function. It is shown in [3] that S_α^β is non-empty if $\beta \ge 0$ and $\alpha+\beta \ge 1$. If these conditions hold then E_α^β is also non-empty, since $f(-x) \in S_\alpha^\beta$ if $f(x) \in S_\alpha^\beta$, and $f(x)+f(-x)$ is always even. If

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u)e^{iux}\,du$$

denotes the Fourier transform of $f(x)$, then a fundamental property of the class S_α^β (see op. cit.) is that $\widehat{S_\alpha^\beta} = S_\beta^\alpha$, where in general $\widehat{U} = \{\widehat{f}(x) : f(x) \in U\}$. Henceforth let $\varphi(x) \in E_{1-\delta}^\delta$ be non-negative, where $\delta > 0$ is a small constant, and set

$$f_\varphi(r) = f_\varphi(r, K) = \frac{r^2+\frac14}{r^2+1000}\left\{\varphi\left(\frac{r+K}{G}\right) + \varphi\left(\frac{r-K}{G}\right)\right\}, \tag{5.3}$$

where

$$C(\log K)^\delta \le G \le \sqrt{K}, \qquad (C = C(\delta) > 0). \tag{5.4}$$

The function $\varphi(x)$ is of fast decay by (5.2), and moreover by the general theory (op. cit.) the analytic continuation of $\varphi(z)$ certainly exists in the strip $|y| = |\Im m\, z| \le C$ $(C > 0)$, where it is of rapid decay, so that $f_\varphi(r)$ satisfies the assumptions of Lemma 1.

Our main task is to show that (2.17) holds with f_+ (cf. (2.2)) relating to $f_\varphi(r)$, as given by (5.3), and G satisfying (5.4), where of course it is the lower bound that is critical. We follow the reasoning given from (2.18)–(2.22) in the proof of Lemma 3, but make the following observations. The reason $G = C\sqrt{\log K}$ was the limit in Lemma 3 (and indirectly in the proof of Theorem 1) is the appearance of $\exp(-(G^2u^2 + 2iKu))$ in (2.21). With $f_\varphi(r)$ replacing f (cf. (2.14)), the integral over r in (2.18) can be truncated at $|r| = \log^2 K$ with negligible error. While the term $2iKu$ in (2.21) (which comes after the change of variable $r = K + Gx$) cannot be avoided, the term $-G^2u^2$ comes from the fact that essentially e^{-x^2} $(\in S_{1/2}^{1/2})$ is the Fourier transform of itself, which is embodied in the formula (2.22). This factor sets the lower bound $G = C\sqrt{\log K}$. However, in this new situation we shall obtain, instead of $\exp(-G^2u^2)$, the function $\hat{\varphi}_f(x) \in S_\delta^{1-\delta}$, which by (5.2) satisfies

$$\hat{\varphi}_f(Gu) \ll \exp(-a|Gu|^{1/\delta}). \tag{5.5}$$

Thus we may truncate the integration in the analogue of (2.21) now at $|u| \le u_0$, provided that $G \ge C(\log K)^\delta, C = C(\delta) > 0$ sufficiently large, and the analogue of (2.17) will hold again.

It only remains to check that the integration over $[K_0, 2K_0]$ in the proof of Theorem 1 will go through. To do this, instead of (3.2) consider

$$w_\varphi(r) := \frac{1}{BG}\int_{K_0}^{2K_0} f_\varphi(r, K)\,dK, \tag{5.6}$$

where $B = \hat{\varphi}(0) = \int_{-\infty}^{\infty} \varphi(x)\,\mathrm{d}x$. Since $\varphi(x) \in E_{1-\delta}^{\delta}$, we have

$$\varphi(x) \ll \exp(-a|x|^{1/(1-\delta)}) \qquad (a > 0).$$

Therefore by using e.g., the inequality

$$\mathrm{e}^{-x} \ \le\ 24(x+1)^{-4} \qquad (x \ge 0),$$

we obtain the analogue of (3.5) for $w_\varphi(r)$. This means that the choice $G = C(\log K_0)^\delta$ is permissible in the proof of Theorem 1, which ends our discussion.

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