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CALABI-YAU THEOREM AND HODGE-LAPLACIAN HEAT EQUATION IN A CLOSED STRICTLY PSEUDOCONVEX CR MANIFOLD

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Abstract

In this paper, we address the Calabi-Lee conjecture for pseudo-Einstein contact structure via the CR Poincaré-Lelong equation. Then we confirm the Calabi-Yau Theorem via Hodge-Laplacian heat flow in a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, θ) for $n \geq 2$. With its applications, we affirm a partial answer of the CR Frankel conjecture in a closed spherical strictly pseudoconvex CR $(2n+1)$-manifold.

Dedicated to the memory of Professor Jianguo Cao

1. Introduction

In his celebrated paper [**21**], Yau established several related results which are of fundamental importance in the study of Kähler manifolds. These results have to do with the existence of Kähler metrics with certain special properties on compact Kähler manifolds. In order to achieve this goal, Yau reduced the problem to questions about some nonlinear partial differential equations of Monge-Ampere type and then solved them by a continuity method involving a priori estimates. More precisely, Yau established the following Calabi-Yau Theorem.

Proposition 1.1. *([**21**]) Let (X, ω_0) be a compact Kähler manifold of complex dimension m with a Kähler class $[\omega_0] \in H^2(X, \mathbf{R}) \cup H^{1,1}(X, \mathbf{C})$. Given any form Ω representing the first Chern class $c_1(X)$, there exists a unique Kähler metric $\omega \in [\omega_0]$ such that*

$$Ric(\omega) = \Omega.$$

In particular if $c_1(X) = 0$, there exists a unique Kähler metric $\omega \in [\omega_0]$ such that $Ric(\omega) = 0$.

By the well-known $\partial\overline{\partial}$-Lemma (see, *e.g.,* [**18**]) in Kähler geometry, this is equivalent to finding a solution φ of the complex Monge-Ampere equation

$$(\omega_0 + \frac{i}{2\pi}\partial\overline{\partial}\varphi)^m = e^f \omega_0^m$$

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where f is unique after normalizing to $\int_X (e^f - 1)\omega_0^m = 0$.

In the present paper, we study the CR analogue of the Calabi-Yau Theorem in a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, θ) (see the next section for basic notions in pseudohermitian geometry). The pseudohermitian Ricci tensor and the torsion tensor on $T^{1,0}M$ are defined by

$$\begin{aligned} Ric(X,Y) &= R_{\alpha\bar{\beta}} X^\alpha Y^{\bar{\beta}}, \\ Tor(X,Y) &= i(A_{\bar{\alpha}\bar{\beta}} X^{\bar{\alpha}} Y^{\bar{\beta}} - A_{\alpha\beta} X^\alpha Y^\beta), \end{aligned}$$

where $X = X^\alpha Z_\alpha,\ Y = Y^\beta Z_\beta,\ R_{\alpha\bar{\beta}} = R_{\gamma}{}^{\gamma}{}_{\alpha\bar{\beta}}$. The Tanaka-Webster scalar curvature is $R = R_\alpha{}^\alpha = h^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$. Before going any further, let us recall some definitions.

Definition 1.1. *([**16**]) A contact form θ on a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, θ) is said to be pseudo-Einstein for $n \geq 2$ if the pseudohermitian Ricci tensor $R_{\alpha\overline{\beta}}$ is proportional to the Levi form $h_{\alpha\overline{\beta}}$, i.e.,*

$$R_{\alpha\overline{\beta}} = \tfrac{R}{n} h_{\alpha\overline{\beta}}$$

where $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ is the Tanaka-Webster scalar curvature of (J, θ).

The pseudo-Einstein condition is less rigid than the Einstein condition in Riemannian geometry. Indeed, the CR contracted Bianchi identity no longer implies R to be a constant due to the presence of pseudohermitian torsion for $n \geq 2$,

$$R_{\alpha\overline{\beta},\beta} = R_\alpha - i(n-1) A_{\alpha\beta,\overline{\beta}}.$$

Note that any contact form on a closed strictly pseudoconvex 3-manifold is actually pseudo-Einstein (since the pseudohermitian Ricci tensor has only one component $R_{1\overline{1}}$).

Next we define the real *first Chern class* $c_1(T^{1,0}(M))$ for the holomorphic subbundle $T^{1,0}M$ in (M, θ).

Definition 1.2. *([**16**]) Let (M, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold. We define the first Chern class $c_1(T^{1,0}M) \in H^2(M, \mathbf{R})$ for the holomorphc tangent bundle $T^{1,0}M$ by*

$$c_1(T^{1,0}M) = \frac{i}{2\pi}[d\omega_\alpha{}^\alpha] = \frac{i}{2\pi}[R_\alpha{}^\alpha{}_{AB}\theta^A \wedge \theta^B] = \frac{i}{2\pi}[\gamma]$$

with

$$\gamma = R_{\alpha\overline{\beta}} \theta^\alpha \wedge \theta^{\overline{\beta}} + A_{\alpha\mu,\overline{\alpha}} \theta^\mu \wedge \theta - A_{\overline{\alpha\mu},\alpha} \theta^{\overline{\mu}} \wedge \theta$$

which is the purely imaginary two-form.

Then we have the following result.

Proposition 1.2. ([**16**]) *For any pseudo-Einstein manifold* (M^{2n+1}, θ) *with* $n \geq 2$*, the first Chern class* $c_1(T^{1,0}M)$ *of* $T^{1,0}M$ *is represented by*

$$\gamma = -\tfrac{i}{n} d(R\theta).$$

Here γ *is globally exact and hence represents the trivial cohomology class:*

$$c_1(T^{1,0}M) = 0.$$

In view of Proposition 1.1 and Proposition 1.2, we have the corresponding CR Calabi-Lee conjecture ([**16**], [**4**]) in a closed strictly pseudoconvex CR $(2n+1)$-manifold (M, θ) for $n \geq 2$ as follows.

Conjecture 1.1. *(Calabi-Lee Conjecture) Given any closed* 2*-form* Φ *representing the first Chern class* $c_1(T^{1,0}M)$ *in a closed strictly pseudoconvex CR* $(2n+1)$*-manifold* (M, θ)*, there exists a unique contact structure* $\widetilde{\theta} \in [\theta]$ *such that*

$$\widetilde{Ric}_{\widetilde{\theta}}(X, Y) = \Phi(X, Y)$$

for all $X, Y \in \ker \theta$*. More precisely, it is equivalent to*

$$\widetilde{Ric}_{\widetilde{\theta}}(X, Y) = \Phi(X, Y) = Ric(X, Y) + d\sigma(X, Y) \tag{1.1}$$

for some purely imaginary 1*-form* $\sigma = (\sigma_{\overline{\beta}}\theta^{\overline{\beta}} - \sigma_\alpha \theta^\alpha) + i\sigma_0\theta$*. In particular if* $c_1(T^{1,0}M) = 0$*, there exists a unique pseudo-Einstein contact structure* $\widetilde{\theta} \in [\theta]$ *such that*

$$\frac{\widetilde{R}}{n}\widetilde{h}(X, Y) = \Phi(X, Y) = Ric(X, Y) + d\sigma(X, Y). \tag{1.2}$$

We observe that as in the Calabi conjecture for compact Kähler manifolds, it is natural to work on a fixed Kähler class due to the $\partial\overline{\partial}$-Lemma. However, we do not have the analogue $\partial_b\overline{\partial}_b$-Lemma in the CR case. Instead, we work on a fixed contact class. More precisely, it is proved (Theorem 3.1) that $\widetilde{\theta} = e^{2u}\theta$ is a pseudo-Einstein contact structure if and only if u is the solution of

$$(n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) = R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}.$$

Some of well-known results (Theorem 4.1) for the CR Calabi-Lee conjecture were derived by J. Lee ([**16**]) and Cao and the second author ([**3**]) via the elliptic method.

In this paper, we derive the following CR Calabi-Yau Theorem via CR Hodge-Laplacian heat equation.

Theorem 1.1. *Let* (M, θ) *be a closed strictly pseudoconvex CR* $(2n+1)$*-manifold with* $c_1(T^{1,0}M) = 0$ *for* $n \geq 2$*. Then there is a smooth real-valued function* u *solving*

$$\Delta u = \frac{1}{n+2}(r - R) \quad \textit{with} \quad r = \frac{\int_M R d\mu}{\int_M d\mu}.$$

u also satisfies the following identities:

$$(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha}) = R_{\alpha\overline{\beta}} - \frac{r}{n}h_{\alpha\overline{\beta}}$$

and

$$(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha}) = R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}. \tag{1.3}$$

Hence $e^{2u}\theta$ is a pseudo-Einstein contact structure. In addition, if R is constant, then u is constant and

$$R_{\alpha\overline{\beta}} = \frac{R}{n}h_{\alpha\overline{\beta}}. \tag{1.4}$$

In general, it is difficult to see when a CR manifold has the vanishing first Chern class $c_1(T^{1,0}M)$. By applying the CR version of Bochner-type identity due to Mok-Siu-Yau [**17**] in the case of Kähler manifolds, we derive the following result (also Corollary 5.1):

Theorem 1.2. *Let (M, J, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature and*

$$A_{\alpha\gamma,\overline{\alpha}} = 0$$

for each α. Then there exists a smooth real-valued function u such that

$$(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha}) = R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}.$$

Hence $e^{2u}\theta$ is a pseudo-Einstein contact structure.

As an application of Theorem 1.1, we can affirm a partial answer of the CR Frankel conjecture in a closed spherical strictly pseudoconvex CR $(2n+1)$-manifold for $n \geq 2$.

Conjecture 1.2. *(CR Frankel Conjecture) A simply connected closed strictly pseudoconvex CR $(2n+1)$-manifold (M, J, θ) of positive pseudohermitian bisectional curvature is globally CR equivalent to a standard CR sphere $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$ in $\mathbf{C}^{n+1}$ with the induced CR structure $\widehat{J}$ and the standard contact form $\widehat{\theta}$.*

Definition 1.3. *([**6**]) Let (M, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold with $n \geq 2$; we call a CR structure J spherical if the Chern curvature tensor*

$$\begin{array}{rcl} C_{\beta\overline{\alpha}\lambda\overline{\sigma}} & = & R_{\beta\overline{\alpha}\lambda\overline{\sigma}} - \frac{1}{n+2}[R_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + R_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}} + \delta^{\alpha}_{\beta}R_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}R_{\beta\overline{\sigma}}] \\ & & + \frac{R}{(n+1)(n+2)}[\delta^{\alpha}_{\beta}h_{\lambda\overline{\sigma}} + \delta^{\alpha}_{\lambda}h_{\beta\overline{\sigma}}] \end{array} \tag{1.5}$$

vanishes identically.

Remark 1.1. *1. Note that $C_{\alpha\overline{\alpha}\lambda\overline{\sigma}} = 0$. Hence $C_{\beta\overline{\alpha}\lambda\overline{\sigma}}$ is always vanishing for $n = 1$.*

2. We observe that the spherical structure is CR invariant and a closed spherical CR $(2n+1)$-manifold (M, J) is locally CR equivalent to $(\mathbf{S}^{2n+1}, \widehat{J})$.

3. ([**14**]) *In general, a spherical CR structure on a $(2n+1)$-manifold is a system of coordinate charts into S^{2n+1} such that the overlap functions are restrictions of elements of $PU(n+1, 1)$. Here $PU(n+1, 1)$ is the group of complex projective automorphisms of the unit ball in $\mathbf{C}^{n+1}$ and the holomorphic isometry group of the complex hyperbolic space $\mathbf{CH}^n$.*

Now we may state the second main theorem in this paper.

Theorem 1.3. *Let (M, J, θ) be a simply connected, closed spherical strictly pseudoconvex CR $(2n+1)$-manifold of positive constant Tanaka-Webster scalar curvature with $c_1(T^{1,0}M) = 0$ for $n \geq 2$. Then M is CR equivalent to the standard CR sphere $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$.*

Note that in [**6**], Chern and Ji proved a generalization of the Riemann mapping theorem: If Ω is a bounded simply connected domain in $\mathbb{C}^{n+1}$ and its connected smooth boundary $\partial\Omega$ has a spherical CR structure, then it is biholomorphic to the unit ball and $M = \partial\Omega$ is the standard CR $(2n+1)$-sphere.

However, it is shown ([**3**, Proposition 3.2 and Lemma 3.1]) that $c_1(T^{1,0}M) = 0$ if M is the boundary of a smooth, bounded strictly pseudoconvex domain in a complete Stein manifold V^{n+1} for $n \geq 2$. Hence Theorem 1.3 implies the following result.

Corollary 1.1. *Let (M, J, θ) be the smooth simply connected spherical boundary of a bounded strictly pseudoconvex domain Ω in a complete Stein manifold V^{n+1} for $n \geq 2$. Assume that (M, J, θ) has positive constant Tanaka-Webster scalar curvature. Then M is CR equivalent to the standard CR sphere $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$. In particular, any simply connected closed spherical CR hypersurface of positive constant Tanaka-Webster scalar curvature in $\mathbb{C}^{n+1}$ is CR equivalent to the standard CR sphere $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$ for $n \geq 2$.*

Furthermore, Proposition 5.1 implies that there is a smooth real-valued function u such that $e^{2u}\theta$ is a pseudo-Einstein contact structure if (M, θ) is a closed spherical CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature for $n \geq 2$. Hence from Theorem 1.3 again, we have

Corollary 1.2. *Any simply connected closed spherical CR $(2n+1)$-manifold (M, J, θ) of positive pseudohermitian bisectional curvature and constant Tanaka-Webster scalar curvature is CR equivalent to the standard CR sphere $(\mathbf{S}^{2n+1}, \widehat{J}, \widehat{\theta})$ for $n \geq 2$.*

It is conjectured that any simply connected closed spherical CR 3-manifold (M, J, θ) of positive constant Tanaka-Webster scalar curvature is CR equivalent to the standard CR sphere $(\mathbf{S}^3, \widehat{J}, \widehat{\theta})$.

The paper is organized as follows. In section 2, we introduce some basic materials in a pseudohermitian $(2n+1)$-manifold. In section 3, we address the Calabi-Lee conjecture for pseudo-Einstein contact structure via Poincaré-Lelong equation. In section 4, we prove the CR Calabi-Yau Theorem via the Hodge-Laplacian heat equation. Finally, some applications on the CR Frankel conjecture for a closed spherical strictly pseudoconvex $(2n+1)$-manifold are derived in section 5.

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2. Preliminary

We first give a brief introduction to pseudohermitian geometry (see [**16**] for more details). Let (M,ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ, $\dim_R \xi = 2n$. A CR structure compatible with ξ is an endomorphism $J : \xi \to \xi$ such that $J^2 = -1$. We also assume that J satisfies the following integrability condition: If X and Y are in ξ, then so is $[JX,Y]+[X,JY]$ and $J([JX,Y]+[X,JY]) = [JX,JY]-[X,Y]$. A CR structure J can extend to $\xi\otimes\mathbb{C}$ and decomposes $\xi\otimes\mathbb{C}$ into the direct sum of $T^{1,0}M$ and $T^{0,1}M$ which are eigenspaces of J with respect to eigenvalues i and $-i$, respectively. A pseudohermitian structure compatible with ξ is a CR structure J compatible with ξ together with a choice of contact form θ. Such a choice determines a unique real vector field T transverse to ξ, which is called the characteristic vector field of θ, such that $\theta(T) = 1$ and $\mathcal{L}_T\theta = 0$ or $d\theta(T,\cdot) = 0$.

Let $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T^{1,0}$, $Z_{\bar{\alpha}} = \overline{Z_\alpha} \in T^{0,1}$, and T is the characteristic vector field. Then $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$, which is the admissible coframe dual to $\{T, Z_\alpha, Z_{\bar{\alpha}}\}$, satisfies

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}},$$

for some hermitian matrix of functions $(h_{\alpha\bar{\beta}})$. We call $\{\theta^\alpha\}$ an admissible coframe for θ. Moreover, we say (M,θ) is a (strictly pseudoconvex) CR manifold if the hermitian matrix $(h_{\alpha\bar{\beta}})$ is positive definite. We always assume it through this paper.

A complex-valued q-form η is said to be of type $(q,0)$ if $T^{0,1}\rfloor\eta = 0$, and type $(0,q)$ if $T^{1,0}\rfloor\eta = 0$. The canonical bundle K_M is the complex line bundle of $(n+1,0)$-forms. The Levi form $L_\theta := \langle\ ,\ \rangle$ is the Hermitian form on $T^{1,0}$ defined by

$$\langle Z, W\rangle = -i\left\langle d\theta, Z \wedge \overline{W}\right\rangle$$

or

$$L_\theta(U^\alpha Z_\alpha, V^\beta Z_\beta) = h_{\alpha\bar\beta} U^\alpha V^{\overline{\beta}}.$$

We use the matrix $h_{\alpha\bar\beta}$ in the usual way to raise and lower indices: $A^\alpha{}_{\overline{\beta}} = h_{\gamma\bar\beta} A^{\alpha\gamma}$. We can extend $\langle\ ,\ \rangle$ to $T^{0,1}$ by defining $\left\langle \overline{Z}, \overline{W}\right\rangle = \overline{\langle Z, W\rangle}$ for all $Z, W \in T^{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T^{1,0}$, also denoted by $\langle\ ,\ \rangle$, and hence on all the induced tensor bundles.

The pseudohermitian connection of (J, θ) is the connection ∇ on $TM \otimes \mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha \in T^{1,0}$ by

$$\nabla Z_\alpha = \omega_\alpha{}^\beta \otimes Z_\beta, \quad \nabla Z_{\bar\alpha} = \omega_{\bar\alpha}{}^{\bar\beta} \otimes Z_{\bar\beta}, \quad \nabla T = 0,$$

where $\omega_\alpha{}^\beta$ are the 1-forms uniquely determined by the following equations:

$$\begin{gathered} d\theta^\beta = \theta^\alpha \wedge \omega_\alpha{}^\beta + \theta \wedge \tau^\beta, \\ \tau_\alpha \wedge \theta^\alpha = 0, \quad \omega_\alpha{}^\beta + \omega_{\bar\beta}{}^{\bar\alpha} = dh_{\alpha\bar\beta}. \end{gathered} \tag{2.1}$$

We can write $\tau_\alpha = A_{\alpha\beta}\theta^\beta$ with $A_{\alpha\beta} = A_{\beta\alpha}$. The curvature of the Tanaka-Webster connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\bar\alpha}\}$, is

$$\begin{aligned} \Pi_\beta{}^\alpha &= \overline{\Pi_{\bar\beta}{}^{\bar\alpha}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha, \\ \Pi_0{}^\alpha &= \Pi_\alpha{}^0 = \Pi_0{}^{\bar\beta} = \Pi_{\bar\beta}{}^0 = \Pi_0{}^0 = 0. \end{aligned}$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + W_\beta{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\bar\rho}\theta^{\bar\rho} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha, \tag{2.2}$$

where the coefficients satisfy

$$R_{\beta\bar\alpha\rho\bar\sigma} = \overline{R_{\alpha\bar\beta\sigma\bar\rho}} = R_{\bar\alpha\beta\bar\sigma\rho} = R_{\rho\bar\alpha\beta\bar\sigma}, \quad W_{\beta\bar\alpha\gamma} = W_{\gamma\bar\alpha\beta}.$$

It is useful to note that contraction of (2.2) yields

$$\Pi_\alpha{}^\alpha = d\omega_\alpha{}^\alpha = R_{\rho\bar\sigma}\theta^\rho \wedge \theta^{\bar\sigma} + W_\alpha{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^{\overline{\alpha}}{}_{\overline{\alpha}\bar\rho}\theta^{\bar\rho} \wedge \theta. \tag{2.3}$$

We will denote components of covariant derivatives with indices preceded by a comma; thus write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \bar\alpha\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\bar\alpha}\}$. For derivatives of a function, we will often omit the comma, for instance,

$$\varphi_\alpha = Z_\alpha\varphi, \ \varphi_{\alpha\bar\beta} = Z_{\bar\beta}Z_\alpha\varphi - \omega_\alpha{}^\gamma(Z_{\bar\beta})Z_\gamma\varphi, \ \varphi_0 = T\varphi$$

for a (smooth) function φ. The tangential Cauchy-Riemann operator ∂_b is defined locally as $\partial_b\varphi = \varphi_\alpha\theta^\alpha$, and $\overline{\partial}_b$ is the complex conjugate of ∂_b such that $\overline{\partial}_b\varphi = \varphi_{\overline{\alpha}}\theta^{\overline{\alpha}}$. Then the formal adjoint of ∂_b on functions (with respect to the Levi form and the volume form $d\mu = \theta \wedge (d\theta)^n$) is $\partial_b^* = -\delta_b$. Here δ_b is the divergence operator that takes $(1,0)$-forms to functions by $\delta_b(\sigma_\alpha\theta^\alpha) = \sigma_\alpha,^\alpha$ and $\bar{\delta}_b(\sigma_{\overline{\alpha}}\theta^{\overline{\alpha}}) = \sigma_{\overline{\alpha}},^{\overline{\alpha}}$. In general, we define an L^2 inner product by

$$(\omega,\ \zeta) = \int_M < \omega,\ \zeta > \theta \wedge (d\theta)^n$$

for any $(0,q)$-form $\omega,\ \zeta$ on M. Here $< \omega,\ \zeta >= \omega_{\overline{\alpha}_1\ldots\overline{\alpha}_q}\zeta^{\overline{\alpha}_1\ldots\overline{\alpha}_q}$ with

$$\omega = \omega_{\overline{\alpha}_1\ldots\overline{\alpha}_q}\theta^{\overline{\alpha}_1} \wedge \ldots \wedge \theta^{\overline{\alpha}_q} \quad \text{and} \quad \zeta = \zeta_{\overline{\alpha}_1\ldots\overline{\alpha}_q}\theta^{\overline{\alpha}_1} \wedge \ldots \wedge \theta^{\overline{\alpha}_q}.$$

The formal adjoint $\overline{\partial}_b^*$ of $\overline{\partial}_b$ is given by

$$(\overline{\partial}_b^*\varpi,\ \zeta) = (\varpi,\ \overline{\partial}_b\zeta)$$

for any $(0,q)$-form ζ and $(0,q+1)$-form ϖ on M.

For a function φ, the subgradient ∇_b is defined locally by $\nabla_b\varphi = \varphi^\alpha Z_\alpha + \varphi^{\overline{\alpha}} Z_{\bar{\alpha}}$. The sub-Laplacian Δ_b on functions is defined as

$$\Delta_b\varphi = -({\varphi_\alpha}^\alpha + {\varphi_{\overline{\alpha}}}^{\overline{\alpha}}).$$

The Kohn-Rossi Laplacian $\Box_b$ on functions is defined by

$$\Box_b\varphi = 2\overline{\partial}_b^*\overline{\partial}_b\varphi = (\Delta_b + inT)\varphi = -2{\varphi_{\overline{\alpha}}}^{\overline{\alpha}}$$

and is defined on $(0,q)$-forms by

$$\Box_b = 2(\overline{\partial}_b^*\overline{\partial}_b + \overline{\partial}_b\overline{\partial}_b^*).$$

3. Pseudo-Einstein contact structure and Poincaré-Lelong equation

In this section, we address the Calabi-Lee conjecture for pseudo-Einstein contact structure via the CR Poincaré-Lelong equation (3.8). Let $\{\theta^\alpha\}$ be an admissible coframe for θ with

$$d\theta = ih_{\alpha\bar{\beta}}\theta^\alpha \wedge \theta^{\bar{\beta}}.$$

Then it is convenient to work with the coframe $\{\widetilde{\theta}^\alpha = \theta^\alpha + 2iu^\alpha\theta\}$ which is admissible for $\widetilde{\theta} = e^{2u}\theta$ with

$$d\widetilde{\theta} = i\widetilde{h}_{\alpha\bar{\beta}}\widetilde{\theta}^\alpha \wedge \widetilde{\theta}^{\bar{\beta}}.$$

Thus

$$\widetilde{h}_{\alpha\overline{\beta}} = e^{2u}h_{\alpha\overline{\beta}}. \tag{3.1}$$

Theorem 3.1. *Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold with a fixed contact class $[\theta]$ and $\widetilde{\theta}\in[\theta]$ with $\widetilde{\theta}=e^{2u}\theta$. Then $\widetilde{\theta}=e^{2u}\theta$ is a pseudo-Einstein contact structure if and only if u is the solution of*

$$(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha})=\frac{1}{(n+2)}\{R_{\alpha\overline{\beta}}-\frac{1}{n}[(n+2)\Delta_b u+R]h_{\alpha\overline{\beta}}\}. \tag{3.2}$$

Proof. Let Φ be a closed 2-form representing the first Chern class $c_1(T^{1,0}M)$ as we have introduced before. We need to find a contact structure $\widetilde{\theta}$ such that

$$\widetilde{Ric}(X,Y)=\Phi(X,Y)$$

for all $X,Y\in\ker\theta$. It is natural to choose $\widetilde{\theta}$ in a fixed contact class $[\theta]$ with $\widetilde{\theta}=e^{2u}\theta$. It follows from [**16**, Lemma 2.4] that

$$\widetilde{R}_{\alpha\overline{\beta}}=R_{\alpha\overline{\beta}}-(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha})+[\Delta_b u-2(n+1)|\nabla u|^2]h_{\alpha\overline{\beta}}. \tag{3.3}$$

Next we choose a purely imaginary 1-form

$$\sigma=-(n+2)(u_{\overline{\beta}}\theta^{\overline{\beta}}-u_\alpha\theta^\alpha)-i[\Delta_b u-2(n+1)|\nabla u|^2]\theta.$$

Then

$$d\sigma=\{-(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha})+[\Delta_b u-2(n+1)|\nabla u|^2]h_{\alpha\overline{\beta}}\}\theta^\alpha\wedge\theta^{\overline{\beta}} \mod \theta$$

and

$$\widetilde{Ric}(X,Y)=Ric(X,Y)+d\sigma(X,Y). \tag{3.4}$$

Now if $\widetilde{\theta}=e^{2u}\theta$ is a pseudo-Einstein contact structure, then

$$\widetilde{R}_{\alpha\overline{\beta}}=\frac{\widetilde{R}}{n}\widetilde{h}_{\alpha\overline{\beta}}. \tag{3.5}$$

But for $\widetilde{\theta}=e^{2u}\theta$, we have ([**16**])

$$\widetilde{R}=e^{-2u}[R+2(n+1)\Delta_b u-2n(n+1)|\nabla u|^2]. \tag{3.6}$$

Therefore, it follows from (3.1), (3.3), (3.5), and (3.6) that

$$(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha})=R_{\alpha\overline{\beta}}-\frac{1}{n}[(n+2)\Delta_b u+R]h_{\alpha\overline{\beta}}.$$

q.e.d.

Remark 3.1. *By* [**16**], *if u is a CR-pluriharmonic function, then there exists a smooth function φ on M such that*

$$u_{\alpha\overline{\beta}}=\varphi h_{\alpha\overline{\beta}}.$$

Note that $u_{\overline{\beta}\alpha}=u_{\alpha\overline{\beta}}-iu_0h_{\alpha\overline{\beta}}$. Hence $u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha}=2u_{\alpha\overline{\beta}}-iu_0h_{\alpha\overline{\beta}}=(2\varphi-iu_0)h_{\alpha\overline{\beta}}$. Now if θ is pseudo-Einstein, it follows from (3.3) *that*

$$\widetilde{R}_{\alpha\overline{\beta}}=e^{-2u}[\frac{R}{n}-(n+2)(2\varphi-iu_0)+(\Delta_b u-2(n+1)|\nabla u|^2)]\widetilde{h}_{\alpha\overline{\beta}}$$

and $\widetilde{\theta}$ is also pseudo-Einstein.

Define

$$d_H = \partial_b + \overline{\partial}_b \quad \text{and} \quad d_H^c = i(\overline{\partial}_b - \partial_b).$$

Thus

$$d_H d_H^c = i(\partial_b \overline{\partial}_b - \overline{\partial}_b \partial_b)$$

and

$$\begin{aligned} d_H d_H^c u &= i[\partial_b(u_{\overline{\beta}}\theta^{\overline{\beta}}) - \overline{\partial}_b(u_\alpha \theta^\alpha)] \\ &= i[(u_{\overline{\beta}\alpha}\theta^\alpha \wedge \theta^{\overline{\beta}}) - (u_{\alpha\overline{\beta}}\theta^{\overline{\beta}} \wedge \theta^\alpha)] \\ &= i(u_{\overline{\beta}\alpha} + u_{\alpha\overline{\beta}})\theta^\alpha \wedge \theta^{\overline{\beta}}. \end{aligned} \tag{3.7}$$

It follows from Theorem 3.1 that

Corollary 3.1. *Let (M, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold with a fixed contact class $[\theta]$ and $\widetilde{\theta} \in [\theta]$ with $\widetilde{\theta} = e^{2u}\theta$. Then $\widetilde{\theta} = e^{2u}\theta$ is a pseudo-Einstein contact structure if and only if u is the solution of the CR Poincaré-Lelong equation*

$$d_H d_H^c u = \frac{i}{(n+2)}\{R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}\}\theta^\alpha \wedge \theta^{\overline{\beta}} \tag{3.8}$$

which is equivalent to

$$d_H d_H^c u = \frac{1}{(n+2)}\{iR_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]d\theta\}. \tag{3.9}$$

4. CR Hodge-Laplacian heat equation

In this section, we will derive the CR analogue of the Calabi-Yau Theorem via the so-called CR Hodge-Laplacian heat equation. We first define the CR Hodge-Laplacian

$$\Delta_H = -(d_H d_H^* + d_H^* d_H)$$

with $d_H^* = \partial_b^* + \overline{\partial}_b^*$. We know that a k-form ω can be decomposed uniquely as

$$\omega = \omega_1 + \theta \wedge \omega_2$$

with $\omega_1 = \sum_{|I|+|I'|=k} f_{I,I'}\theta^I \wedge \theta^{I'}$ and $\omega_2 = \sum_{|I|+|I'|=k-1} g_{I,I'}\theta^I \wedge \theta^{I'}$. Thus

$$d(\omega_1 + \theta \wedge \omega_2) = (d_H\omega_1 + (d\theta) \wedge \omega_2) + \theta \wedge (T\omega_1 - d_H\omega_2). \tag{4.1}$$

Furthermore

$$\partial_b^2 = \overline{\partial}_b^2 = \partial_b \overline{\partial}_b^* + \overline{\partial}_b^* \partial_b = 0$$

and

$$d_H^2 = \partial_b \overline{\partial}_b + \overline{\partial}_b \partial_b = -Te(d\theta)$$

with $e(d\theta)\omega = d\theta \wedge \omega$. Straightforward computation ([**19**]) yields

$$\Delta_H = -\frac{1}{2}(\Box_b + \overline{\Box}_b).$$

Lemma 4.1. *For any* $(1,1)$*-form* ψ *with* $\psi = \psi_{\alpha\overline{\beta}}\theta^{\alpha} \wedge \theta^{\overline{\beta}}$*, we have*

$$(\Delta_H \psi)_{\alpha\overline{\beta}} = -\Delta_b \psi_{\alpha\overline{\beta}} + 2R_{\alpha\overline{\gamma}\mu\overline{\beta}}\psi_{\gamma\overline{\mu}} - (R_{\gamma\overline{\beta}}\psi_{\alpha\overline{\gamma}} + R_{\alpha\overline{\gamma}}\psi_{\gamma\overline{\beta}}).$$

Here $\Delta_b \psi_{\alpha\overline{\beta}} = -(\psi_{\alpha\overline{\beta},\gamma\overline{\gamma}} + \psi_{\alpha\overline{\beta},\overline{\gamma}\gamma})$.

Proof. Direct computation ([**8**], [**15**]) shows that

$$\frac{1}{2}(\Box_b \psi)_{\alpha\overline{\beta}} = -\psi_{\alpha\overline{\beta},\overline{\gamma}\gamma} - 2i\psi_{\alpha\overline{\beta},0} - R_{\alpha\overline{\gamma}\mu\overline{\beta}}\psi_{\gamma\overline{\mu}} + R_{\gamma\overline{\beta}}\psi_{\alpha\overline{\gamma}}$$

and

$$\frac{1}{2}(\overline{\Box}_b \psi)_{\alpha\overline{\beta}} = -\psi_{\alpha\overline{\beta},\gamma\overline{\gamma}} + 2i\psi_{\alpha\overline{\beta},0} - R_{\alpha\overline{\gamma}\mu\overline{\beta}}\psi_{\gamma\overline{\mu}} + R_{\alpha\overline{\gamma}}\psi_{\gamma\overline{\beta}}.$$

The conclusion of this lemma follows immediately. q.e.d.

We will work on the so-called CR Hodge-Laplacian heat equation on $M \times [0, \infty)$

$$\frac{\partial}{\partial t}\eta(x,t) = \Delta_H \eta(x,t). \tag{4.2}$$

It follows from Lemma 4.1 that the equation (4.2) is equivalent to the CR Lichnerowicz-Laplacian heat equation:

$$(\frac{\partial}{\partial t} + \Delta_b)\eta_{\alpha\overline{\beta}}(x,t) = 2R_{\alpha\overline{\gamma}\mu\overline{\beta}}\eta_{\gamma\overline{\mu}} - (R_{\gamma\overline{\beta}}\eta_{\alpha\overline{\gamma}} + R_{\alpha\overline{\gamma}}\eta_{\gamma\overline{\beta}}) \tag{4.3}$$

for any $(1,1)$-form $\eta(x,t) = \eta_{\alpha\overline{\beta}}\theta^{\alpha} \wedge \theta^{\overline{\beta}}$.

Lemma 4.2. *For any* d_H*-closed* $(1,1)$*-form* η*, we have*

$$d_H d_H^c(\wedge \eta) = \Delta_H \eta.$$

Here $\wedge$ *is the trace operator.*

Proof. We first compute

$$\begin{aligned} d_H d_H^c(\wedge\eta) &= \tfrac{1}{2}(d_H d_H^c \wedge \eta - d_H^c d_H \wedge \eta) \\ &= \tfrac{1}{2} i[(\partial_b\overline{\partial}_b - \overline{\partial}_b\partial_b) \wedge \eta - (\overline{\partial}_b\partial_b - \partial_b\overline{\partial}_b) \wedge \eta]. \end{aligned}$$

Recall that ([**19**])

$$[\wedge,\ \partial_b] = i\overline{\partial}_b^* \quad \text{and} \quad [\wedge,\ \overline{\partial}_b] = -i\partial_b^*.$$

Thus

$$\wedge\partial_b\eta - \partial_b \wedge \eta = i\overline{\partial}_b^*\ \eta$$

and since $\partial_b\eta = 0 = \overline{\partial}_b\eta$,

$$\partial_b \wedge \eta = -i\overline{\partial}_b^*\ \eta.$$

Similarly, we have

$$\overline{\partial}_b \wedge \eta = i\partial_b^*\ \eta.$$

Finally, these computations imply that

$$\begin{aligned} d_H d_H^c(\wedge\eta) &= -\tfrac{1}{2}[(\overline{\partial}_b\overline{\partial}_b^* + \partial_b\partial_b^*)\eta + (\overline{\partial}_b\overline{\partial}_b^* + \partial_b\partial_b^*)\eta] \\ &= -(\overline{\partial}_b\overline{\partial}_b^*\eta + \partial_b\partial_b^*\eta) \\ &= -[(\overline{\partial}_b\overline{\partial}_b^* + \overline{\partial}_b^*\overline{\partial}_b)\eta + (\partial_b\partial_b^* + \partial_b^*\partial_b)\eta] \\ &= -\tfrac{1}{2}(\Box_b\eta + \overline{\Box}_b\eta) \\ &= \Delta_H\eta. \end{aligned}$$

This completes the proof of the lemma. q.e.d.

Now we are in a position to discuss the heat equation for the CR Hodge-Laplacian. This approach was initiated by Ni and Tam ([**20**]) for Kähler manifolds. Here we generalize it to the sub-Riemannian setting. The heat equation associated with a subelliptic differential operator is a very interesting subject which was studied by many mathematicians extensively in the past 20 years; see, *e.g.,* Beals, Greiner, and Stanton ([**1**]) and Beals, Gaveau, and Greiner ([**2**]).

Theorem 4.1. *Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold. Suppose the following are true:*

(i) There is a real $(1,1)$-form $\eta(x,t)$ satisfying

$$\begin{cases} \frac{\partial}{\partial t}\eta(x,t) = \Delta_H\eta(x,t), & M\times[0,\infty) \\ \eta(x,0) = \rho(x). \end{cases}$$

Here $\rho(x) = i\rho_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}}$ is a real d_H-closed $(1,1)$-form with $\rho_{\alpha\overline{\beta}} = \frac{1}{(n+2)}\{R_{\alpha\overline{\beta}} - \frac{r}{n}h_{\alpha\overline{\beta}}\}$ and $r = \int_M R d\mu \big/ \int_M d\mu$ such that $\eta(x,t)$ is d_H-closed and

$$\lim_{t\to\infty}\eta(x,t) = 0. \tag{4.4}$$

(ii) There is a smooth real-valued function u solving $\Delta_b u = -tr(\rho_{\alpha\overline{\beta}}) = \frac{1}{n+2}(r-R)$ and a solution $v(x,t)$ of

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) = -\Delta_b v(x,t), & M\times[0,\infty) \\ v(x,0) = u(x) \end{cases}$$

and

$$\lim_{t\to\infty} d_H d_H^c v(x,t) = 0. \tag{4.5}$$

Then

$$d_H d_H^c u = \frac{i}{(n+2)}\{R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}\}\theta^\alpha\wedge\theta^{\overline{\beta}}.$$

Proof. Define

$$\phi(x,t) = tr(\eta_{\alpha\overline{\beta}}(x,t)).$$

Then

$$\begin{cases} \frac{\partial}{\partial t}\phi(x,t) = tr(\frac{\partial}{\partial t}\eta_{\alpha\overline{\beta}}(x,t)) = tr(\Delta_H\eta)_{\alpha\overline{\beta}} = -\Delta_b\phi(x,t), \\ \phi(x,0) = tr(\eta_{\alpha\overline{\beta}}(x,0)) = tr(\rho_{\alpha\overline{\beta}}(x)). \end{cases}$$

Consider

$$w(x,t) = -\int_0^t \phi(x,s)ds.$$

Thus

$$\frac{\partial}{\partial t} w(x,t) = -\phi(x,t) = -tr(\eta_{\alpha\overline{\beta}}(x,t)) \tag{4.6}$$

and

$$\begin{aligned}\Delta_b w(x,t) &= -\int_0^t \Delta_b \phi(x,s)ds = \int_0^t \frac{\partial}{\partial s}\phi(x,s)ds \\ &= tr(\eta_{\alpha\overline{\beta}}(x,t)) - tr(\rho_{\alpha\overline{\beta}}(x)).\end{aligned}$$

Hence

$$\left\{\begin{array}{lcl} \frac{\partial}{\partial t} w(x,t) & = & -\Delta_b w(x,t) - tr(\rho_{\alpha\overline{\beta}}(x)), \\ w(x,0) & = & 0. \end{array}\right. \tag{4.7}$$

Let

$$\widetilde{v}(x,t) = u(x) - w(x,t).$$

Then

$$\begin{array}{rcl} \frac{\partial}{\partial t}\widetilde{v}(x,t) & = & \Delta_b w(x,t) + tr(\rho_{\alpha\overline{\beta}}(x)) \\ & = & -\Delta_b \widetilde{v}(x,t) + \Delta_b u(x) + tr(\rho_{\alpha\overline{\beta}}(x)) \\ & = & -\Delta_b \widetilde{v}(x,t). \end{array}$$

It follows that

$$\left\{\begin{array}{lcl} \frac{\partial}{\partial t}\widetilde{v}(x,t) & = & -\Delta_b \widetilde{v}(x,t), \\ \widetilde{v}(x,0) & = & u(x). \end{array}\right. \tag{4.8}$$

Finally by maximum principle, we have

$$v(x,t) = \widetilde{v}(x,t) = u(x) - w(x,t). \tag{4.9}$$

By applying Lemma 4.2, we compute

$$\begin{array}{rcl} \frac{d}{dt}(\eta + d_H d_H^c w) & = & \Delta_H \eta + d_H d_H^c(\frac{\partial}{\partial t} w) \\ & = & \Delta_H \eta - d_H d_H^c(\wedge \eta) \\ & = & 0. \end{array}$$

But at $t = 0$,

$$\eta + d_H d_H^c w = \rho.$$

This implies

$$\eta + d_H d_H^c w = \rho$$

and

$$\eta + d_H d_H^c u - d_H d_H^c v = \rho$$

for all $t > 0$. But

$$\lim_{t\to\infty} \eta(x,t) = 0 = \lim_{t\to\infty} d_H d_H^c v(x,t).$$

Therefore

$$d_H d_H^c u = \rho. \tag{4.10}$$

Note that

$$d_H d_H^c u = i(u_{\overline{\beta}\alpha} + u_{\alpha\overline{\beta}})\theta^\alpha \wedge \theta^{\overline{\beta}}$$

and

$$\rho = \frac{i}{(n+2)}[R_{\alpha\overline{\beta}} - \frac{r}{n} h_{\alpha\overline{\beta}}]\theta^\alpha \wedge \theta^{\overline{\beta}}.$$

It follows from (4.10) that

$$r = (n+2)\Delta_b u + R$$

and then

$$d_H d_H^c u = \frac{i}{(n+2)}\{R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}\}\theta^\alpha \wedge \theta^{\overline{\beta}}.$$

The proof of the theorem is therefore complete. q.e.d.

In general, if condition (4.4) does not hold, we still have a real $(1,1)$-form

$$\lim_{t\to\infty} \eta(x,t) := \frac{i}{(n+2)}(\eta^\infty_{\alpha\overline{\beta}}(x) + \eta^\infty_{\overline{\beta}\alpha}(x))\theta^\alpha \wedge \theta^{\overline{\beta}}$$

with the following property:

Corollary 4.1. *Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold. Then there is a smooth real-valued function u solving $\Delta_b u = \frac{1}{n+2}(r-R)$ such that*

$$(n+2)(u_{\overline{\beta}\alpha} + u_{\alpha\overline{\beta}}) = R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}} - (\eta^\infty_{\alpha\overline{\beta}} + \eta^\infty_{\overline{\beta}\alpha})$$

with

$$\eta^\infty_{\alpha\overline{\alpha}} + \eta^\infty_{\overline{\alpha}\alpha} = 0.$$

In the following, we will investigate situations when $\eta^\infty_{\alpha\overline{\beta}} + \eta^\infty_{\overline{\beta}\alpha} = 0$. More precisely, we are able to derive the following CR analogue of Calabi-Yau Theorem via the Hodge-Laplacian parabolic equation which recaptures well-known results by the elliptic method due to Cao-Chang ([**4**]) and Lee ([**16**]).

Theorem 4.2. *Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold for $n \geq 2$. Assume that*

(i) (M,θ) is the smooth boundary of a bounded strongly pseudo-convex domain Ω in a complete Stein manifold V^{n+1}, or

(ii) (M,θ) has positive pseudohermitian Ricci curvature with $c_1(T^{1,0}M) = 0$.

Then there is a smooth real-valued function u solving $\Delta_b u = \frac{1}{n+2}(r - R)$ and $r = \int_M R d\mu \big/ \int_M d\mu$ such that

$$(n+2)d_H d_H^c u = i\{R_{\alpha\overline{\beta}} - \frac{r}{n} h_{\alpha\overline{\beta}}\}\theta^\alpha \wedge \theta^{\overline{\beta}}$$

and

$$(n+2)d_Hd_H^cu = i\{R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}\}\theta^\alpha \wedge \theta^{\overline{\beta}}.$$

Hence $e^{2u}\theta$ *is a pseudo-Einstein contact structure.*

Proof. In order to apply Theorem 4.1, we need to justify (4.4) and (4.5). We first note that

$$\rho(x) = \frac{1}{(n+2)}\{iR_{\alpha\overline{\beta}}\theta^\alpha \wedge \theta^{\overline{\beta}} - \frac{r}{n}d\theta\}$$

is an d_H-closed $(1,1)$-form. Since M is closed, then $\eta(x,t)$ is also d_H-closed. If $c_1(T^{1,0}M) = 0$, there exists a global imaginary one-form

$$\sigma(x) = \sigma_{\overline{\beta}}(x)\theta^{\overline{\beta}} - \sigma_\alpha(x)\theta^\alpha + i\sigma_0(x)\theta \tag{4.11}$$

on M such that

$$d{\omega_\alpha}^\alpha(x) = d\sigma(x). \tag{4.12}$$

Thus ([**16**])

$$R_{\alpha\overline{\beta}}(x) = \sigma_{\overline{\beta},\alpha}(x) + \sigma_{\alpha,\overline{\beta}}(x) - \sigma_0(x)h_{\alpha\overline{\beta}}(x). \tag{4.13}$$

It follows from (4.1) that

$$(n+2)\rho(x) = id_H\widetilde{\sigma} - (\sigma_0 + \frac{r}{n})d\theta = id_H\widetilde{\sigma} - d_H((\sigma_0 + \frac{r}{n})\theta)$$

with

$$\widetilde{\sigma}(x) = (\sigma_{\overline{\beta}}\theta^{\overline{\beta}} - \sigma_\alpha\theta^\alpha).$$

Thus

$$\eta(x,t) \quad = \quad d_H[\delta(x,t) + \kappa(x,t)]$$

with

$$\delta(x,t) = i[l_{\overline{\beta}}(x,t)\theta^{\overline{\beta}} - l_\alpha(x,t)\theta^\alpha)] \quad \text{and} \quad \kappa(x,t) = k(x,t)\theta.$$

Here

$$\eta(x,0) = \rho(x) = d_H[\delta(x,0) + \kappa(x,0)]$$

and

$$\delta(x,0) = i[\sigma_{\overline{\beta}}\theta^{\overline{\beta}} - \sigma_\alpha\theta^\alpha)] \ ; \ \kappa(x,0) = -(\sigma_0 + \frac{r}{n})\theta. \tag{4.14}$$

Now

$$\Delta_H\eta(x,t) \quad = \quad \tfrac{\partial}{\partial t}\eta(x,t) = d_H[\tfrac{\partial}{\partial t}\delta(x,t)) + \tfrac{\partial}{\partial t}\kappa(x,t)]$$

and since $d_H\eta(x,t) = 0$,

$$\Delta_H\eta(x,t) = -(d_Hd_H^* + d_H^*d_H)\eta(x,t) = -d_Hd_H^*\eta(x,t).$$

Hence

$$\frac{\partial}{\partial t}\delta(x,t) = -d_H^*d_H\delta(x,t) + \widetilde{\delta} \tag{4.15}$$

with $d_H\widetilde{\delta}=0$ and $\widetilde{\delta}=\widetilde{\delta}^{(1,0)}+\widetilde{\delta}^{(0,1)}$ for real-valued $(1,0)$-form $\widetilde{\delta}^{(1,0)}$ and $(0,1)$-form $\widetilde{\delta}^{(0,1)}$ which implies

$$\overline{\partial}_b\widetilde{\delta}^{(0,1)}=0 \quad \text{and} \quad \partial_b\widetilde{\delta}^{(1,0)}=0. \tag{4.16}$$

On the other hand, if $d_H(f(x,t)\theta)=0$, then we have $0=d(f(x,t)\theta)=f(x,t)d\theta$ which implies $f(x,t)=0$. Thus

$$\frac{\partial}{\partial t}(\kappa(x,t))=-d_H^*d_H\kappa(x,t). \tag{4.17}$$

(i) (M,θ) is the smooth boundary of a bounded strongly pseudoconvex domain Ω in a complete Stein manifold V^{n+1} for $n\geq 2$. By a theorem of Kohn ([**15**], [**3**]), (4.16) implies that there is a smooth real-valued function g on M such that

$$\widetilde{\delta}=i(g_{\overline{\beta}}\theta^{\overline{\beta}}-g_\alpha\theta^\alpha).$$

Since $d_H\widetilde{\delta}=0$, we have

$$g_{\overline{\beta}\alpha}+g_{\alpha\overline{\beta}}=0$$

and then

$$\Delta_b g=0.$$

Therefore g is constant and

$$\widetilde{\delta}=0. \tag{4.18}$$

(ii) Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold of positive pseudohermitian Ricci curvature. It follows from ([**15**]) that

$$\widetilde{\delta}^{(0,1)}=ig_{\overline{\beta}}\theta^{\overline{\beta}}+\gamma^{(0,1)} \tag{4.19}$$

with $\Box_b\gamma^{(0,1)}=0$. Since the pseudohermitian Ricci curvature is positive, by Lee's result ([**16**, Proposition 6.4]), we have

$$\gamma^{(0,1)}=0.$$

Then

$$\widetilde{\delta}^{(0,1)}=ig_{\overline{\beta}}\theta^{\overline{\beta}}.$$

Similarly, we have

$$\widetilde{\delta}^{(1,0)}=-ig_\alpha\theta^\alpha.$$

Hence

$$\widetilde{\delta}=i(g_{\overline{\beta}}\theta^{\overline{\beta}}-g_\alpha\theta^\alpha) \tag{4.20}$$

and again $\widetilde{\delta}=0$ as in (i). Now from (4.15) and (4.17), we have

$$\frac{\partial}{\partial t}\delta(x,t)=-d_H^*d_H\delta(x,t)$$

and

$$\frac{\partial}{\partial t}\kappa(x,t)=-d_H^*d_H\kappa(x,t).$$

Thus

$$\begin{aligned}\tfrac{d}{dt}\textstyle\int_M ||\delta(x,t)||^2 d\mu &= -2(d_H^* d_H \delta(x,t), \delta(x,t)) \\ &= -2(d_H \delta(x,t), d_H \delta(x,t)) \\ &= -2\textstyle\int_M ||d_H \delta(x,t)||^2 d\mu\end{aligned}$$

and

$$\begin{aligned}\tfrac{d}{dt}\textstyle\int_M ||\kappa(x,t)||^2 d\mu &= -2(d_H^* d_H \kappa(x,t), \kappa(x,t)) \\ &= -2\textstyle\int_M ||d_H \kappa(x,t)||^2 d\mu.\end{aligned}$$

Therefore

$$\lim_{t\to\infty} \eta(x,t) = 0.$$

Similarly,

$$\frac{d}{dt}\int_M ||v||^2 d\mu = -2\int_M ||d_H v||^2 d\mu$$

and then

$$\lim_{t\to\infty} d_H d_H^c v(x,t) = 0.$$

The proof of the theorem is therefore complete. q.e.d.

Now we will express $\widetilde{\delta}$ in the following general form.

Lemma 4.3. *Let (M, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold for $n \geq 2$. Assume that $\widetilde{\delta}$ is a smooth real-valued one-form with*

$$d_H \widetilde{\delta} = 0.$$

Then

$$\widetilde{\delta} = i(\gamma_{\overline{\beta}} \theta^{\overline{\beta}} - \gamma_\alpha \theta^\alpha)$$

and

$$\gamma_{\alpha,\overline{\beta}} + \gamma_{\overline{\beta},\alpha} = 0. \tag{4.21}$$

Here

$$\Box_b(\gamma_{\overline{\beta}} \theta^{\overline{\beta}}) = 0 = \overline{\Box}_b(\gamma_\alpha \theta^\alpha).$$

Proof. Since $d_H \widetilde{\delta} = 0$, it follows from (4.19) and (4.20) that

$$\widetilde{\delta} = i(g_{\overline{\beta}} \theta^{\overline{\beta}} - g_\alpha \theta^\alpha) + i(\gamma_{\overline{\beta}} \theta^{\overline{\beta}} - \gamma_\alpha \theta^\alpha)$$

with $\Box_b \gamma^{(0,1)} = 0 = \overline{\Box}_b \gamma^{(1,0)}$. Again from $d_H \widetilde{\delta} = 0$ and (4.16), we have

$$\partial_b \widetilde{\delta}^{(0,1)} + \overline{\partial}_b \widetilde{\delta}^{(1,0)} = 0.$$

But $\widetilde{\delta}^{(0,1)} = i g_{\overline{\beta}} \theta^{\overline{\beta}} + i\gamma_{\overline{\beta}} \theta^{\overline{\beta}}$ and $\widetilde{\delta}^{(1,0)} = -i g_\alpha \theta^\alpha - i\gamma_\alpha \theta^\alpha$. Then

$$0 = \partial_b \widetilde{\delta}^{(0,1)} + \overline{\partial}_b \widetilde{\delta}^{(1,0)} = i[(g_{\overline{\beta}\alpha} + g_{\alpha\overline{\beta}}) + (\gamma_{\overline{\beta},\alpha} + \gamma_{\alpha,\overline{\beta}})]\theta^\alpha \wedge \theta^{\overline{\beta}}.$$

Therefore

$$(g_{\overline{\beta}\alpha} + g_{\alpha\overline{\beta}}) + (\gamma_{\overline{\beta},\alpha} + \gamma_{\alpha,\overline{\beta}}) = 0$$

and

$$-\Delta_b g + \gamma_{\overline{\alpha},\alpha} + \gamma_{\alpha,\overline{\alpha}} = 0.$$

On the other hand, $\Box_b\gamma^{(0,1)} = 0$ implies that

$$\gamma_{\overline{\alpha},\alpha} = 0 = \gamma_{\alpha,\overline{\alpha}}.$$

Thus

$$\Delta_b g = 0$$

and g is constant. Moreover, we have

$$\gamma_{\alpha,\overline{\beta}} + \gamma_{\overline{\beta},\alpha} = 0.$$

This is exactly (4.21) and the conclusion of the lemma follows immediately. q.e.d.

Remark 4.1. *Theorem* 4.2 *is a special case of Lemma* 4.3 *where* $\gamma^{(0,1)} = 0 = \gamma^{(1,0)}$.

In general, if $\widetilde{\delta}$ is non-vanishing with $d_H\widetilde{\delta} = 0$, then

$$\begin{aligned}\frac{d}{dt}(\delta(x,t),\ \delta(x,t)) &= \frac{d}{dt}\int_M ||\delta(x,t)||^2 d\mu \\ &= -2\int_M ||d_H\delta(x,t)||^2 d\mu - 2(\widetilde{\delta}(x,t),\ \delta(x,t)).\end{aligned} \tag{4.22}$$

We will work on the case of

$$(\widetilde{\delta}(x,t),\ \delta(x,t)) = 0$$

with $d_H\widetilde{\delta} = 0$.

We first consider the special case with

$$\sigma_\alpha = \widetilde{\varphi}_\alpha$$

for a smooth real-valued function $\widetilde{\varphi}$. Then

$$d\omega_\alpha{}^\alpha = d\sigma$$

with

$$\sigma = \widetilde{\varphi}_{\overline{\beta}}\theta^{\overline{\beta}} - \widetilde{\varphi}_\alpha\theta^\alpha + i\sigma_0\theta = i(-d^c_H\widetilde{\varphi} + \sigma_0\theta).$$

On the other hand, it follows from (4.14) and (4.13) that

$$(n+2)\rho(x) = d_H[\delta(x,0) - \gamma(x,0)] = d_H d^c_H\widetilde{\varphi} - (\sigma_0 + \frac{r}{n})\theta.$$

Hence

$$\delta(x,t) = d^c_H\varphi(x,t) \tag{4.23}$$

with $\varphi(x,0)=\widetilde{\varphi}$. By Lemma 4.3 and $\Box_b\gamma^{(0,1)}=0=\overline{\Box}_b\gamma^{(1,0)}$, we have $\overline{\partial}_b^*\gamma^{(0,1)}=0=\partial_b^*\gamma^{(1,0)}$ and then

$$\begin{aligned}(\widetilde{\delta}(x,t),\quad \delta(x,t)) &= (i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}}-\gamma_\alpha\theta^\alpha),\ i(\overline{\partial}_b\varphi-\partial_b\varphi))\\ &= -([\overline{\partial}_b^*(\gamma_{\overline{\beta}}\theta^{\overline{\beta}})-\overline{\partial}_b^*(\gamma_\alpha\theta^\alpha)-\partial_b^*(\gamma_{\overline{\beta}}\theta^{\overline{\beta}})+\partial_b^*(\gamma_\alpha\theta^\alpha)],\varphi)\\ &= -([\overline{\partial}_b^*(\gamma_{\overline{\beta}}\theta^{\overline{\beta}})+\partial_b^*(\gamma_\alpha\theta^\alpha)],\varphi)\\ &= 0.\end{aligned}$$

On the other hand, it follows from (4.14) and (4.13) that

$$R_{\alpha\overline{\beta}}=\widetilde{\varphi}_{\overline{\beta}\alpha}+\widetilde{\varphi}_{\alpha\overline{\beta}}-\sigma_0 h_{\alpha\overline{\beta}}$$

and then

$$R=-\Delta_b\widetilde{\varphi}-n\sigma_0.$$

Hence

$$\begin{aligned}\widetilde{\varphi}_{\overline{\beta}\alpha}+\widetilde{\varphi}_{\alpha\overline{\beta}} &= R_{\alpha\overline{\beta}}+\sigma_0 h_{\alpha\overline{\beta}}\\ &= R_{\alpha\overline{\beta}}-\tfrac{1}{n}(\Delta_b\widetilde{\varphi}+R)h_{\alpha\overline{\beta}}.\end{aligned}$$

Taking $u=\frac{\widetilde{\varphi}}{N+2}$, we have

$$(n+2)(u_{\alpha\overline{\beta}}+u_{\overline{\beta}\alpha})=R_{\alpha\overline{\beta}}-\frac{1}{n}[(n+2)\Delta_b u+R]h_{\alpha\overline{\beta}}$$

and $e^{2u}\theta$ is a pseudo-Einstein contact structure.

Let $\mathcal{H}^{(0,q)}$ be the space of smooth harmonic $(0,q)$-forms with $\mathcal{H}^{(0,q)}=Ker(\Box_b)$. It is of finite dimension in a closed strictly pseudoconvex CR $(2n+1)$-manifold (M,θ) with $1\le q\le n-1$ and $n\ge 2$. Then for any $(0,1)$-form ω, we have the following Hodge-type decomposition ([**15**], [**5**]):

$$\omega=\overline{\partial}_b\overline{\partial}_b^*\zeta\oplus\overline{\partial}_b^*\overline{\partial}_b\zeta\oplus\mathcal{H}^{(0,1)}\omega \tag{4.24}$$

and

$$\overline{\omega}=\partial_b\partial_b^*\overline{\zeta}\oplus\partial_b^*\partial_b\overline{\zeta}\oplus\overline{\mathcal{H}^{(0,1)}\omega} \tag{4.25}$$

for some $(0,1)$-form ζ.

Lemma 4.4. *Let (M,θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold for $n\ge 2$. Assume that*

$$d\omega_\alpha{}^\alpha=d\sigma\quad\text{with}\quad \sigma=i(\sigma_{\overline{\beta}}\theta^{\overline{\beta}}-\sigma_\alpha\theta^\alpha)+i\sigma_0\theta$$

and

$$i(\sigma_{\overline{\beta}}\theta^{\overline{\beta}})\in(\mathcal{H}^{(0,1)})^\perp.$$

Then there is a smooth real-valued function u such that $e^{2u}\theta$ is a pseudo-Einstein contact structure.

Proof. Again, it suffices to show that

$$(\widetilde{\delta}(x,t),\quad \delta(x,t)) = 0$$

if

$$i(\sigma_{\overline{\beta}}\theta^{\overline{\beta}}) = \overline{\partial}_b\overline{\partial}_b^*\zeta_0 + \overline{\partial}_b^*\overline{\partial}_b\zeta_0$$

with $(0,1)$-form ζ_0.

Now from Lemma 4.3, (4.24), (4.25), and $\Box_b\gamma^{(0,1)} = 0 = \overline{\Box}_b\gamma^{(1,0)}$,

$$\begin{aligned}
&(\widetilde{\delta}(x,t),\quad \delta(x,t))\\
&= (i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}} - \gamma_\alpha\theta^\alpha),\quad i(\delta_{\overline{\beta}}\theta^{\overline{\beta}} - \delta_\alpha\theta^\alpha))\\
&= i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}} - \gamma_\alpha\theta^\alpha,\quad \overline{\partial}_b\overline{\partial}_b^*\zeta + \overline{\partial}_b^*\overline{\partial}_b\zeta\ + \partial_b\partial_b^*\overline{\zeta} + \partial_b^*\partial_b\overline{\zeta})\\
&= i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}},\quad \overline{\partial}_b\overline{\partial}_b^*\zeta + \overline{\partial}_b^*\overline{\partial}_b\zeta) - i(\gamma_\alpha\theta^\alpha,\quad \partial_b\partial_b^*\overline{\zeta} + \partial_b^*\partial_b\overline{\zeta})\\
&= i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}},\quad \overline{\partial}_b\overline{\partial}_b^*\zeta) + i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}},\quad \overline{\partial}_b^*\overline{\partial}_b\zeta) - i(\gamma_\alpha\theta^\alpha,\quad \partial_b\partial_b^*\overline{\zeta}) - i(\gamma_\alpha\theta^\alpha,\quad \partial_b^*\partial_b\overline{\zeta})\\
&= 0.
\end{aligned}$$

Here we have used the fact that $(i\delta_{\overline{\beta}}\theta^{\overline{\beta}}) = \overline{\partial}_b\overline{\partial}_b^*\zeta + \overline{\partial}_b^*\overline{\partial}_b\zeta$ and $(-i\delta_\alpha\theta^\alpha) = \partial_b\partial_b^*\overline{\zeta} + \partial_b^*\partial_b\overline{\zeta}$ with $\zeta(x,0) = \zeta_0(x)$ and $\delta_{\overline{\beta}}(x,0) = \sigma_{\overline{\beta}}(x)$. q.e.d.

Now we are in a position to prove the *main theorem*, Theorem 1.1 :

Proof. It suffices to show that

$$(\widetilde{\delta}(x,t),\quad \delta(x,t)) = 0$$

under

$$\frac{\partial}{\partial t}\delta(x,t) = -d_H^* d_H\delta(x,t) + \widetilde{\delta} \tag{4.26}$$

with $d_H\widetilde{\delta} = 0$. We first define H be the subspace of $\mathcal{H}^{(0,1)}$ as

$$H = \{i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}}) \in \mathcal{H}^{(0,1)} \mid\ \gamma_{\alpha,\overline{\beta}} + \gamma_{\overline{\beta},\alpha} = 0\}.$$

It follows from Lemma 4.3 that for $i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}}) \in H$, the real-valued one-form

$$\widetilde{\delta} = i(\gamma_{\overline{\beta}}\theta^{\overline{\beta}} - \gamma_\alpha\theta^\alpha)$$

satisfying

$$d_H\widetilde{\delta} = 0.$$

Note that if $\overline{\delta}(x,t) \in H$, then

$$d_H\frac{\partial}{\partial t}\overline{\delta}(x,t) = 0$$

and

$$\frac{\partial}{\partial t}\overline{\delta}(x,t) \in H.$$

Hence, for the real one-form $\delta(x,t) = i(\delta_{\overline{\beta}}(x,t)\theta^{\overline{\beta}} - \delta_\alpha(x,t)\theta^\alpha)$ with $\delta_{\overline{\beta}}(x,0) = \sigma_{\overline{\beta}}(x)$ as in (4.26), we may assume that

$$(i\delta_{\overline{\beta}}(x,t)\theta^{\overline{\beta}}) = \overline{\partial}_b\overline{\partial}_b^*\zeta \oplus \overline{\partial}_b^*\overline{\partial}_b\zeta \oplus \mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}})$$

with

$$\mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}}) \perp H. \tag{4.27}$$

This also implies

$$-(i\delta_\alpha(x,t)\theta^\alpha) = \partial_b\partial_b^*\overline{\zeta} \oplus \partial_b^*\partial_b\overline{\zeta} \oplus \mathcal{H}^{(1,0)}(-i\delta_\alpha(x,t)\theta^\alpha)$$

with

$$\mathcal{H}^{(1,0)}(-i\delta_\alpha(x,t)\theta^\alpha) \perp \overline{H}. \tag{4.28}$$

Thus from Lemma 4.4, (4.27), and (4.28), we have

$$\begin{aligned}
&(\widetilde{\delta}(x,t), \quad \delta(x,t)) \\
&= (i\gamma_{\overline{\beta}}\theta^{\overline{\beta}} - i\gamma_\alpha\theta^\alpha, \quad i(\delta_{\overline{\beta}}(x,t)\theta^{\overline{\beta}} - \delta_\alpha(x,t)\theta^\alpha)) \\
&= (i\gamma_{\overline{\beta}}\theta^{\overline{\beta}}, \quad \overline{\partial}_b\overline{\partial}_b^*\zeta + \overline{\partial}_b^*\overline{\partial}_b\zeta \ + \mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}}) + \partial_b\partial_b^*\overline{\zeta} + \partial_b^*\partial_b\overline{\zeta} \\
&\quad +\mathcal{H}^{(1,0)}(i\delta_\alpha\theta^\alpha)) - (i\gamma_\alpha\theta^\alpha, \quad \overline{\partial}_b\overline{\partial}_b^*\zeta + \overline{\partial}_b^*\overline{\partial}_b\zeta \ + \mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}}) \\
&\quad +\partial_b\partial_b^*\overline{\zeta} + \partial_b^*\partial_b\overline{\zeta} + \mathcal{H}^{(1,0)}(i\delta_\alpha\theta^\alpha)) \\
&= (i\gamma_{\overline{\beta}}\theta^{\overline{\beta}}, \ \mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}}) + \mathcal{H}^{(1,0)}(i\delta_\alpha\theta^\alpha)) \\
&\quad -(i\gamma_\alpha\theta^\alpha, \ \mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}}) + \mathcal{H}^{(1,0)}(i\delta_\alpha\theta^\alpha)) \\
&= (i\gamma_{\overline{\beta}}\theta^{\overline{\beta}}, \quad \mathcal{H}^{(0,1)}(i\delta_{\overline{\beta}}\theta^{\overline{\beta}})) - (i\gamma_\alpha\theta^\alpha, \ \mathcal{H}^{(1,0)}(i\delta_\alpha\theta^\alpha)) \\
&= 0.
\end{aligned}$$

This completes the proof of Theorem 1.1. q.e.d.

5. The CR Frankel conjecture

In this section, we provide a partial answer of the CR Frankel conjecture in a closed spherical strictly pseudoconvex CR $(2n+1)$-manifold.

We start with the proof of Theorem 1.3:

Proof. It follows from the contracted Bianchi identity ([**16**, (2.11)])

$$R_{\alpha\overline{\beta}},^{\overline{\beta}} = R_\alpha - i(n-1)A_{\alpha\beta},^{\beta}. \tag{5.1}$$

On the other hand, from (1.4) of Theorem 1.1, we have

$$R_{\alpha\overline{\beta}} = \tfrac{R}{n}h_{\alpha\overline{\beta}} \tag{5.2}$$

if R is the positive constant Tanaka-Webster scalar curvature. This and (5.1) imply

$$A_{\alpha\beta},^{\beta} = 0 \tag{5.3}$$

for all α and $n \geq 2$. Since J is spherical, it follow from (1.5) and (5.2) that

(5.4)

$$\begin{aligned}
R_{\beta\overline{\alpha}\lambda\overline{\sigma}} &= \tfrac{k}{n+2}[h_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + h_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}} + \delta_\beta^\alpha h_{\lambda\overline{\sigma}} + \delta_\lambda^\alpha h_{\beta\overline{\sigma}}] \\
&\quad -\tfrac{nk}{(n+1)(n+2)}[\delta_\beta^\alpha h_{\lambda\overline{\sigma}} + \delta_\lambda^\alpha h_{\beta\overline{\sigma}}] \\
&= \tfrac{k}{n+2}[h_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + h_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}}] + \tfrac{k}{(n+1)(n+2)}[\delta_\beta^\alpha h_{\lambda\overline{\sigma}} + \delta_\lambda^\alpha h_{\beta\overline{\sigma}}].
\end{aligned}$$

Now by [**16**, (2.7)] and (5.4),

(5.5)
$$iA_{\alpha\gamma,\overline{\beta}}h_{\rho\overline{\sigma}}+iA_{\alpha\gamma,\overline{\sigma}}h_{\rho\overline{\beta}}-iA_{\alpha\rho,\overline{\beta}}h_{\gamma\overline{\sigma}}-iA_{\alpha\rho,\overline{\sigma}}h_{\gamma\overline{\beta}}=R_{\alpha\overline{\beta}\rho\overline{\sigma},\gamma}-R_{\alpha\overline{\beta}\gamma\overline{\sigma},\rho}=0.$$

Contracting both sides by $h^{\rho\overline{\sigma}}$,

$$inA_{\alpha\gamma,\overline{\beta}}+iA_{\alpha\gamma,\overline{\sigma}}\delta^{\sigma}_{\beta}-iA_{\alpha\rho,\overline{\beta}}\delta^{\rho}_{\gamma}-iA_{\alpha\rho}{}^{,\rho}h_{\gamma\overline{\beta}}=0. \tag{5.6}$$

But from (5.3),

$$A_{\alpha\gamma,\overline{\sigma}}\delta^{\sigma}_{\beta}-A_{\alpha\rho,\overline{\beta}}\delta^{\rho}_{\gamma}-A_{\alpha\rho}{}^{,\rho}h_{\gamma\overline{\beta}}=A_{\alpha\gamma,\overline{\beta}}-A_{\alpha\gamma,\overline{\beta}}=0. \tag{5.7}$$

Hence

$$A_{\alpha\gamma,\overline{\beta}}=0 \tag{5.8}$$

for all α,γ,β. Again by [**16**, (2.15)],

$$A_{\alpha\rho,\beta\overline{\gamma}}-A_{\alpha\rho,\overline{\gamma}\beta}=ih_{\beta\overline{\gamma}}A_{\alpha\rho,0}+R_{\alpha}{}^{\kappa}{}_{\beta\overline{\gamma}}A_{\kappa\rho}+R_{\rho}{}^{\kappa}{}_{\beta\overline{\gamma}}A_{\alpha\kappa}$$

and from (5.8),

$$A_{\alpha\rho,\beta\overline{\gamma}}=ih_{\beta\overline{\gamma}}A_{\alpha\rho,0}+R_{\alpha}{}^{\kappa}{}_{\beta\overline{\gamma}}A_{\kappa\rho}+R_{\rho}{}^{\kappa}{}_{\beta\overline{\gamma}}A_{\alpha\kappa}.$$

Contracting both sides by $h^{\beta\overline{\gamma}}$,

$$\begin{aligned}A_{\alpha\rho,\gamma}{}^{\gamma} &= inA_{\alpha\rho,0}+R_{\alpha}{}^{\kappa}{}_{\gamma}{}^{\gamma}A_{\kappa\rho}+R_{\rho}{}^{\kappa}{}_{\gamma}{}^{\gamma}A_{\alpha\kappa}\\ &= inA_{\alpha\rho,0}+R_{\alpha\overline{\kappa}}A^{\overline{\kappa}}{}_{\rho}+R_{\rho\overline{\kappa}}A^{\overline{\kappa}}{}_{\alpha}\\ &= inA_{\alpha\rho,0}+kh_{\alpha\overline{\kappa}}A^{\overline{\kappa}}{}_{\rho}+kh_{\rho\overline{\kappa}}A^{\overline{\kappa}}{}_{\alpha}\\ &= inA_{\alpha\rho,0}+2kA_{\alpha\rho}.\end{aligned}$$

Here $k:=\frac{R}{n}$. That is,

$$A_{\alpha\gamma,\sigma}{}^{\sigma}=inA_{\alpha\gamma,0}+2kA_{\alpha\gamma} \tag{5.9}$$

for all α,γ. Next we claim that

$$inA_{\alpha\gamma,0}=-\tfrac{nk}{n+1}A_{\alpha\gamma}. \tag{5.10}$$

Again from [**16**, (2.9)],

$$A_{\alpha\rho,\overline{\beta}\gamma}-A_{\alpha\gamma,\overline{\beta}\rho}=ih_{\rho\overline{\beta}}A_{\alpha\gamma,0}-ih_{\gamma\overline{\beta}}A_{\alpha\rho,0}+R_{\alpha\overline{\beta}\rho\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma}-R_{\alpha\overline{\beta}\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho}$$

and from (5.8),

$$ih_{\rho\overline{\beta}}A_{\alpha\gamma,0}-ih_{\gamma\overline{\beta}}A_{\alpha\rho,0}+R_{\alpha\overline{\beta}\rho\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma}-R_{\alpha\overline{\beta}\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho}=0.$$

Contracting both sides by $h^{\rho\overline{\beta}}$,

$$inA_{\alpha\gamma,0}-i\delta^{\rho}_{\gamma}A_{\alpha\rho,0}+R_{\alpha}{}^{\rho}{}_{\rho\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma}-R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho}=0.$$

Hence

$$i(n-1)A_{\alpha\gamma,0}+R_{\alpha\overline{\sigma}}A^{\overline{\sigma}}{}_{\gamma}-R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho}=0$$

and thus

$$i(n-1)A_{\alpha\gamma,0}+kA_{\alpha\gamma}-R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho}=0.$$

On the other hand,

$$\begin{aligned}R_{\alpha}{}^{\rho}{}_{\gamma\overline{\sigma}}A^{\overline{\sigma}}{}_{\rho} &= \frac{k}{n+2}[h_{\alpha\overline{\rho}}h_{\gamma\overline{\sigma}} + h_{\gamma\overline{\rho}}h_{\alpha\overline{\sigma}}]A^{\overline{\sigma}}{}_{\rho} \\ &\quad + \frac{k}{(n+1)(n+2)}[\delta^{\rho}_{\alpha}h_{\gamma\overline{\sigma}} + \delta^{\rho}_{\gamma}h_{\alpha\overline{\sigma}}]A^{\overline{\sigma}}{}_{\rho} \\ &= \frac{2k}{n+1}A_{\alpha\gamma}.\end{aligned}$$

All these imply

$$i(n-1)A_{\alpha\gamma,0} + \tfrac{n-1}{n+1}kA_{\alpha\gamma} = 0$$

for $n \geq 2$. Thus (5.10) follows. Next, from (5.9) and (5.10), we obtain

$$A_{\alpha\gamma,\sigma}{}^{\sigma} = inA_{\alpha\gamma,0} + 2kA_{\alpha\gamma} = \tfrac{n+2}{n+1}kA_{\alpha\gamma}.$$

We integrate both sides with $A^{\alpha\gamma}$ to get

$$\tfrac{n+2}{n+1}k\int_M \sum_{\alpha,\gamma}|A_{\alpha\gamma}|^2 d\mu + \int_M \sum_{\alpha,\gamma,\sigma}|A_{\alpha\gamma,\sigma}|^2 d\mu = 0.$$

Thus

$$A_{\alpha\gamma} = 0.$$

Moreover, it follows from (5.4) that

$$R_{\beta\overline{\alpha}\lambda\overline{\sigma}} = \frac{R}{n(n+1)}[h_{\beta\overline{\alpha}}h_{\lambda\overline{\sigma}} + h_{\lambda\overline{\alpha}}h_{\beta\overline{\sigma}}].$$

Hence (M,θ) is a simply connected, closed spherical CR $(2n+1)$-manifold of positive constant pseudohermitian bisectional curvature with vanishing torsion. It follows from ([**14**]) that M is CR equivalent to the standard CR sphere. q.e.d.

In general, it is difficult to determine if a manifold has the vanishing first Chern class $c_1(T^{1,0}M)$. By applying the CR version of Bochner-type identity due to Mok-Siu-Yau ([**17**]) in case of Kähler manifolds, we are able to characterize it for a closed spherical CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature.

Proposition 5.1. *Let (M,J,θ) be a closed spherical CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature for $n \geq 2$. There is a smooth real-valued function u solving*

$$\Delta_b u = \frac{1}{n+2}(r-R) \quad \textit{with} \quad r = \frac{\int_M R d\mu}{\int_M d\mu}.$$

u also satisfies the following identities:

$$\begin{aligned}(n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) &= R_{\alpha\overline{\beta}} - \frac{r}{n}h_{\alpha\overline{\beta}} \\ (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) &= R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}.\end{aligned}$$

Hence $\widetilde{\theta} = e^{2u}\theta$ *is a pseudo-Einstein contact structure. In addition, if* R *is constant, then* u *is constant and*

$$R_{\alpha\overline{\beta}} = \frac{R}{n} h_{\alpha\overline{\beta}}.$$

Proof. Define

$$v_{\alpha\overline{\beta}} := R_{\alpha\overline{\beta}} - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) - \frac{r}{n} h_{\alpha\overline{\beta}} \tag{5.11}$$

for r with $\Delta_b u = \frac{1}{n+2}(r - R)$. Note that $tr(v_{\alpha\overline{\beta}}) = v_{\alpha\overline{\alpha}} = 0 = \overline{v}_{\overline{\alpha}\alpha}$.

Next we apply the CR version of Bochner-type identity to estimate the following:

(5.12)

$$\begin{aligned}
\Delta_b ||v_{\alpha\overline{\beta}}||^2 &= \Delta_b(v_{\alpha\overline{\beta}}\overline{v}_{\overline{\alpha}\beta}) \\
&= (v_{\alpha\overline{\beta}}\overline{v}_{\overline{\alpha}\beta})_{\lambda\overline{\lambda}} + (v_{\alpha\overline{\beta}}\overline{v}_{\overline{\alpha}\beta})_{\overline{\lambda}\lambda} \\
&= v_{\alpha\overline{\beta}\lambda\overline{\lambda}}\overline{v}_{\overline{\alpha}\beta} + v_{\alpha\overline{\beta}\overline{\lambda}\lambda}\overline{v}_{\overline{\alpha}\beta} + \overline{v}_{\overline{\alpha}\beta\lambda\overline{\lambda}}v_{\alpha\overline{\beta}} + \overline{v}_{\overline{\alpha}\beta\overline{\lambda}\lambda}v_{\alpha\overline{\beta}} + 2v_{\alpha\overline{\beta}\lambda}\overline{v}_{\overline{\alpha}\beta\overline{\lambda}} \\
&\quad + 2v_{\alpha\overline{\beta}\overline{\lambda}}\overline{v}_{\overline{\alpha}\beta\lambda} \\
&\geq v_{\alpha\overline{\beta}\lambda\overline{\lambda}}\overline{v}_{\overline{\alpha}\beta} + v_{\alpha\overline{\beta}\overline{\lambda}\lambda}\overline{v}_{\overline{\alpha}\beta} + \text{Conj}.
\end{aligned}$$

Note that

$$\rho(x) = \frac{i}{(n+2)}\{R_{\alpha\overline{\beta}} - \frac{r}{n}h_{\alpha\overline{\beta}}\}\theta^{\alpha} \wedge \theta^{\overline{\beta}}$$

is a real-valued d_H-closed $(1,1)$-form. We may rewrite $v = iv_{\alpha\overline{\beta}}\theta^{\alpha} \wedge \theta^{\overline{\beta}}$ as a real-valued $(1,1)$-form. Locally $v_{\alpha\overline{\beta}} = w_{,\alpha\overline{\beta}} + w_{,\overline{\beta}\alpha} - \frac{r}{n}h_{\alpha\overline{\beta}}$ for some smooth function w. But $w_{,\alpha\overline{\beta}} + w_{,\overline{\beta}\alpha} = 2w_{,\alpha\overline{\beta}} - iw_0 h_{\alpha\overline{\beta}}$, which implies

$$\begin{aligned}
v_{\alpha\overline{\beta}}\overline{v}_{\overline{\alpha}\beta} &= (2w_{,\alpha\overline{\beta}} - iw_0 h_{\alpha\overline{\beta}} - \frac{r}{n}h_{\alpha\overline{\beta}})\overline{v}_{\overline{\alpha}\beta} \\
&= 2w_{,\alpha\overline{\beta}}\overline{v}_{\overline{\alpha}\beta} - (iw_0 + \frac{r}{n})\overline{v}_{\overline{\alpha}\alpha} = 2w_{,\alpha\overline{\beta}}\overline{v}_{\overline{\alpha}\beta}.
\end{aligned}$$

To apply CR Bochner-type identity to estimate

$$v_{\alpha\overline{\beta}\lambda\overline{\lambda}}\overline{v}_{\overline{\alpha}\beta} + v_{\alpha\overline{\beta}\overline{\lambda}\lambda}\overline{v}_{\overline{\alpha}\beta} + \text{Conj}$$

in the last term of the right hand side of (5.12), we may assume that $v_{\alpha\overline{\beta}} = v_{,\alpha\overline{\beta}}$ for some smooth function v (say the same notation).

More precisely, we will derive the following pointwise estimates: (5.13), (5.14), (5.15), and (5.16).

(i) First from ([**16**, Lemma 2.3.]),

$$\begin{aligned}
v_{\alpha\overline{\beta}\lambda\overline{\lambda}} &= (v_{\alpha\lambda\overline{\beta}} - ih_{\lambda\overline{\beta}}v_{\alpha 0} - R_{\alpha\overline{\rho}\lambda\overline{\beta}}v_{\rho})_{\overline{\lambda}} \\
&= v_{\lambda\alpha\overline{\beta}\overline{\lambda}} - ih_{\lambda\overline{\beta}}v_{\alpha 0\overline{\lambda}} - R_{\alpha\overline{\rho}\lambda\overline{\beta},\overline{\lambda}}v_{\rho} - R_{\alpha\overline{\rho}\lambda\overline{\beta}}v_{\rho\overline{\lambda}}
\end{aligned}$$

and

$$\begin{aligned}
v_{\lambda\alpha\overline{\beta}\overline{\lambda}} &= v_{\lambda\alpha\overline{\lambda}\overline{\beta}} + ih_{\lambda\overline{\beta}}A_{\overline{\lambda}\overline{\rho}}v_{\rho\alpha} - ih_{\lambda\overline{\lambda}}A_{\overline{\beta}\overline{\rho}}v_{\rho\alpha} \\
&\quad + ih_{\alpha\overline{\beta}}A_{\overline{\lambda}\overline{\rho}}v_{\rho\lambda} - ih_{\alpha\overline{\lambda}}A_{\overline{\beta}\overline{\rho}}v_{\rho\lambda}.
\end{aligned}$$

Hence

$$\begin{aligned}v_{\alpha\overline{\beta}\lambda\overline{\lambda}} &= v_{\lambda\alpha\overline{\lambda\beta}} - ih_{\lambda\overline{\beta}}v_{\alpha 0\overline{\lambda}} - R_{\alpha\overline{\rho}\lambda\overline{\beta},\overline{\lambda}}v_\rho - R_{\alpha\overline{\rho}\lambda\overline{\beta}}v_{\rho\overline{\lambda}}\\ &\quad + ih_{\lambda\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\alpha} - ih_{\lambda\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\alpha} + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\lambda} - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\lambda}.\end{aligned}$$

Since $Tr(v_{\alpha\overline{\beta}}) = 0$

$$\begin{aligned}v_{\lambda\alpha\overline{\lambda\beta}} &= v_{\lambda\overline{\lambda}\alpha\overline{\beta}} + (ih_{\alpha\overline{\lambda}}v_{\lambda 0} + R_{\lambda\overline{\rho}\alpha\overline{\lambda}}v_\rho)_{\overline{\beta}}\\ &= ih_{\alpha\overline{\lambda}}v_{\lambda 0\overline{\beta}} + R_{\lambda\overline{\rho}\alpha\overline{\lambda},\overline{\beta}}v_\rho + R_{\lambda\overline{\rho}\alpha\overline{\lambda},\overline{\beta}}v_{\rho\overline{\beta}},\end{aligned}$$

and we obtain

(5.13)

$$\begin{aligned}v_{\alpha\overline{\beta}\lambda\overline{\lambda}} &= ih_{\alpha\overline{\lambda}}v_{\lambda 0\overline{\beta}} - ih_{\lambda\overline{\beta}}v_{\alpha 0\overline{\lambda}} + R_{\lambda\overline{\rho}\alpha\overline{\lambda},\overline{\beta}}v_\rho + R_{\lambda\overline{\rho}\alpha\overline{\lambda}}v_{\rho\overline{\beta}} - R_{\alpha\overline{\rho}\lambda\overline{\beta},\overline{\lambda}}v_\rho\\ &\quad - R_{\alpha\overline{\rho}\lambda\overline{\beta}}v_{\rho\overline{\lambda}} + ih_{\lambda\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\alpha} - ih_{\lambda\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\alpha} + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\lambda}\\ &\quad - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\lambda}.\end{aligned}$$

(ii) Again from ([**16**, Lemma 2.3.]),

$$\begin{aligned}&v_{\alpha\overline{\beta\lambda}\lambda}\\ &= (v_{\alpha\overline{\lambda\beta}} + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_\rho - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_\rho)_\lambda\\ &= (v_{\overline{\lambda}\alpha\overline{\beta}} + ih_{\alpha\overline{\lambda}}v_{0\overline{\beta}} + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_\rho - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_\rho)_\lambda\\ &= v_{\overline{\lambda}\alpha\overline{\beta}\lambda} + ih_{\alpha\overline{\lambda}}v_{0\overline{\beta}\lambda} + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho},\lambda}v_\rho + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\lambda} - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho},\lambda}v_\rho\\ &\qquad - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\lambda}\end{aligned}$$

and

$$\begin{aligned}v_{\overline{\lambda}\alpha\overline{\beta}\lambda} &= v_{\overline{\lambda}\alpha\lambda\overline{\beta}} - ih_{\lambda\overline{\beta}}v_{\overline{\lambda}\alpha 0} - R_{\overline{\lambda}\rho\overline{\beta}\lambda}v_{\overline{\rho}\alpha} - R_{\alpha\overline{\rho}\overline{\beta}\lambda}v_{\rho\overline{\lambda}}\\ &= v_{\overline{\lambda}\lambda\alpha\overline{\beta}} + (-ih_{\alpha\overline{\lambda}}A_{\lambda\rho}v_{\overline{\rho}} + ih_{\lambda\overline{\lambda}}A_{\alpha\rho}v_{\overline{\rho}})_{\overline{\beta}}\\ &\qquad - ih_{\lambda\overline{\beta}}v_{\overline{\lambda}\alpha 0} - R_{\overline{\lambda}\rho\overline{\beta}\lambda}v_{\overline{\rho}\alpha} - R_{\alpha\overline{\rho}\overline{\beta}\lambda}v_{\rho\overline{\lambda}}.\end{aligned}$$

We have

(5.14)

$$\begin{aligned}v_{\alpha\overline{\beta\lambda}\lambda} &= ih_{\alpha\overline{\lambda}}v_{0\overline{\beta}\lambda} - ih_{\lambda\overline{\beta}}v_{\overline{\lambda}\alpha 0} - R_{\overline{\lambda}\rho\overline{\beta}\lambda}v_{\overline{\rho}\alpha} - R_{\alpha\overline{\rho}\overline{\beta}\lambda}v_{\rho\overline{\lambda}}\\ &\quad - ih_{\alpha\overline{\lambda}}A_{\lambda\rho,\overline{\beta}}v_{\overline{\rho}} - ih_{\alpha\overline{\lambda}}A_{\lambda\rho}v_{\overline{\rho\beta}} + ih_{\lambda\overline{\lambda}}A_{\alpha\rho,\overline{\beta}}v_{\overline{\rho}} + ih_{\lambda\overline{\lambda}}A_{\alpha\rho}v_{\overline{\rho\beta}}\\ &\quad + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho},\lambda}v_\rho + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\lambda} - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho},\lambda}v_\rho - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\lambda}\\ &= ih_{\alpha\overline{\lambda}}v_{0\overline{\beta}\lambda} - ih_{\lambda\overline{\beta}}v_{0\overline{\lambda}\alpha} - R_{\overline{\lambda}\rho\overline{\beta}\lambda}v_{\overline{\rho}\alpha} - R_{\alpha\overline{\rho}\overline{\beta}\lambda}v_{\rho\overline{\lambda}}\\ &\quad - ih_{\alpha\overline{\lambda}}A_{\lambda\rho,\overline{\beta}}v_{\overline{\rho}} - ih_{\alpha\overline{\lambda}}A_{\lambda\rho}v_{\overline{\rho\beta}} + ih_{\lambda\overline{\lambda}}A_{\alpha\rho,\overline{\beta}}v_{\overline{\rho}} + ih_{\lambda\overline{\lambda}}A_{\alpha\rho}v_{\overline{\rho\beta}}\\ &\quad + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho},\lambda}v_\rho + ih_{\alpha\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\lambda} - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho},\lambda}v_\rho - ih_{\alpha\overline{\lambda}}A_{\overline{\beta\rho}}v_{\rho\lambda}\\ &\quad + ih_{\lambda\overline{\beta}}A_{\overline{\lambda\rho},\alpha}v_\rho + ih_{\lambda\overline{\beta}}A_{\overline{\lambda\rho}}v_{\rho\alpha} + ih_{\lambda\overline{\beta}}A_{\alpha\rho}v_{\overline{\lambda\rho}} + ih_{\lambda\overline{\beta}}A_{\alpha\rho,\overline{\lambda}}v_{\overline{\rho}}.\end{aligned}$$

Here we have used the following commutation relation:

$$\begin{aligned}v_{\overline{\lambda}\alpha 0} &= v_{\overline{\lambda}0\alpha} - A_{\alpha\rho}v_{\overline{\lambda\rho}} - A_{\alpha\rho,\overline{\lambda}}v_{\overline{\rho}}\\ &= (v_{0\overline{\lambda}} - A_{\overline{\lambda\rho}}v_\rho)_\alpha - A_{\alpha\rho}v_{\overline{\lambda\rho}} - A_{\alpha\rho,\overline{\lambda}}v_{\overline{\rho}}\\ &= v_{0\overline{\lambda}\alpha} - A_{\overline{\lambda\rho},\alpha}v_\rho - A_{\overline{\lambda\rho}}v_{\rho\alpha} - A_{\alpha\rho}v_{\overline{\lambda\rho}} - A_{\alpha\rho,\overline{\lambda}}v_{\overline{\rho}}.\end{aligned}$$

(iii) By the same method as in (i) and (ii), we have

$$\begin{aligned}\overline{v}_{\overline{\alpha}\beta\lambda\overline{\lambda}} &= (\overline{v}_{\overline{\alpha}\lambda\beta} - ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}})_{\overline{\lambda}}\\ &= \overline{v}_{\lambda\overline{\alpha}\beta\overline{\lambda}} - ih_{\lambda\overline{\alpha}}\overline{v}_{0\beta\overline{\lambda}} + (-ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}})_{\overline{\lambda}}\\ &= \overline{v}_{\lambda\overline{\alpha}\overline{\lambda}\beta} + ih_{\beta\overline{\lambda}}\overline{v}_{\lambda\overline{\alpha}0} + R_{\lambda\overline{\rho}\beta\overline{\lambda}}\overline{v}_{\rho\overline{\alpha}} + R_{\overline{\alpha}\rho\beta\overline{\lambda}}\overline{v}_{\lambda\overline{\rho}}\\ &\qquad - ih_{\lambda\overline{\alpha}}\overline{v}_{0\beta\overline{\lambda}} + (-ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}})_{\overline{\lambda}}\end{aligned}$$

and

$$\begin{aligned}\overline{v}_{\lambda\overline{\alpha}\overline{\lambda}\beta} &= (\overline{v}_{\lambda\overline{\lambda}\alpha} + (ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho}}\overline{v}_{\rho} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho}}\overline{v}_{\rho})_{\beta}\\ &= ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho},\beta}\overline{v}_{\rho} + ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho}}\overline{v}_{\rho\beta} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho},\beta}\overline{v}_{\rho} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho}}\overline{v}_{\rho\beta}.\end{aligned}$$

Hence

(5.15)
$$\begin{aligned}\overline{v}_{\overline{\alpha}\beta\lambda\overline{\lambda}} &= ih_{\beta\overline{\lambda}}\overline{v}_{\lambda\overline{\alpha}0} - ih_{\lambda\overline{\alpha}}\overline{v}_{0\beta\overline{\lambda}} + R_{\lambda\overline{\rho}\beta\overline{\lambda}}\overline{v}_{\rho\overline{\alpha}} + R_{\overline{\alpha}\rho\beta\overline{\lambda}}\overline{v}_{\lambda\overline{\rho}}\\ &\quad + ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho},\beta}\overline{v}_{\rho} + ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho}}\overline{v}_{\rho\beta} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho},\beta}\overline{v}_{\rho} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho}}\overline{v}_{\rho\beta}\\ &\quad - ih_{\beta\overline{\alpha}}A_{\lambda\rho,\overline{\lambda}}\overline{v}_{\overline{\rho}} - ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\lambda}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho,\overline{\lambda}}\overline{v}_{\overline{\rho}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}\overline{\lambda}}\\ &= ih_{\beta\overline{\lambda}}\overline{v}_{0\lambda\overline{\alpha}} - ih_{\lambda\overline{\alpha}}\overline{v}_{0\beta\overline{\lambda}} + R_{\lambda\overline{\rho}\beta\overline{\lambda}}\overline{v}_{\rho\overline{\alpha}} + R_{\overline{\alpha}\rho\beta\overline{\lambda}}\overline{v}_{\lambda\overline{\rho}}\\ &\quad + ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho},\beta}\overline{v}_{\rho} + ih_{\lambda\overline{\alpha}}A_{\overline{\lambda}\overline{\rho}}\overline{v}_{\rho\beta} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho},\beta}\overline{v}_{\rho} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho}}\overline{v}_{\rho\beta}\\ &\quad - ih_{\beta\overline{\alpha}}A_{\lambda\rho,\overline{\lambda}}\overline{v}_{\overline{\rho}} - ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\lambda}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho,\overline{\lambda}}\overline{v}_{\overline{\rho}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}\overline{\lambda}}\\ &\quad - ih_{\beta\overline{\lambda}}A_{\lambda\rho,\overline{\alpha}}\overline{v}_{\overline{\rho}} - ih_{\beta\overline{\lambda}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\alpha}} - ih_{\beta\overline{\lambda}}A_{\overline{\alpha}\overline{\rho}}\overline{v}_{\lambda\rho} - ih_{\beta\overline{\lambda}}A_{\overline{\alpha}\overline{\rho},\lambda}\overline{v}_{\rho}.\end{aligned}$$

Here we have used the following commutation relation:

$$\overline{v}_{\lambda\overline{\alpha}0} = \overline{v}_{0\lambda\overline{\alpha}} - A_{\lambda\rho,\overline{\alpha}}\overline{v}_{\overline{\rho}} - A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\alpha}} - A_{\overline{\alpha}\overline{\rho}}\overline{v}_{\lambda\rho} - A_{\overline{\alpha}\overline{\rho},\lambda}\overline{v}_{\rho}.$$

(iv) Finally,

$$\begin{aligned}\overline{v}_{\overline{\alpha}\beta\overline{\lambda}\lambda} &= (\overline{v}_{\overline{\alpha}\overline{\lambda}\beta} + ih_{\beta\overline{\lambda}}\overline{v}_{\overline{\alpha}0} - R_{\overline{\alpha}\rho\overline{\lambda}\beta}\overline{v}_{\overline{\rho}})_{\lambda}\\ &= \overline{v}_{\overline{\lambda}\overline{\alpha}\beta\lambda} + ih_{\beta\overline{\lambda}}\overline{v}_{\overline{\alpha}0\lambda} - R_{\overline{\alpha}\rho\overline{\lambda}\beta,\lambda}\overline{v}_{\overline{\rho}} - R_{\overline{\alpha}\rho\overline{\lambda}\beta}\overline{v}_{\overline{\rho}\lambda}\end{aligned}$$

and

$$\overline{v}_{\overline{\lambda}\overline{\alpha}\beta\lambda} = \overline{v}_{\overline{\lambda}\overline{\alpha}\lambda\beta} - ih_{\beta\overline{\lambda}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\alpha}} + ih_{\lambda\overline{\lambda}}A_{\beta\rho}\overline{v}_{\overline{\rho}\overline{\alpha}} - ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\lambda}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}\overline{\lambda}}.$$

But

$$\overline{v}_{\overline{\lambda}\overline{\alpha}\lambda\beta} = \overline{v}_{\overline{\lambda}\lambda\overline{\alpha}\beta} - ih_{\lambda\overline{\alpha}}\overline{v}_{\overline{\lambda}0\beta} + (R_{\overline{\lambda}\rho\overline{\alpha}\lambda}\overline{v}_{\overline{\rho}})_{\beta}.$$

Hence

(5.16)
$$\begin{aligned}\overline{v}_{\overline{\alpha}\beta\overline{\lambda}\lambda} &= ih_{\beta\overline{\lambda}}\overline{v}_{\overline{\alpha}0\lambda} - ih_{\lambda\overline{\alpha}}\overline{v}_{\overline{\lambda}0\beta} - R_{\overline{\alpha}\rho\overline{\lambda}\beta,\lambda}\overline{v}_{\overline{\rho}} - R_{\overline{\alpha}\rho\overline{\lambda}\beta}\overline{v}_{\overline{\rho}\lambda}\\ &\quad + R_{\overline{\lambda}\rho\overline{\alpha}\lambda,\beta}\overline{v}_{\overline{\rho}} + R_{\overline{\lambda}\rho\overline{\alpha}\lambda}\overline{v}_{\overline{\rho}\beta}\\ &\quad - ih_{\beta\overline{\lambda}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\alpha}} + ih_{\lambda\overline{\lambda}}A_{\beta\rho}\overline{v}_{\overline{\rho}\overline{\alpha}} - ih_{\beta\overline{\alpha}}A_{\lambda\rho}\overline{v}_{\overline{\rho}\overline{\lambda}} + ih_{\lambda\overline{\alpha}}A_{\beta\rho}\overline{v}_{\overline{\rho}\overline{\lambda}}.\end{aligned}$$

Since $v_{\alpha\overline{\beta}}$ is hermitian symmetric, after a unitary change of the admissible coframe, $v_{i\overline{j}} = a_i\delta_{\alpha\beta}$, a_i is real and $v_{ij} = 0$ at a point. Now combing all computations as in (5.13), (5.14), (5.15), and (5.16) at that particular point, we have

(5.17)
$$
\begin{aligned}
& v_{\alpha\overline{\beta}\lambda\overline{\lambda}}\overline{v}_{\overline{\alpha}\beta} + v_{\alpha\overline{\beta}\overline{\lambda}\lambda}\overline{v}_{\overline{\alpha}\beta} + \overline{v}_{\overline{\alpha}\beta\lambda\overline{\lambda}}v_{\alpha\overline{\beta}} + \overline{v}_{\overline{\alpha}\beta\overline{\lambda}\lambda}v_{\alpha\overline{\beta}} \\
= & [R_{\overline{\lambda}\rho\overline{\alpha}\lambda,\beta}\overline{v}_{\overline{\rho}} - R_{\overline{\lambda}\rho\overline{\alpha}\beta,\lambda}\overline{v}_{\overline{\rho}}]v_{\alpha\overline{\beta}} + [R_{\lambda\overline{\rho}\alpha\overline{\lambda},\overline{\beta}}v_{\rho} - R_{\lambda\overline{\rho}\alpha\overline{\beta},\overline{\lambda}}v_{\rho}]\overline{v}_{\overline{\alpha}\beta} \\
& + [R_{\lambda\overline{\rho}\alpha\overline{\lambda}}v_{\rho\overline{\beta}}\overline{v}_{\overline{\alpha}\beta} + R_{\lambda\overline{\rho}\beta\overline{\lambda}}\overline{v}_{\rho\overline{\alpha}}v_{\alpha\overline{\beta}}] - [R_{\rho\overline{\alpha}\beta\overline{\lambda}}\overline{v}_{\overline{\rho}\lambda}v_{\alpha\overline{\beta}} + R_{\rho\overline{\alpha}\beta\overline{\lambda}}\overline{v}_{\overline{\rho}\lambda}v_{\alpha\overline{\beta}}] \\
& + [ih_{\lambda\overline{\lambda}}A_{\alpha\rho,\overline{\beta}}v_{\overline{\rho}}\overline{v}_{\overline{\alpha}\beta} - ih_{\lambda\overline{\lambda}}A_{\overline{\alpha}\overline{\rho},\beta}\overline{v}_{\rho}v_{\alpha\overline{\beta}}] \\
= & [inA_{\beta\rho,\overline{\alpha}} - ih_{\beta\overline{\alpha}}A_{\lambda\rho,\overline{\lambda}}]\overline{v}_{\overline{\rho}}v_{\alpha\overline{\beta}} + \mathrm{Conj} \\
& + [inA_{\alpha\rho,\overline{\beta}}v_{\overline{\rho}}\overline{v}_{\overline{\alpha}\beta}] + \mathrm{Conj} + \sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}}(a_{\alpha} - a_{\beta})^2 \\
= & \sum_{\alpha}[inA_{\alpha\rho,\overline{\alpha}} - iA_{\lambda\rho,\overline{\lambda}}]\overline{v}_{\overline{\rho}}a_{\alpha} + \mathrm{Conj} \\
& + \sum_{\alpha}[inA_{\alpha\rho,\overline{\alpha}}v_{\overline{\rho}}a_{\alpha}] + \mathrm{Conj} + \sum_{\alpha,\beta} R_{\alpha\overline{\alpha}\beta\overline{\beta}}(a_{\alpha} - a_{\beta})^2.
\end{aligned}
$$

Here we have used $R_{\alpha\overline{\rho}\lambda\overline{\lambda}} = R_{\lambda\overline{\rho}\alpha\overline{\lambda}}$, $h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}}$, and $tr(v_{\alpha\overline{\beta}}) = 0$.

At a point, it follows from ([**13**]) that there exists a contact form $\widetilde{\theta}$ which is conformal to θ with

$$A_{\alpha\beta} = \widetilde{A}_{\alpha\beta} \quad \text{and} \quad R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}$$

such that

$$\widetilde{A}_{\alpha\beta,\overline{\alpha}} = 0 \tag{5.18}$$

for each α (we will prove this claim later).

All together, we have at any fixed point

$$\Delta_b||v_{\alpha\overline{\beta}}||^2 \geq \sum R_{i\overline{i}j\overline{j}}(a_i - a_j)^2 \geq 0 \tag{5.19}$$

if (M, θ) is a closed strictly pseudoconvex CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature. Now applying the maximal principle to (5.19), we have

$$a_i = a_j$$

at a point for all i, j. However, since $Tr(v_{\alpha\overline{\beta}}) = 0$, this implies $v_{\alpha\overline{\beta}} = 0$ and then

$$R_{\alpha\overline{\beta}} - (n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) - \frac{r}{n}h_{\alpha\overline{\beta}} = 0$$

on M. Furthermore, we have

$$r = (n+2)\Delta_b u + R.$$

Hence

$$(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) = \frac{1}{(n+2)}\{R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}\}.$$

Finally, we give a proof of (5.18). First it follows from [**13**, Theorem 3.1 and Lemma 3.11] for $m = 3$ that there exists a contact form $\widetilde{\theta}$ which is conformal to θ with

$$R = \widetilde{R} \quad \text{and} \quad A_{\alpha\beta} = \widetilde{A}_{\alpha\beta} \quad \text{and} \quad R_{\alpha\overline{\beta}} = \widetilde{R}_{\alpha\overline{\beta}} \tag{5.20}$$

such that

$$\widetilde{R}_{,\alpha} = 0, \quad \widetilde{R}_{\alpha\overline{\beta},\beta} = 0 \text{ and } \widetilde{A}_{\alpha\rho,\overline{\rho}} = 0 \tag{5.21}$$

at a point. Since M is spherical, it follows from (5.20) that

$$R_{\alpha\overline{\beta}\gamma\overline{\delta}} = \widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}}$$

at a point as well. In the following, we drop $\sim$ without any confusion.

By [**16**, (2.7)] ,

$$iA_{\alpha\gamma,\overline{\beta}}h_{\rho\overline{\sigma}} + iA_{\alpha\gamma,\overline{\sigma}}h_{\rho\overline{\beta}} - iA_{\alpha\rho,\overline{\beta}}h_{\gamma\overline{\sigma}} - iA_{\alpha\rho,\overline{\sigma}}h_{\gamma\overline{\beta}} = R_{\alpha\overline{\beta}\rho\overline{\sigma},\gamma} - R_{\alpha\overline{\beta}\gamma\overline{\sigma},\rho}.$$

Contracting both sides by $h^{\rho\overline{\sigma}}$,

$$inA_{\alpha\gamma,\overline{\beta}} + iA_{\alpha\gamma,\overline{\sigma}}\delta^{\sigma}_{\beta} - iA_{\alpha\rho,\overline{\beta}}\delta^{\rho}_{\gamma} - iA_{\alpha\rho}{}^{,\rho}h_{\gamma\overline{\beta}} = h^{\rho\overline{\sigma}}(R_{\alpha\overline{\beta}\rho\overline{\sigma},\gamma} - R_{\alpha\overline{\beta}\gamma\overline{\sigma},\rho})$$

and from (5.21),

$$inA_{\alpha\gamma,\overline{\beta}} = h^{\rho\overline{\sigma}}(R_{\alpha\overline{\beta}\rho\overline{\sigma},\gamma} - R_{\alpha\overline{\beta}\gamma\overline{\sigma},\rho}) \tag{5.22}$$

at a point. Next for each $\alpha = \beta$, since M is spherical, it follows from (5.21) that

$$R_{\alpha\overline{\alpha}\rho\overline{\sigma},\gamma} = \tfrac{1}{n+2}[R_{\alpha\overline{\alpha},\gamma}h_{\rho\overline{\sigma}} + R_{\rho\overline{\alpha},\gamma}h_{\alpha\overline{\sigma}} + \delta^{\alpha}_{\alpha}R_{\rho\overline{\sigma},\gamma} + \delta^{\alpha}_{\rho}R_{\alpha\overline{\sigma},\gamma}]$$

and

$$h^{\rho\overline{\sigma}}R_{\alpha\overline{\alpha}\rho\overline{\sigma},\gamma} = \frac{1}{n+2}[nR_{\alpha\overline{\alpha},\gamma} + R_{\alpha\overline{\alpha},\gamma} + R_{,\gamma} + R_{\alpha\overline{\sigma},\gamma}] = R_{\alpha\overline{\alpha},\gamma}. \tag{5.23}$$

By similar computation we have

$$h^{\rho\overline{\sigma}}R_{\alpha\overline{\alpha}\gamma\overline{\sigma},\rho} = \frac{1}{n+2}[R_{\alpha\overline{\alpha},\gamma} + R_{\gamma\overline{\alpha},\alpha}]. \tag{5.24}$$

Hence from (5.22) and (5.24),

$$inA_{\alpha\gamma,\overline{\alpha}} = R_{\alpha\overline{\alpha},\gamma} - \frac{1}{n+2}[R_{\alpha\overline{\alpha},\gamma} + R_{\gamma\overline{\alpha},\alpha}].$$

But from the Bianchi identity ([**16**, (2.10)]) and (5.21),

$$R_{\alpha\overline{\alpha},\gamma} - R_{\gamma\overline{\alpha},\alpha} = iA_{\gamma\rho}{}^{,\rho}h_{\alpha\overline{\alpha}} - iA_{\alpha\rho}{}^{,\rho}h_{\gamma\overline{\alpha}} = 0$$

and then

$$inA_{\alpha\gamma,\overline{\alpha}} = \frac{n}{n+2}R_{\alpha\overline{\alpha},\gamma} \tag{5.25}$$

for each α. But again from [**13**, (3.12)], for each α,

$$0 = i(n+2)A_{\alpha\gamma,\overline{\alpha}} + R_{\alpha\overline{\alpha},\gamma} + R_{\gamma\overline{\alpha},\alpha}$$

and then

$$i(n+2)A_{\alpha\gamma,\overline{\alpha}} + 2R_{\alpha\overline{\alpha},\gamma} = 0. \tag{5.26}$$

Thus from (5.25) and (5.26),

$$nA_{\alpha\gamma,\overline{\alpha}} = -\frac{n}{2}A_{\alpha\gamma,\overline{\alpha}}$$

and

$$A_{\alpha\gamma,\overline{\alpha}} = 0$$

for each α. This is (5.18). q.e.d.

Then *Corollary* 1.2 *follows from Proposition* 5.1 *and Theorem* 1.3.

As a byproduct of Proposition 5.1, we have

Corollary 5.1. *Let (M, J, θ) be a closed strictly pseudoconvex CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature and*

$$A_{\alpha\gamma,\overline{\alpha}} = 0$$

for each α. There is a smooth real-valued function u solving $\Delta u = \frac{1}{n+2}(r - R)$ and $r = \int_M R d\mu \diagup \int_M d\mu$ such that

$$(n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) = R_{\alpha\overline{\beta}} - \frac{r}{n}h_{\alpha\overline{\beta}}$$

and then

$$(n+2)(u_{\alpha\overline{\beta}} + u_{\overline{\beta}\alpha}) = R_{\alpha\overline{\beta}} - \frac{1}{n}[(n+2)\Delta_b u + R]h_{\alpha\overline{\beta}}.$$

Hence $e^{2u}\theta$ is a pseudo-Einstein contact structure.

Proof. In fact, it follows from (5.17) that (5.19) holds if (M, θ) is a closed strictly pseudoconvex CR $(2n+1)$-manifold of positive pseudohermitian bisectional curvature and

$$A_{\alpha\gamma,\overline{\alpha}} = 0$$

for each α. Then we finish the proof of the lemma. q.e.d.

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