J. DIFFERENTIAL GEOMETRY
49 (1998) 547-576

HITCHIN'S AND WZW CONNECTIONS ARE THE SAME

YVES LASZLO

1. Introduction

Let X be an algebraic curve over the field $\mathbf{C}$ of complex numbers, which is assumed to be smooth, connected and projective. For simplicity, we assume that the genus of X is > 2. Let G be a simple simply connected group and $M_G(X)$ the coarse moduli scheme of semistable G-bundles on X. Any linear representation determines a line bundle Θ on M and some nonnegative integer l (the Dynkin index of the representation, cf [12], [13]). It is known that the choice of a (closed) point $x \in X(\mathbf{C})$ (and, a priori, of a formal coordinate near x) of X determines an isomorphism (see 5) between the projective space of conformal blocks $\mathbf{P}B_l(X)$ (for G) of level l and the space $\mathbf{P}H^0(M_G(X), \Theta)$ of generalized theta functions (see [3], [7],[12], [13]). In fact, it is observed in [20] that there is a coordinate free description of $B_l(X)$.

When the pointed curve (X, x) runs over the moduli stack $\mathcal{M}_{g,1}$ of genus g pointed curve, these 2 projective spaces organize in 2 projective bundles $\mathbf{P}\Theta$ and $\mathbf{PB}_l$. We first explain (see 5.7) how to identify these 2 projective bundles (this is a global version of the identification above). The projective bundle $\mathbf{P}\Theta$ has a canonical flat connection: the Hitchin connection [9] and $\mathbf{PB}_l$ has a flat connection, which we call the WZW connection coming from the conformal field theory (see [21] or [18]). In the rest of the paper, we prove that this canonical identification 5.7

$$\kappa : \ \mathbf{P}\Theta \overset{\sim}{\to} \mathbf{PB}_l$$

Received April 4, 1997, and, in revised form, March 25, 1998. Author was partially supported by the European HCM Project "Algebraic Geometry in Europe" (AGE).

is *flat* (Theorem 9).

1.1. Let me roughly explain how to prove the flatness. Let M be the smooth open subvariety of $M_G(X)$ parameterizing regularly stable bundles E (such that $\mathrm{Aut}_G(E) = Z(G)$, the center of G). The cup-product

$$H^1(X, T_X) \otimes H^0(X, \mathrm{ad}(E) \otimes \omega_X) \to H^1(X, \mathrm{ad}(E))$$

defines a morphism $T_{[X]}\mathcal{M}_g \to \mathbf{S}^2 T_{[E]}M$ which globalizes in

$$(*) \qquad T_{[X]}\mathcal{M}_g \to H^0(M, \mathbf{S}^2 TM).$$

Let s be a generalized theta function, and $d_i s$ the length 1 complex

$$d_i s : \ \mathcal{D}^i(\Theta) \stackrel{D \mapsto Ds}{\longrightarrow} \Theta,$$

which evaluates the differential operator D of order $\leq i$ on s. The symbol exact sequence

$$0 \to d_1 s \to d_2 s \to \mathbf{S}^2 TM \to 0$$

defines a Bockstein operator $\delta : H^0(\mathbf{S}^2 TM) \to \mathbf{H}^1(d_1 s)$. Let w_s be the composite morphism

$$w_s = H^1(X, T_X) \to H^0(\mathbf{S}^2 TM) \to \mathbf{H}^1(d_1 s).$$

Let $\bar{t}$ be the image of a tangent vector on $\mathcal{M}_G$ by w_s. The main ingredient in the computation of Hitchin's connection is the computation of $w_s(\bar{t})$. If (U_α) is an affine cover of M, the class $w_s(\bar{t})$ can be represented by a pair $(D_\alpha - D_\beta, -D_\alpha s)$ where s is some second order differential operator defined on U_α. It is well known that G-bundles trivialized on punctured curve $X^* = X \backslash x$ are parameterized by an infinite dimensional homogeneous ind-scheme $\mathcal{Q} = G(\mathrm{Frac}(\hat{\mathcal{O}}_x))/G(\hat{\mathcal{O}}_x)$ (see [13]). Let $\mathcal{Q}^0$ be the open sublocus of $\mathcal{Q}$ parameterizing regularly stable G-bundles. The crucial point (cf. [6]) is that $\mathcal{Q}^0 \to M$ is a locally trivial torsor (for the étale topology). The idea of the paper is to use the cover $\mathcal{Q}^0 \longrightarrow\!\!\!\!\rightarrow M$ to compute some representative of $w_s(\bar{t})$, even though the author does not control all second order differential operators on $\mathcal{Q}^0$. Let t be a meromorphic tangent vector on D^* projecting on $\bar{t}$, 7.3. To avoid too much abstract nonsense on differential operators on ind-schemes, we use an étale quasi-section (cf. 8)

$$\begin{array}{ccc} & & \mathcal{Q}^0 \\ & \overset{\sigma}{\nearrow} & \\ N & \overset{r}{\twoheadrightarrow} & M \end{array}$$

of $\mathcal{Q}^0 \to M$ to construct a second order differential operator $\theta(t) \in H^0(N, \mathcal{D}^2(r^*\Theta))$ computing $w_s(\bar{t})$. In a certain sense, $\theta(t)$ is the "pull-back" of the Sugawara tensor $T(t)$ (see definition 8.12). The theorem follows easily, because the only nontrivial term in the formula defining the WZW connection is the Sugawara tensor (9.1).

1.2. Under the hypothesis $\mathrm{codim}_{M_G}(M_G \backslash M_G^0) > 2$, Hitchin constructs the connection not only for the bundle $\mathbf{P}\Theta$ of theta functions coming from determinantal line bundles on M_G, but also for the bundle $\mathbf{P}p_*\mathcal{L}$ where $\mathcal{L}$ is any line bundle on $\underline{M}^0$, and $p : \underline{M}^0 \to \mathcal{M}_g$ is the universal family of coarse moduli spaces of regularly stable bundles. The codimension assumption is used to identify $H^i(M_G, F)$ with $H^i(M_G^0, F)$, $i = 0, 1$ for any vector bundle F on M_G. This identification shows that the formation of the direct image $p_*\mathcal{L}$ commutes with the base change. The flatness result is written in this context.

1.3. For completeness, we compute the Picard group of the universal moduli stack of G-bundles over $\mathcal{M}_{g,1}$. This allows us to compare a determinantal line bundle and the line bundle $\mathcal{L}$ (Section 5).

Notation. We work over the field $\mathbf{C}$ of complex numbers, and fix a simple Lie algebra $\mathfrak{g}$ with a Borel subalgebra $\mathfrak{b}$. Let θ be the longest root (relative to $\mathfrak{b}$), and $\mathfrak{sl}_2(\theta) = (X_\theta, X_{-\theta}, H_\theta)$ a corresponding $\mathfrak{sl}_2$-triple. Finally $(,)$ will be the Cartan-Killing form normalized such that $(\theta, \theta) = 2$. If ρ is half of the sum of the positive roots, the dual Coxeter number is $h^\vee = 1 + <\rho, \theta^\vee>$. Let G be the simply connected algebraic group of Lie algebra $\mathfrak{g}$. The symbol X (resp. x) will always define a smooth, connected and projective complex curve of genus $g > 2$ (resp. a point of $X(\mathbf{C})$). If $\mathcal{X} \to S$ is a family of genus g pointed curve, we'll denote by $\hat{\mathcal{X}}$ the formal neighborhood of the marked section $S \to \mathcal{X}$.

Conformal blocks and theta functions over $\mathcal{M}_{g,1}$

We want to identify over $\mathcal{M}_{g,1}$ the projective bundle of conformal blocks $\mathbf{P}\Theta$ and the projective bundle generalized theta function $\mathbf{PB}_l$ as done in [3] in the absolute case. The precise statement is in 5.7.

2. Residues

We denote by K the field of fractions of $\mathcal{O} = \mathcal{O}_{\hat{X},x}$. The dualizing sheaf ϖ of $\hat{X}$ is the biggest quotient of $\Omega_{\hat{X}/\mathbf{C}}$ which is separated for the x-adic topology. Let me denote by $\mathrm{d} : \mathcal{O} \longrightarrow\!\!\!\!\rightarrow \varpi$ the projection of the universal derivation $\mathcal{O} \to \Omega_{\hat{X}/\mathbf{C}}$ on ϖ. If z is a formal coordinate at x, the $\mathcal{O} = \mathbf{C}[[z]]$-module ϖ is the free module $\mathbf{C}[[z]].\mathrm{d}z$, and $\omega = K \otimes_{\mathcal{O}} \varpi$ is $\mathbf{C}((z)).\mathrm{d}z$. Recall that there exists a residue map $\mathrm{res} : \omega \to \mathbf{C}$ which is given in coordinates by $\mathrm{res}(\sum_{n \geq N} a_n z^n \mathrm{d}z) = a_{-1}$.

2.1. Let $\pi : (\mathcal{X}, x) \to S$ be a pointed curve over an affine $\mathbf{C}$-scheme $S = \mathrm{Spec}(R)$, and ϖ^π (resp. ω^π) be the relative dualizing sheaf of $\hat{\mathcal{X}} \to S$ (resp. $\hat{\mathcal{X}}^* \to S$). Because formal coordinates along x exists Zariski locally in S, the residue is defined as a (functorial) R-morphism $\mathrm{res} : \omega^\pi \to R$. Let $A_{\mathcal{X}}$ be the algebra $\Gamma(S, \pi_* \mathcal{O}_{\mathcal{X} \backslash x})$ which is embedded in $\mathcal{K} = \Gamma(S, \pi_* \mathcal{O}_{\hat{\mathcal{X}}^*})$ by the Taylor expansion.

Lemma 2.2. *Let $f \in A_{\mathcal{X}}$. Then,* $\mathrm{res}(f) = 0$.

Proof. Because $\mathcal{M}_{g,1}$ is a smooth $\mathbf{C}$-stack, one can assume that S is a least reduced of finite type over $\mathbf{C}$. The residue theorem says that $\mathrm{res}(f)(r) = 0$ for all $r \in S(\mathbf{C})$ which implies that $\mathrm{res}(f) = 0$. q.e.d.

3. Loop algebras

We start with our pointed curve (X, x) and the simple algebra $\mathfrak{g}$. Let l be a positive integer. We would like to give an explicit coordinate free description of the vector spaces $B_l(X)$ of conformal blocks of level l on (X, x), which coincide with the usual one, once a coordinate has been chosen and which globalizes when the pointed curve moves.

3.1. The loop algebra $\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{C}.c$ of $\mathfrak{g}$ is the universal central extension of $L\mathfrak{g} = \mathfrak{g} \otimes K$ by $\mathbf{C} = \mathbf{C}.c$ with bracket

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X \mid Y)\mathrm{res}(g\mathrm{d}f).$$

Let me denote by $\widehat{L^+\mathfrak{g}}$ the Lie subalgebra $L^+\mathfrak{g} \oplus \mathbf{C}.c$ of $L\mathfrak{g}$, where $L^+\mathfrak{g} = \mathfrak{g} \otimes \mathcal{O}$.

Let λ be a dominant weight of level l (ie $(\lambda, \theta) \leq l$), and M be the simple $\mathfrak{g}$-module with highest weight λ and highest weight vector v_λ.

Let M_l be the $\widehat{L^+\mathfrak{g}}$-module structure on M where the action of $L^+\mathfrak{g}$ (resp. c) is induced by $L^+\mathfrak{g} \to \mathfrak{g}$ (resp. is the multiplication by l). We denote by $V_{\lambda,l}$ the Verma module of weight (λ, l)

$$V_{\lambda,l} = U(\widehat{L\mathfrak{g}}) \otimes_{U(\widehat{L^+\mathfrak{g}})} M_l,$$

and by $v_{\lambda,l}$ the highest weight vector $1 \otimes v_\lambda$.

Lemma 3.2. *Let z be a formal coordinate of X at x. Then the line $\mathbf{C}.(X_\theta \otimes z^{-1})^{l+1-(\lambda,\theta)} v_l$ of $V_{\lambda,l}$ does not depend on the choice of z.*

Proof. Let $u(z)$ (with $u(0) = 0$ and $u'(0) \neq 0$) be another coordinate (set $a = \frac{1}{u'(0)}$). Then

$$X_\theta \otimes u(z)^{-1} = aX_\theta \otimes z^{-1} \bmod \mathbf{C}X_\theta \oplus L^{>0}\mathfrak{g},$$

where $L^{>0}\mathfrak{g}$ is the kernel of $\mathfrak{g} \otimes \mathcal{O} \to \mathfrak{g}$. Thus,

$$\begin{aligned}&(X_\theta \otimes u(z)^{-1})^{l+1-(\lambda,\theta)}\\ &\quad = a^{l+1-(\lambda,\theta)}(X_\theta \otimes z^{-1})^{l+1-(\lambda,\theta)} \bmod U(\widehat{L\mathfrak{g}})(\mathbf{C}X_\theta \oplus L^{>0}\mathfrak{g}),\end{aligned}$$

(because $l + 1 - (\lambda, \theta)$ is positive) and the lemma follows because X_θ kills v_λ and $L^{>0}\mathfrak{g}$ kills even the whole M. q.e.d.

In the most interesting case for us, namely when $\lambda = 0$ (i.e., $M = \mathbf{C}$), we denote $V_{(\lambda,l)}$ simply by V_l.

Definition 3.3. We denote by Z_l the $U(\widehat{L\mathfrak{g}})$-submodule generated by $\mathbf{C}.(X_\theta \otimes z^{-1})^{l+1}$ (z is any formal coordinate at x) and by H_l the quotient V_l/Z_l.

The usual theory of representation of affine algebras says that H_l is the fundamental representation of level l of $\widehat{L\mathfrak{g}}$ (see [1]). In particular, the canonical embedding of $\mathfrak{g}$-modules $\mathbf{C} \hookrightarrow H_l$ has image the annihilator of $L^+\mathfrak{g}$.

By the residue theorem, the embedding $L_X\mathfrak{g} = \mathfrak{g} \otimes A_X \hookrightarrow L\mathfrak{g}$ lifts canonically to an embedding $L_X\mathfrak{g} \hookrightarrow \widehat{L\mathfrak{g}}$.

Definition 3.4 ([21]). The (finite dimensional) vector space

$$B_l(X) = \mathrm{Hom}_{L_X\mathfrak{g}}(H_l, \mathbf{C}) = (H_l/L_X\mathfrak{g}H_l)^*$$

is the space of vacua (or conformal blocks) of level l.

3.5. Let $\pi : (\mathcal{X}, x) \to S = \mathrm{Spec}(R)$ be a family of genus g pointed curve. One has the relative version

$$(\widehat{L_\pi \mathfrak{g}}, \widehat{L_\pi^+ \mathfrak{g}}, L_{\mathcal{X}}\mathfrak{g}, \mathcal{V}_l(\pi))$$

of $(\widehat{L\mathfrak{g}}, \widehat{L^+\mathfrak{g}}, L_X\mathfrak{g}, V_l)$ is exactly the same as before. Now, because formal coordinates along x exists Zariski locally in S, one defines as in definition 3 the submodule $\mathcal{Z}_l(\pi)$ of $\mathcal{V}_l(\pi)$ and correspondingly the $\widehat{L_\pi\mathfrak{g}}$-modules

$$\mathcal{H}_l(\pi) = \mathcal{V}_l(\pi)/\mathcal{Z}_l(\pi).$$

The Lie algebra $L_{\mathcal{X}}\mathfrak{g}$ embeds canonically in $\widehat{L\mathfrak{g}}$, 2.2. One defines the module (which is in fact a projective R-modules by [21]) of covacua by the equality

$$\mathbf{B}_l{}^*(\pi) = \mathcal{H}_l(\pi)/L_{\mathcal{X}}\mathfrak{g}.\mathcal{H}_l(\pi),$$

and the module of vacua by

$$\mathbf{B}_l(\pi) = \mathrm{Hom}_R(\mathbf{B}_l(\pi), R).$$

The construction $\pi \longmapsto \mathbf{B}_l{}^*(\pi)$ (resp. $\pi \longmapsto \mathbf{B}_l(\pi)$) is functorial in π; this defines two vector bundles $\mathbf{B}_l{}^*$ and $\mathbf{B}_l$ on $\mathcal{M}_{g,1}$ which are dual to each other. If π is the fixed curve $(X, x) \to \mathrm{Spec}(\mathbf{C})$, the fiber $\mathbf{B}_l(\pi)$ is $B_l(X)$ [21].

4. Loop groups

Let us first recall the construction of the Kac-Moody group $\widehat{LG}$ (of Lie algebra $\widehat{L\mathfrak{g}}$) in the absolute case, and of the corresponding generator $\mathcal{L}$ of the Picard group of $\mathcal{Q} = \widehat{LG}/\widehat{L^+G}$ (see [13]).

4.1. The adjoint action of $L\mathfrak{g}$ on $\widehat{L\mathfrak{g}}$ can be integrated explicitly as follows. Let LG be the loop group of G (whose R-points are $G(\hat{X}_R^*)$ or simply $G(R((z)))$ once a formal coordinate z at x has been chosen). Let γ be a point of $LG(R)$; the cotangent morphism of the morphism

$$\gamma : \hat{X}_R^* \to G$$

defines a morphism

$$\mathfrak{g}^* \otimes H^0(\hat{X}_R^*, \mathcal{O}_{\hat{X}_R^*}) \to \Omega_{\hat{X}_R^*/R} \to \omega^\pi.$$

Let me denote by $\gamma^{-1}\mathrm{d}\gamma$ the corresponding element of $\mathfrak{g} \otimes \omega^\pi$.

Remark 4.2. Suppose that G is embedded in some $\mathbf{GL}_N$ and that a coordinate z has been chosen. Then, γ is some invertible matrix $\gamma(z)$ of rank N with coefficients in $R((z))$, and $\gamma^{-1}d\gamma$ is the matrix product $\gamma(z)^{-1}\gamma'(z)dz \in \omega^{\pi} = \mathfrak{g} \otimes_{\mathbf{C}} R((z))dz$.

Let $\alpha \in L\mathfrak{g}(R)$ and $r \in R$. Then, γ acts on $\alpha + r.c \in \widehat{L\mathfrak{g}}(R)$ by

$$\mathrm{Ad}(\gamma).(\alpha + r.c) = \mathrm{Ad}(\gamma).\alpha + (s + \mathrm{res}(\gamma^{-1}\mathrm{d}\gamma \mid \alpha)).c. \tag{4.1}$$

4.3. Let me recall the following integrability property (result which is due to Faltings, see [3, Lemma A.3]) of the basic integrable representation $\rho : \widehat{L\mathfrak{g}} \to \mathrm{End}(H_1)$:

Proposition 4.4 (Faltings). *Let R be a $\mathbf{C}$-algebra and $\gamma \in LG(R)$. Then locally over* $\mathrm{Spec}(R)$*, there exists an automorphism u of $H_1 \otimes R$, uniquely determined up to R^*, satisfying*

$$u\rho_R(\alpha)u^{-1} = \rho_R(\mathrm{Ad}(\gamma).\alpha)$$

for any $\alpha \in \widehat{L\mathfrak{g}(R)}$.

This proves that the representation $\widehat{L\mathfrak{g}} \to \mathrm{End}(H_1)/\mathbf{C}.\mathrm{Id}$ is the derivative of an algebraic (i.e., morphism of $\mathbf{C}$-groups) representation $\bar{\rho} : LG \to \mathbf{PGL}(H_1)$.

4.5. Let

$$1 \to G_m \to \widehat{LG} \to LG \to 1$$

be the pull back of the extension

$$1 \to G_m \to \mathbf{GL}(H_1) \to \mathbf{PGL}(H_1) \to 1.$$

The corresponding central extension of Lie algebras

$$0 \to \mathbf{C} \to \mathrm{Lie}(\widehat{LG}) \to L\mathfrak{g} \to 0 \tag{4.2}$$

is the pull-back pull-back of

$$0 \to \mathbf{C} \to \mathrm{End}(H_1) \to \mathrm{End}(H_1)/\mathbf{C}.\mathrm{Id} \to 0$$

by $\mathrm{d}\bar{\rho}$.

Lemma 4.6. *The central extension (4.2) is the universal central extension*

$$0 \to \mathbf{C} \to \widehat{L\mathfrak{g}} \to L\mathfrak{g} \to 0$$

of 3.1.

Proof. As a vector space, $\widehat{L\mathfrak{g}} = L\mathfrak{g} \oplus \mathbf{C}.c$. Let Φ be the morphism $\Phi : L\mathfrak{g} \to \mathrm{Lie}(\widehat{LG})$ defined by $\Phi(a, b.c) = [a, \mathrm{d}\bar{\rho}(a) + b.c]$ for $a \in \widehat{L\mathfrak{g}}$ and $b \in \mathbf{C}$. By construction, Φ is a Lie algebra isomorphism. q.e.d.

With the identification of the above lemma, the derivative of

$$\bar{\rho} : \widehat{LG} \to \mathbf{GL}(H_1)$$

is ρ.

4.7. Let $L^+G \hookrightarrow LG$ be the $\mathbf{C}$-space whose R-points are $G(\hat{X}_R)$. Notice that L^+G is an (infinite dimensional) affine $\mathbf{C}$-scheme.

Lemma 4.8. *There exists a unique splitting $\chi : \widehat{L^+G} \to G_m$ of*

$$1 \to G_m \to \widehat{LG} \to LG \to 1$$

over L^+G.

Proof. By construction, the line $\mathbf{C}.v_1$ of H_1 is stable by $\widehat{L^+G}$ and therefore defines the character χ which is a splitting. Because every character of LG is trivial, this splitting is unique. q.e.d.

4.9. If now we allow the pointed curve (X, x) to move, i.e., if we consider our family π of pointed curve over a finite type basis $S = \mathrm{Spec}(R)$ (which is possible because $\mathcal{M}_{g,1}$ is locally of finite type), one can construct the relative version $\widehat{L_\pi G}$ of $\widehat{LG}$ by integration of the representation $\mathcal{H}_l(\pi)$ as in Lemma 4.4. First of all, by unicity of the representation $\bar{\rho}$, the problem is local in S. One can therefore assume that a formal coordinate $z \in \Gamma(\hat{\mathcal{X}}_R, \mathcal{O})$ identifies $\hat{\mathcal{X}}$ with $\hat{X}_R$ and $\mathcal{H}_l$ with $H_l \otimes_{\mathbf{C}} R$, reducing the problem to the absolute case. The details are left to the reader.

5. The universal Verlinde's isomorphism

Let us first recall in the absolute case how loop groups allow to uniformize the moduli stack $\mathcal{M}_G$ of G-bundle over X and accordingly to describe generalized theta functions in terms of conformal blocks (see [13]).

5.1. Let $\mathcal{Q} = LG/L^+G$ be the grassmannian parameterizing families of pairs (E, ρ), where E is a G-bundle over X and ρ is a trivialization of E over X^*. Let $L_XG \hookrightarrow LG$ be the ind-group parameterizing automorphisms of the trivial G-bundle $X^* \times G$. Then, the forgetful morphism

$$\begin{cases} \mathcal{Q} & \rightarrow & \mathcal{M}_G \\ (E,\rho) & \longmapsto & E \end{cases}$$

is a $L_X G$-torsor. The character $\chi : \widehat{L^+G} \rightarrow G_m$ of Lemma 4 defines a $\widehat{LG}$-linearized line bundle $\mathcal{L}$ on $\mathcal{Q} = \widehat{LG}/\widehat{L^+G}$ which is a generator of $\mathrm{Pic}(\mathcal{Q})$ (see [13]).

The line bundle $\mathcal{L}$ is associated to χ^{-1} (cf. Example 3.9 of [3]). Sections of $\mathcal{L}$ are functions f on $\widehat{LG}$ such that

$$f(gh) = \chi(h)f(g),\ \ g \in \widehat{LG}(R), h \in \widehat{L^+G}(R).$$

With this section, $\mathcal{L}$ is the positive generator of $\mathcal{Q}$.

5.2. Let us recall the argument of [19] proving that $L_X G$ is a subgroup of $\widehat{LG}$. The fibred product

$$\widehat{L_X G} = \widehat{LG} \times_{LG} L_X G$$

certainly acts on the finite dimensional vector space of level-1 conformal blocks

$$B_1(X) = (H_1/L_X\mathfrak{g}H_1)^*.$$

The differential at the origin of the projective action

$$L_X G \rightarrow \mathbf{PGL}(B_1(X))$$

is the natural morphism

$$L_X\mathfrak{g} \rightarrow \mathrm{End}(B_1(X))/\mathbf{C}.\mathrm{Id}$$

and is therefore trivial. Because $L_X G$ is integral (see [13]), $\widehat{L_X G}$ acts by a character on $B_1(X)$ defining the embedding $L_X G \hookrightarrow \widehat{LG}$.

5.3. In particular, $\mathcal{L}$ is $L_X G$-linearized and defines a line bundle still denoted by $\mathcal{L}$ on $\mathcal{M}_G = L_X G \backslash \mathcal{Q}$ which generates $\mathrm{Pic}(\mathcal{M}_G)$. Let $\mathcal{M}_G^0$ be the open substack of $\mathcal{M}_G$ parameterizing regularly stable bundles (bundles E such that $\mathrm{Aut}_G(E) = Z(G)$, the center of G). Because $Z(G)$ acts trivially on V_1, the center $Z(G)$ acts trivially on the restriction of $\mathcal{L}$ to $\mathcal{Q}^0$, and $\mathcal{L}$ is therefore $L_X G/Z(G)$-linearized. Thus, $\mathcal{L}$ comes from a line bundle, still denoted by $\mathcal{L}$, on the smooth and quasi-projective coarse moduli space $M = M_G^0$ of regularly stable bundles since $\mathcal{Q}^0 \rightarrow M$ is an isotrivial $L_X G/Z(G)$-torsor.

5.4. The space of generalized theta functions of level l is by definition

$$H^0(\mathcal{M}_G, \mathcal{L}^l) = H^0(\mathcal{Q}, \mathcal{L}^l)^{L_X G}.$$

By a codimension argument, it is also $H^0(\mathcal{M}_G^0, \mathcal{L}^l)$ which is in turn $H^0(M_G^0, \mathcal{L}^l)$ (see [3], [12], [13]). By [11], [14], the $L\mathfrak{g}$-module $H^0(\mathcal{Q}, \mathcal{L}^l)$ is the (algebraic) dual H_l^* of H_l, the isomorphism being unique up to nonzero scalar by Schur's lemma. Let us explicit by give the associated Verlinde isomorphism (see [3], [7], [12], [13])

$$\kappa : \mathbf{P}B_l(X) \xrightarrow{\sim} \mathbf{P}H^0(\mathcal{M}_G, \mathcal{L}^l) = \mathbf{P}H^0(M_G^0, \mathcal{L}^l).$$

Let $u \in B_l(X)$ be a $L_X G$-invariant form on H_l. After an eventual étale base change, any smooth morphism $S \to M_G^0$ can be defined by a family of bundles. Therefore, let us consider $S \to \mathcal{M}_G$ a smooth morphism where S is a $\mathbf{C}$-scheme of finite type defined by a family of G-bundles E. Étale locally in S, let us choose a formal cocycle $\gamma \in LG(S)$ defining E. The multivalued function u_E

$$s \longmapsto u(\gamma(s).v_l) \tag{5.1}$$

defines a divisor on the smooth scheme S; E is generic by assumption and therefore u_E is generically nonzero. The gluing of these divisors defines $\kappa(u)$.

5.5. If now the curve $\pi : (\mathcal{X}, x) \to S = \mathrm{Spec}(R)$ is nonconstant, the family of ind-groups $(L_{\mathcal{X}_s} G)_{s\in S}$ glues to give an ind-group $L_{\mathcal{X}} G$ over S, which is a subgroup of $L_\pi G$. As in 5.2, the action of $\widehat{L_{\mathcal{X}} G}$ on the vector bundle of level-1 vacua $\mathbf{B}_1$ defines a character $\widehat{L_{\mathcal{X}} G} \to G_{m,S}$ and therefore an embedding (over S)

$$\begin{tikzcd} L_{\mathcal{X}} G \arrow[r, hook] \arrow[dr] & \widehat{L_\pi G} \arrow[d] \\ & S \end{tikzcd}$$

Recall that the action of $\widehat{L_\pi^+ G}$ on the trivial line bundle $\mathcal{O}_S.v_1 \hookrightarrow \mathcal{H}_1(\pi)$ defines a character

$$\chi : \begin{cases} \widehat{L_\pi^+ G} \to G_{m,S} \\ \searrow \quad \swarrow \\ \quad S \end{cases}$$

Of course, this construction is functorial in π, and all the above constructions are universal over $\mathcal{M}_{g,1}$.

5.6. The relative version of 5 goes as follows. Consider the relative grassmannian $\mathcal{Q}_\pi = \widehat{L_\pi G}/\widehat{L_\pi^+ G}$ over S and the line bundle $\mathcal{L}$ on $\mathcal{Q}_\pi$ defined by χ^{-1}. Because $L_\mathcal{X} G$ embeds in $\widehat{L_\pi G}$, the line bundle $\mathcal{L}$ is $L_\mathcal{X} G$-linearized and therefore defines a line bundle $\mathcal{L}$ on the universal moduli stack $L_\mathcal{X} G\backslash\mathcal{Q}$. The projection

$$q_\pi : \ \mathcal{Q}_\pi \to S$$

is locally trivial for the Zariski topology; the choice of a formal coordinate along x defines such a trivialization. Formula (5.1) defines a morphism

$$\iota_\pi : \mathcal{H}_l(\pi)^* \to q_{\pi,*}\mathcal{L}^l.$$

Because q is locally trivial, it follows that ι_π is an isomorphism and therefore that $\iota_\pi \otimes \mathbf{C}(s)$ is so for every $s \in S(\mathbf{C})$, which is the above theorem of [11], [14]. As in 5, let me consider the $L_\mathcal{X} G$-torsor

$$r_\pi : \ \mathcal{Q}_\pi \to L_\mathcal{X} G\backslash \mathcal{Q}_\pi = \mathcal{M}_{G,\pi}.$$

If p_π denotes the projection $\mathcal{M}_{G,\pi} \to S$, the sheaf $p_{\pi,*}\mathcal{L}^l$ of global sections of $\mathcal{L}^l$ is the invariant sheaf

$$(q_{\pi,*}\mathcal{L}^l)^{L_\mathcal{X} G} = (\mathcal{H}_l(\pi)^*)^{L_\mathcal{X} G}.$$

5.7. These constructions are functorial in π. Let $\underline{M}_G^0$ (resp. $\underline{\mathcal{M}}_G^0$) be the universal coarse moduli space (resp. moduli stack) of regularly stable bundles. Let $p : \ \underline{M}_G^0 \to \mathcal{M}_{g,1}$ be the projection, $\mathcal{X}$ be the universal curve and $\mathcal{H}_l$ the universal family of basic level l representations. As in the absolute case, the restriction of $\mathcal{L}$ to $\underline{\mathcal{M}}_G^0$ defines a line bundle $\mathcal{L}$ on $\underline{M}_G^0$. By the above discussions, the global Verlinde's isomorphism is the isomorphism

$$\kappa : \ \mathbf{PB}_l = \mathbf{P}(\mathcal{H}_l^*)^{L_\mathcal{X} G} \xrightarrow{\sim} \mathbf{P}p_*\mathcal{L}^l,$$

which is explicitly described by formula (5.1).

Computation of the connections

We choose a positive integer l. We denote by M the regularly stable locus of $M_G(X)$, and by Θ the line bundle $\mathcal{L}^l$ on M, 5.3. As explained above, the line bundle Θ exists over $\mathcal{M}_{g,1}$.

6. Deformations of global sections and connections

Let $U_i,\ i \in I$ be an affine open cover of any smooth variety V. Let s be a global section of the line bundle L on V. For the convenience of the reader, let me first recall some deformation theory of the triple (V, L, s) (see [22]). We denote by $(V_\epsilon, L_\epsilon, s_\epsilon)$ a deformation of (V, L, s) over the length 2-scheme $D_\epsilon = \mathrm{Spec}(\mathbf{C}[\epsilon])$ with $\epsilon^2 = 0$.

6.1. The restriction $U_{i,\epsilon}$ of V_ϵ to U_i is trivial, because U_i is smooth and affine. Let us choose an isomorphism

$$\iota_i : \ \mathcal{O}_{U_i}[\epsilon] = \mathcal{O}_{U_i} \boxtimes \mathbf{C}[\epsilon] \xrightarrow{\sim} \mathcal{O}_{U_{i,\epsilon}},$$

which restricts to Id when $\epsilon = 0$. The matrix of $\iota_j^{-1} \circ \iota_i$ is of the form

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \xi_{i,j} & \mathrm{Id} \end{pmatrix},$$

where $\xi_{i,j}$ is a derivation of $\mathcal{O}_{U_i \cap U_j}$. The image of the cocycle $(\xi_{i,j})$ in $H^1(V, T_V)$ is the Kodaira-Spencer class of the deformation V_ϵ. One checks that this procedure identifies isomorphism classes of infinitesimal deformations of V and $H^1(V, T_V)$.

6.2. As above, the restriction $L_{U_i,\epsilon}$ of L_ϵ to U_i is trivial. Let us therefore choose a morphism

$$\phi_i : \ L_{U_i}[\epsilon] = L_{U_i} \boxtimes \mathbf{C}[\epsilon] \to L_{U_i,\epsilon},$$

which restricts to Id when $\epsilon = 0$. The morphism ϕ_i is an isomorphism, and the matrix of $\phi_j^{-1} \circ \phi_i$ is of the form

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \eta_{i,j} & \mathrm{Id} \end{pmatrix} m$$

where $\eta_{i,j}$ is a first order differential operator of symbol $\eta_{i,j}$ of $L_{U_i \cap U_j}$. Let $\mathcal{D}^i(L), i \in \mathbf{N}$ be the sheaf of differential operators of order $\leq i$ on L. The image of the cocycle $(\eta_{i,j})$ in $H^1(V, \mathcal{D}^1(L))$ is the Kodaira-Spencer class of the deformation (V_ϵ, L_ϵ). One checks that this procedure identifies isomorphism classes of infinitesimal deformations of (V, L) and $H^1(V, \mathcal{D}^1(L))$.

6.3. There exists a (uniquely defined) section σ_i of L_{U_i} such that the restriction $s_{U_i,\epsilon}$ of s_ϵ to U_i can be written,

$$s_{U_i,\epsilon} = \phi_i(s_{U_i} + \epsilon\sigma_i).$$

One has the tautological relation $s_{U_i} = s_{U_j}$ on $U_i \cap U_j$ and, by definition of η, one has the equality

$$(*) \qquad \sigma_j - \sigma_i = \eta_{i,j}(s).$$

Let $d_i s, i \in \mathbf{N}$ be the complex:

$$d_i s = \begin{cases} \mathcal{D}^i(L) & \xrightarrow{ev_s} \quad L \\ \deg(0) & \qquad \deg(1). \end{cases}$$

The equality (*) means that

$$(\eta_{i,j}, \sigma_i) \in \mathcal{C}^1(\{U_i\}, d_1 s) = \mathcal{C}^1(\{U_i\}, \mathcal{D}^1(L)) \oplus \mathcal{C}^0(\{U_i\}, L)$$

is a cocycle and therefore defines a class in $\mathbf{H}^1(d_1 s)$. One checks that this procedure identifies isomorphism classes of infinitesimal deformations of (V, L, s) and $\mathbf{H}^1(d_1 s)$.

7. How to compute Hitchin's connection

Let us first explain why it is enough to compute the covariant derivative.

7.1. Let E be a vector bundle on a (smooth) variety V, and ∇ be a connection on the projective bundle $\mathbf{P}E$ of lines of E. Let $\bar{v}$ be a vector field defined on some open subset U of V, and let s be a section of E on U. Let u be a point of $U(\mathbf{C})$, and v be the tangent vector $\bar{v}(u)$. Let us denote by $(u, \bar{v}(u))^\nabla$ the tangent vector of $\mathbf{P}E$ at $s(u)$, which is the horizontal lifting of v. Then, the difference

$$(7.1) \qquad \nabla_{\bar{v}}(s)[u] = ds(v) - (u, \bar{v}(u))^\nabla \in T_{s(u)}\mathbf{P}E$$

is tangent to the fiber $\mathbf{P}E_u$ and therefore lives in $T_{s(u)}\mathbf{P}E_u = E \otimes \mathbf{C}(u)/\mathbf{C}.s(u)$. Because the space of connection is an affine space under $H^0(V, \Omega_V \otimes \mathrm{End}(E)/\mathcal{O}_V.\mathrm{Id})$, and V is reduced, the collection $\nabla_{\bar{v}}(s)[u], u \in U(\mathbf{C})$ determines the connection ∇.

7.2. Let $\bar{t} \in H^1(X, T_X)$ and $\bar{t}_\epsilon : D_\epsilon \to \mathcal{M}_g$ be the corresponding morphism. Let us denote the pull-back $\bar{t}_\epsilon^*(\underline{M}_G^0, \mathcal{X}, \Theta)$ of the universal data simply by $(X_\epsilon, M_\epsilon, \Theta_\epsilon)$, and its restriction to $(\epsilon = 0)$ by (X, M, Θ).

Remark 7.3. Recall that for any vector bundle F on X, the Cech complex $\mathfrak{C}_F$

$$H^0(D, F) \oplus H^0(X^*, F) \xrightarrow{\delta_F} H^0(D^*, F)$$

associated to the flat cover $D \sqcup X^* \longrightarrow\!\!\!\!\rightarrow X$ of X calculates the cohomology $H^*(X, F)$. In particular, the complex $\mathfrak{C}_{TX}$ defines a projection from the vector space of meromorphic vector fields T_{D^*} on D^* onto $H^1(X, T_X)$. If t is a meromorphic vector field on D, which projects to $\bar{t}$, then the infinitesimal deformation X_ϵ of X over D_ϵ can be described in the following manner: one glues the 2 trivial deformations $X^*[\epsilon]$ and $D[\epsilon]$ of X^* and D respectively along $D^*[\epsilon]$ thanks to the automorphism of $D^*[\epsilon]$ defined by

$$\begin{cases} \mathcal{O}_{D^*}[\epsilon] \to \mathcal{O}_{D^*}[\epsilon] \\ \\ f \longmapsto f + \epsilon < t, \mathrm{d}f > , \end{cases}$$

In particular, a formal coordinate z on X lifts canonically to a formal coordinate on X_ϵ.

7.4. Let s be a global section of Θ. To construct Hitchin's connection, one has to lift s to a global section s^∇ of Θ_ϵ. The basic observation of Hitchin's construction is that the cup-product pairing

$$H^1(X, T_X) \otimes H^0(X, \mathrm{Ad}(E) \otimes \omega_X) \to H^1(X, \mathrm{Ad}(E)),$$

where E is a regularly stable bundle on X, induces by Serre duality a morphism

$$\tau : H^1(X, T_X) \to \mathbf{S}^2\, H^1(\mathrm{ad}(E)) = (\mathbf{S}^2\, T_M)_{[E]}, \tag{7.2}$$

which globalizes when $[E]$ runs over $M(\mathbf{C})$ to give the quadratic differential

$$\tau : H^1(X, T_X) \to H^0(M, \mathbf{S}^2\, T_M). \tag{7.3}$$

The short exact sequence of complexes

$$0 \to d_1 s \to d_2 s \to \mathbf{S}^2\, T_M[0] \to 0 \tag{7.4}$$

gives a morphism

$$\delta : H^0(M, \mathbf{S}^2\, T_M) \to \mathbf{H}^1(d_1 s).$$

Let

$$w_s : \ H^1(X, T_X) \to \mathbf{H}^1(d_1 s)$$

be the composition $\delta \circ \tau$.

Lemma 7.5 ([9]). *The deformation of (M,Θ) defined by the projection of $-w_s(\bar{t})/(2l+2h^\vee)$ in $H^1(\mathcal{D}^1(\Theta))$ is isomorphic to $(M_\epsilon,\Theta_\epsilon)$.*

Proof. Let Λ be the integer defined by $\omega_{M_G}=\mathcal{O}(-\Lambda)$ where $\mathcal{O}(1)$ is the determinant bundle. One has the equality $\Lambda=2h^\vee$ (see [12] for instance). By Theorem 3.6 of [9], the projection $-w_s(\bar{t})/(2l+\Lambda)$ in $H^1(M,T_M)$ is the Kodaira-Spencer class of M_ϵ. Because the codimension of the nonregularly stable locus is at least 2 (see the appendix), $H^1(M,\mathcal{O}_M)$ is zero and the symbol map $H^1(M,\mathcal{D}^1(\Theta))\to H^1(M,T_M)$ is injective. Because the image of $(M_\epsilon,\Theta_\epsilon)$ in $H^1(M,T_M)$ is (tautologically) the Kodaira-Spencer class of M_ϵ, the lemma follows. q.e.d.

Remark 7.6. Strictly speaking, only the case where $G=\mathbf{SL}_r$ is treated in [9]. But the proof in [9] can be straightforwardly adapted to the general case if Λ is defined by the equality $\omega_{M_G}=\mathcal{O}(-\Lambda)$ as above.

7.7. By the lemma, $-w_s(\bar{t})/(2l+2h^\vee)$ defines a section over $\bar{t}$ of Θ_ϵ denoted by $(s,\bar{t})^\nabla$ well-defined up to $\mathrm{Ker}(\mathrm{Aut}(\Theta_\epsilon)\longrightarrow\!\!\!\!\rightarrow \mathrm{Aut}(\Theta))=1+\epsilon\mathbf{C}$ which is the horizontal lifting (for Hitchin's connection) of $\bar{t}$ through s. If s_ϵ is a section of Θ_ϵ restricting on s when $\epsilon=0$, the difference $s_\epsilon-(s,\bar{t})^\nabla$ lives in $\epsilon H^0(M,\Theta)/\mathbf{C}.s$ and one has the equality (cf. (7.1))

$$\epsilon(\nabla_{\bar{t}}s_\epsilon)(0)=s_\epsilon-(s,\bar{t})^\nabla. \tag{7.5}$$

7.8. The explicit Cech calculation (relative to the covering U_i) of $w_s(\bar{t})$ goes as follows. Choose second order differential operators D_i on Θ_{U_i} whose symbols are $\tau(\bar{t})$ on U_i. The differential of $\{D_i\}\in\mathcal{C}^0(d_2s)$ is

$$(D_j-D_i,D_is)\in\mathcal{C}^1(d_2s)=\mathcal{C}^1(\mathcal{D}^2\Theta)\oplus\mathcal{C}^0(\Theta).$$

Because the symbol of D_j-D_i vanishes, D_j-D_i is of order one and (D_j-D_i,D_is) is a cocycle of $\mathcal{C}^1(d_1s)$ (as it has to be). By definition of the connecting homomorphism, in $\mathbf{H}^1(d_1s)$ one has the equality one has the equality

$$w_s(\bar{t})=[D_j-D_i,D_is] \tag{7.6}$$

(compare with (3.17) of [9] and [22, p. 187]). With the notation above, one has

$$\eta_{i,j}=\mathrm{symbol}(D_j-D_i)\text{ and }\sigma_i=D_is.$$

7.9. Suppose that the diagram

$$N \times_M N = \sqcup_{i,j} U_i \cap U_j \Longrightarrow N = \sqcup U_i \longrightarrow\!\!\!\!\!\rightarrow M$$

is replaced by

$$N_1 = N \times_M N \underset{p}{\overset{q}{\Longrightarrow}} N \xrightarrow{r}\!\!\!\!\!\rightarrow M,$$

where $N \longrightarrow\!\!\!\!\!\rightarrow M$ is any étale epimorphism such that $r^*(M_\epsilon, \Theta_\epsilon)$ is trivial. We suppose also that the pull-back of the quadratic differential $\tau(\bar{t})$ is the image of a second order differential operator $\theta(t) \in H^0(N, \mathcal{D}^2(r^*\Theta))$ by the composite

$$H^0(N, \mathcal{D}^2(r^*\Theta)) \xrightarrow{\text{symbol}} H^0(N, \mathbf{S}^2\, T_N) \xrightarrow{r_*} H^0(N, r^*\, \mathbf{S}^2\, T_M).$$

The degree-one piece $\mathfrak{C}^1(r, d_1 s)$ of the Cech complex of r is

$$\mathfrak{C}^1(r, d_1 s) = H^0(N_1, \rho^*\mathcal{D}^1(\Theta)) \oplus H^0(N, r^*\Theta),$$

where $\rho = r \circ p = r \circ q$. Because the coherent cohomology can be calculated using the étale topology, one has a canonical morphism

$$\mathfrak{C}^1(r, d_1 s) \to \mathbf{H}^1(d_1 s).$$

Then, as in (7.6), one has the equality in $\mathbf{H}^1(d_1 s)$

$$w_s(t) = [p^* r_* \theta(t) - q^* r_* \theta(t), \theta(t).r^* s], \tag{7.7}$$

and the infinitesimal lifting $(s, \bar{t})^\nabla$ defined by the class $-w_s(\bar{t})/(2l+2h^\vee)$ is given on N by

$$(s, \bar{t})^\nabla = r^* s - \frac{\epsilon}{(2l + 2h^\vee)} \theta.(t) r^* s. \tag{7.8}$$

Suppose that the global section s_ϵ of Θ_ϵ is given on N by

$$s_\epsilon = u + \epsilon v,\ \ u, v \in H^0(N, r^*\Theta).$$

Then, the formula (7.5) gives

$$\nabla_{\bar{t}} s_\epsilon(0) = v + \frac{\theta(t).u}{2l + 2h^\vee} \text{ in } H^0(N, r^*\Theta)/\mathbf{C}.u. \tag{7.9}$$

8. Sugawara tensors and differential operators

Recall that $r^*\Theta$ is the homogeneous line bundle $\mathcal{L}_\lambda$ where λ is the character χ^{-l} of $\widehat{L^+G}$. If $\widehat{LG}$ were finite dimensional, one would have a morphism

$$U(\widehat{L\mathfrak{g}})^{\mathrm{opp}} \to H^0(\mathcal{Q}, \mathcal{D}(\mathcal{L}_\lambda)),$$

and the Sugawara tensor $T(t)$ would define a second order differential operator on $\mathcal{L}_\lambda$, a natural candidate for $\theta(\bar{t})$ (see 7.9). Let $\widehat{LG}^0$ (resp. $\mathcal{Q}^0$) be the regularly stable locus of $\widehat{LG}$ (resp. $\mathcal{Q}$). To avoid too much abstract nonsense about differential operators on ind-schemes, one uses quasi-section

$$\begin{array}{ccc} & & \mathcal{Q}^0 \\ & \sigma \nearrow & \\ N & \xrightarrow{r} & M \end{array}$$

(cf. [6]) of $\pi : \mathcal{Q}^0 \to M$ to construct the differential operator $\theta(t)$ using $T(t)$ (formally, one just pull-back $T(t)$ by σ). By convention, all cohomology groups of any coherent sheaf on N are endowed with the discrete topology.

8.1. Let us first define the "differential"

$$\sigma^* d\pi : \widehat{L\mathfrak{g}} \to H^0(N, r^*TM) \xrightarrow{\sim} H^0(N, TN).$$

Let $n \in N(R)$ and x be an element of $\widehat{L\mathfrak{g}}$. The image of

$$\exp(\epsilon x).\sigma(n(\epsilon)) \in \mathcal{Q}^0(R[\epsilon])$$

by π is a point $m(\epsilon)$ of $M[\epsilon]$ which restricts to $r(n)$ when $\epsilon = 0$ (recall that $\mathcal{Q}^0$ is open in $\mathcal{Q}$). Because r is étale, there exists a unique point $\nu(\epsilon)$ of $N[\epsilon]$ such that $\nu(0) = n$ and $r(\nu(\epsilon)) = m(\epsilon)$. If f is a regular function defined near n, the expansion

$$f(\nu(\epsilon)) = f(n) + \epsilon x.f(n)$$

defines a regular function near n. The corresponding vector field is denoted by $\sigma^* d\pi(x)$. One checks that

$$\sigma^* d\pi : \widehat{L\mathfrak{g}}^{\mathrm{opp}} \to H^0(N, T_N)$$

is a morphism of Lie algebras and therefore induces a morphism of filtered algebras

$$U(\widehat{L\mathfrak{g}})^{\mathrm{opp}} \to H^0(N, \mathcal{D}(\mathcal{O}_N)). \tag{8.1}$$

8.2. We want to extend (8.1) to a completion of $\bar{U}(\widehat{L\mathfrak{g}})$ in which lives the Sugawara tensors. Let U be the enveloping algebra of $\widehat{\mathfrak{g}\otimes K}$. For $n \geq 0$, let U^n be the subspace of $u \in U$ which is of order $\leq n$. We define a filtration $F^iU^n, i > 0$ by

$$F^iU^n = U.\mathfrak{g}_i \cap U^n,$$

where $\mathfrak{g}_i$ is the kernel of the projection $\mathfrak{g}\otimes\mathcal{O} \longrightarrow\!\!\!\rightarrow \mathfrak{g}\otimes\mathcal{O}_{ix}$. The family $F^iU_n, i > 0$ defines a topology of U^n; let $\bar{U}^n$ be the corresponding completion, and $\bar{U} = \cup_{n\in\mathbf{N}}\bar{U}_n$ be our completion of U. It is a complete associative algebra which is filtered by definition and acts on every integrable representation. Let us choose a formal coordinate z at x. For $x \in \mathfrak{g}$ and $i \in \mathbf{Z}$, let me denote the vector $X \otimes z^i$ by $x(i)$.

Lemma 8.3. *There exists an integer* i *such that*

$$\sigma^* d\pi(x(j)) = 0$$

for all $x \in \mathfrak{g}$ *and* $j \geq i$.

Proof. Because N is of finite type, there exists i such that

$$\mathrm{Ad}(\gamma).\exp(\epsilon x(j)) \in L_+G(R[\epsilon]/(\epsilon^2)) \text{ for all } j \geq i \text{ and } \gamma \in \sigma(N(R)).$$

The lemma follows since π is right L_+G-invariant. q.e.d.

In particular, we get continuous morphisms (see 8.2 for the definition of the completion $\bar{U}^{n,\mathrm{opp}}(\widehat{L\mathfrak{g}})$)

$$\bar{U}^{n,\mathrm{opp}}(\widehat{L\mathfrak{g}}) \to H^0(N, \mathcal{D}^n(\mathcal{O}_N)). \tag{8.2}$$

8.4. Let n be a point of N. Let us consider $\sigma(n)$ as a pair (E,ρ) where ρ is a trivialization of $E_{|X^*}$. The geometric interpretation of

$$\sigma^* d\pi_n : \widehat{L\mathfrak{g}} \to T_nN = H^1(X, \mathrm{Ad}(E))$$

goes as follows. Let $x \in \widehat{L\mathfrak{g}}$ and let E_ϵ be the underlying G-bundle on $X[\epsilon]$ of $\exp(\epsilon)\sigma(n)$. The family E_ϵ defines a Kodaira-Spencer map

$$T_0D_\epsilon \to H^1(X, \mathrm{Ad}(E)).$$

Then, the image of $d/d\epsilon \in T_0D_\epsilon$ is $\sigma^* d\pi_n(x)$ by the Kodaira-Spencer map.

8.5. One can of course explicitly calculate this map. The trivialization ρ defines an isomorphisms between $\mathfrak{C}_{\mathrm{Ad}(E)}$ (cf. 7.3) and

$$H^0(D, \mathrm{Ad}(E)) \oplus \mathfrak{g} \otimes A_X \to \mathfrak{g} \otimes K.$$

The corresponding surjection

$$\widehat{L\mathfrak{g}} \longrightarrow\!\!\!\!\rightarrow \mathfrak{g} \otimes K \longrightarrow\!\!\!\!\rightarrow H^1(X, \mathrm{Ad}(E)) \tag{8.3}$$

is the differential $\sigma^* d\pi_n$.

8.6. Let $t \in T_{D^*}$ which projects to $\bar{t} \in H^1(X, T_X)$ 7.3 and $\tau(\bar{t}) \in H^0(M, \mathbf{S}^2 T_M)$ the corresponding quadratic tensor (7.3). One can compute the value

$$r^*\tau(\bar{t})_n \in \mathbf{S}^2\, T_n N = \mathbf{S}^2\, H^1(X, \mathrm{Ad}(E))$$

of $r^*\tau(\bar{t}) \in H^0(S^2 TN)$ at n as follows. The Killing form of $\mathfrak{g}$ defines an isomorphism between $\mathrm{Ad}(E)$ and its dual. The residue theorem says that the residue $\mathrm{res} : \Omega_{D^*} \to \mathbf{C}$ factors through

$$\Omega_{D^*}/(\Omega_{X^*} \oplus \Omega_D) = H^1(\mathfrak{C}_{\omega_X}) \xrightarrow{\sim} H^1(X, \omega_X)$$

to give the canonical isomorphism $H^1(X, \omega_X) \xrightarrow{\sim} \mathbf{C}$ defined by the meromorphic form dt/t. By Serre duality, $r^*\tau(\bar{t})_n$ is therefore a quadratic form on $H^0(X, \mathrm{Ad}(E) \otimes \omega_X)$. By 7, $r^*\tau(\bar{t})_n$ is induced by the cup-product

$$H^1(X, T_X) \otimes H^0(X, \mathrm{ad}(E) \otimes \omega_X) \to H^1(X, \mathrm{Ad}(E)).$$

The trivialization ρ defines an injection

$$H^0(X, \mathrm{Ad}(E) \otimes \omega_X) \hookrightarrow \mathfrak{g} \otimes \Omega_{X^*}.$$

The Killing form defines a pairing

$$\mathrm{tr} :\ [\mathfrak{g} \otimes \Omega_{X^*}] \otimes [\mathfrak{g} \otimes K] \to K \otimes \Omega_{X^*} \xrightarrow{\sim} \Omega_{D^*}. \tag{8.4}$$

The tensor $\tau(\bar{t})_n \in S^2 H^1(X, \mathrm{Ad}(E))$ of 7 is characterized by the formula

$$\tau(\bar{t})(\phi \otimes \phi) = \mathrm{res}\ \mathrm{tr}(\bar{\phi} \otimes t.\bar{\phi}) \tag{8.5}$$

for every $\phi \in H^0(X, \mathrm{Ad}(E) \otimes \omega_X)$ mapping to $\bar{\phi} \in \mathfrak{g} \otimes \Omega_{X^*}$ and $t \in T_{D^*}$; the contraction $t.\bar{\phi}$ is thought as an element of $\mathfrak{g} \otimes K$.

8.7. The twisted version is analogous. Consider the commutative diagram with a cartesian square

$$\begin{tikzcd} \hat{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widehat{LG}^0 \arrow[d] \\ N \arrow[r, "\sigma"] \arrow[dr, "r"'] & \mathcal{Q}^0 \arrow[d, "\pi"'] \\ & M \end{tikzcd}$$

The morphism of $\mathbf{C}$-space $\hat{N} \to N$ is a $\widehat{L^+G}$-torsor, and sections of $r^*\Theta = \sigma^*\mathcal{L}_\lambda$ are functions on $\hat{N}$ which are λ-equivariant. Let f be such a function, and let $\hat{n} = (n, \gamma)$ be a point of $\hat{N}(R)$. With the notation above,

$$\exp(\epsilon x)\hat{n} := (\nu(\epsilon), \exp(\epsilon x)\gamma)$$

is a point of $\hat{N}(R[\epsilon])$ restricting to $\hat{n}$ when $\epsilon = 0$. The expansion

$$f(\exp(\epsilon x)\hat{n}) = f(\hat{n}) + \epsilon x.f(n)$$

defines a morphism of Lie algebras

$$\begin{cases} \widehat{L\mathfrak{g}}^{\mathrm{opp}} & \to & H^0(N, r^*\Theta), \\ x & \longmapsto & (f \longmapsto x.f). \end{cases}$$

As above, Lemma 8 allows us to define continuous morphisms

$$\bar{U}^{n,\mathrm{opp}}(\widehat{L\mathfrak{g}}) \to H^0(N, \mathcal{D}^n(r^*\Theta)). \tag{8.6}$$

The arrows (8.6) and (8.2) are compatible, meaning that the symbol diagram

$$\begin{tikzcd} & H^0(N, \mathcal{D}^n(r^*\Theta)) \arrow[dr, "\text{symbol}"] & \\ \bar{U}^{n,\mathrm{opp}}(\widehat{L\mathfrak{g}}) \arrow[ur] \arrow[dr] & & H^0(N, \mathbf{S}^n T_N) \\ & H^0(N, \mathcal{D}^n\mathcal{O}_N) \arrow[ur, "\text{symbol}"'] & \end{tikzcd} \tag{8.7}$$

is commutative.

8.8. Let me recall the definition of $T_n \in \bar{U}$ (see [10, (12.8.4)]). Let x_i be an orthonormal basis of $\mathfrak{g}$ (for the Killing form). The sequence of operators

$$(8.8)\qquad \begin{aligned} T_0 &= \sum\nolimits_i x_i x_i + 2\sum_{n=1}^{\infty} x_i(-n)x_i(n), \\ T_n &= \sum\nolimits_{m\in\mathbf{Z}} \sum_i x_i(-m)x_i(m+n) \qquad \text{if } n\neq 0 \end{aligned}$$

is well defined and does not depend on the choice of the x_i's.

Remark 8.9. The notation is not standard. Usually, $1/(2l+2h^\vee)T_n$ is denoted by L_n and the formal power series $\sum L_n u^{-n-2}$ is denoted by $T(u)$ (for instance in [21]); notice the opposite convention in [18], giving a change of sign in the definition of the WZW connection.

8.10. Suppose that n is positive. Because $x_i(-m)$ and $x_i(n+m)$ commute in $U(\widehat{L\mathfrak{g}})$, one then has $x_i(-n)x_i(n+m)\in \mathbf{F}^{[n/2]}U^2(\widehat{L\mathfrak{g}})$ for every integer m. Therefore,

$$T_n \in \mathbf{F}^{[n/2]}U^2(\widehat{L\mathfrak{g}}) \text{ and } \lim_{n\in\mathbf{N}} T_n = 0.$$

Let d_n be the meromorphic tangent vector $z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}$.

Definition 8.12. Let $t=\sum_{n\geq -N} t_n d_n$ be a meromorphic vector field on D^*. The Sugawara tensor $T(t)\in \bar{U}^2(\widehat{L\mathfrak{g}})$ is defined by the equality

$$T(t) = \sum_{n\geq -N} t_n T_n.$$

The second order differential operator $\theta(t)\in H^0(N,\mathcal{D}^2(r^*\Theta)$ is the image of $T(t)$ by the morphism

$$\bar{U}^{2,\mathrm{opp}}(\widehat{L\mathfrak{g}}) \to H^0(N,\mathcal{D}^2(r^*\Theta))$$

of (8.6).

8.12. Let $\phi\in H^0(X,\mathrm{Ad}(E)\otimes\omega_X)$ be a mapping to $\bar{\phi}\in\mathfrak{g}\otimes\Omega_{X^*}$, and $t\in T_{D^*}$. The series

$$\sum\nolimits_{m\in\mathbf{Z}} \sum_i <\bar{\phi}, x_i(-m)><\bar{\phi}, x_i(m+n)>$$

has finite support which allows us to define

$$(8.9)\quad <\bar{\phi}\otimes\bar{\phi}, T_n> = \sum\nolimits_{m\in\mathbf{Z}} \sum_i <\bar{\phi}, x_i(-m)><\bar{\phi}, x_i(m+n)>.$$

One defines $<\bar\phi\otimes\bar\phi, T_0>$ by the analogous formula. By (8.3) and 8.7, the symbol of $\theta(t)$ evaluated at

$$\phi\otimes\phi\in \mathbf{S}^2\, T_n^* N = \mathbf{S}^2\, H^0(X, \mathrm{Ad}(E)\otimes\omega_X)$$

is equal to the finite sum

$$\sum_{n\in\mathbf{Z}} t_n <\bar\phi\otimes\bar\phi, T_n> = \sum_{n\leq 2|\mathrm{val}(\bar\phi)|} t_n <\bar\phi\otimes\bar\phi, T_n> .$$

Proposition 8.13. *The symbol of $\theta(t)$ is the quadratic differential $\tau(\bar t)$ of (7.3).*

Proof. By (8.5) (keeping the notation above), one has to prove the equality

$$\mathrm{res}\ \mathrm{tr}(\bar\phi\otimes t.\bar\phi) = <\bar\phi\otimes\bar\phi, T^{\mathrm{symb}}(t)> .$$

Observe that the preceding expression still makes sense if $\bar\phi$ lives in $\mathfrak{g}\otimes\Omega_{D^*}$. Now, if the valuation $\mathrm{val}(\bar\phi)$ is big enough, both the scalars $<\bar\phi\otimes\bar\phi, T^{\mathrm{symb}}(t)>$ and $\mathrm{res}\ \mathrm{tr}(\bar\phi\circ t.\bar\phi)$ are zero. One can therefore assume that $\bar\phi = x_j(l)\mathrm{d}z$ for some $l\in\mathbf{Z}$, and also that $t = d_n, n\in\mathbf{Z}$. Now, we compute

$$< x_j(l)\mathrm{d}z\otimes x_j(l)\mathrm{d}z, T_n> = \delta_{n+2l,-2} = \mathrm{res}(z^{n+1+2l}\mathrm{d}z)$$

(even in the case where $n=0$), and obtain

$$\begin{aligned}\mathrm{res}\ \mathrm{tr}(x_j(l)\mathrm{d}z\circ d_n.x_j(l)\mathrm{d}z) &= \mathrm{res}\ \mathrm{tr}(x_j\otimes z^l\mathrm{d}z\circ z^{n+1}\frac{\mathrm{d}}{\mathrm{d}z}.x_j\otimes z^l\mathrm{d}z)\\ &= \mathrm{res}(z^{n+1+2l}\mathrm{d}z).\end{aligned}$$

q.e.d.

8.14. The computation of the Hitchin's covariant derivative $\nabla_{\bar t}s_\epsilon(0), s_\epsilon\in H^0(D_\epsilon,\Xi)$ is now easy. Let us choose a local coordinate on X, which lifts to a local coordinate on X_ϵ along x (see Remark 7.3), identifying the universal pair $(\mathcal{Q}^0,\Theta)$ over D_ϵ to the trivial deformation $(\mathcal{Q}^0[\epsilon],\Theta[\epsilon])$. We pick quasi-section

$$\begin{array}{ccc} & & \mathcal{Q}^0\\ & \sigma\nearrow & \\ N & \xrightarrow{r} & M\end{array}$$

of $\pi: \mathcal{Q}^0 \to M$. We define $\theta(t)$ as in 8.12; one is under the hypothesis of 7.9. By 5.7, there exists 2 linear forms U, V on H_l such that

$$\kappa(U + \epsilon V) = s_\epsilon.$$

With the notation of 8.7, the pull-back $\sigma^* s_\epsilon$ can be decomposed as

$$\sigma^* s_\epsilon = u + \epsilon v,$$

where the section u of $r^*\Theta$ can be thought of as a λ-equivariant function on $\hat{N}$ defined by (5.1)

$$\hat{n} = (n, \gamma) \longmapsto u(\hat{n}) = U(\gamma.v_\kappa),$$

v_κ being the highest weight vector of H_{d_κ}. The action 8.7 of $x \in \widehat{L\mathfrak{g}}$ on u is defined by the ϵ-derivative of

$$u(\exp(\epsilon x).\hat{n}) = U(\gamma.v_\kappa) - \epsilon x.U(\gamma.v_\kappa).$$

Therefore, one has the equality

$$x.u = -\sigma^*\kappa(x.U) \text{ and } T(t).u = \sigma^*\kappa(T(t).U).$$

Formula (7.9) thus becomes

$$\begin{aligned} r^*\nabla_{\bar{t}} s_\epsilon(0) &= v + T(t)/(2l + 2h^\vee).u \\ &= \sigma^*\kappa(V + T(t)/(2l + 2h^\vee).U) \quad \mod u. \end{aligned} \tag{8.10}$$

9. WZW connection

Let me recall how the WZW connection on V_l can be explicitly computed (see [21, Definition 5.1.2]).

9.1. We start with a versal deformation $\mathcal{X} \to S$ of the pointed curve $\mathcal{X}_0$. Let t be a meromorphic vector field on $\mathcal{D}$, which projects to the image by the Kodaira-Spencer map of some tangent vector $\tau \in T_0 S$. If f is a function on S, and u is a linear form on $\mathcal{H}_l$, the WZW-connection Δ on V_l^* is defined by the formula

$$\Delta_\tau(u \otimes f) = u \otimes t.f + T(t)/(2l + 2h^\vee)u \otimes f \quad \mod (u \otimes f) \tag{9.1}$$

(see [18, Definition 2.7.4]) and Remark 8.9.

9.2. The tangent vector τ defines a morphism $D_\epsilon \to (S,0)$ such that $\partial/\partial\epsilon$ maps to τ. Let us pull-back the situation by this morphism. The first order expansion of (9.1) then gives

$$\Delta_{\partial/\partial\epsilon}(u+\epsilon v) = v + T(t)/(2l+2h^\vee).u \mod u, \tag{9.2}$$

which is precisely $\nabla_{\partial/\partial\epsilon}\kappa(u+\epsilon v)$ (see (8.10). We endow $\mathbf{P}(\mathbf{V}_{\mathrm{d}_\kappa})^*$ with the WZW connection, and $\mathbf{P}p_*\Theta_\kappa$ with the Hitchin's connection. Comparing (8.10) and 9, we have proved

Theorem 9.3. *With the notation of 5.7, the morphism κ*

$$\mathbf{PB}_l \xrightarrow{\sim} p_*\mathcal{L}^l$$

is a flat isomorphism of flat projective bundles over $\mathcal{M}_{g,1}$.

Remark 9.4. In fact, the result remains true if $g=2$, at least if G is not $\mathbf{S}L_2$ or $\mathbf{S}P_4$ (see the appendix below).

10. The Picard group of $\mathcal{Q}$

We know that the Picard group of each fiber $q^{-1}(s)$ (s a complex point of $\mathcal{M}_{g,1}$) is $\mathbf{Z}.\mathcal{L}_s$ (see the appendix); this defines an integer $\deg(L)$ of every line bundle on $\mathcal{Q}$, which is the exponent e such that $L_s = \mathcal{L}_s^{\otimes e}$ (recall that $\mathcal{M}_{g,1}$ is connected).

Proposition 10.1. *The sequence*

$$0 \to \mathrm{Pic}(\mathcal{M}_{g,1}) \xrightarrow{q^*} \mathrm{Pic}(\mathcal{Q}) \xrightarrow{\deg} \mathbf{Z} \to 0$$

is exact and the morphism $\begin{cases} \mathbf{Z} & \to & \mathrm{Pic}(\mathcal{Q}) \\ e & \longmapsto & \mathcal{L}^{\otimes e} \end{cases}$ *is a splitting.*

Proof. The grassmannian $\mathcal{Q}$ is the direct limit $\varinjlim \mathcal{Q}_w$ where $w \in W_{\mathrm{aff}}/W = Q(\mathcal{R}^\vee)$ and $\mathcal{Q}_w$ is the relative Schubert variety of index w which can be geometrically described as follows. Let $L^{>0}G$ be the inverse image of 1 by the evaluation $L^+G \twoheadrightarrow T$, and let $\bar{w}$ be the direct image of the G_m-torsor $V(\mathcal{O}(-x)) \setminus 0$ by $w : G_m \to G$; because $\mathcal{O}(-x)$ is canonically trivial on X^*, the G-bundle $\bar{w}$ is trivialized on X^* and therefore defines a point of $\mathcal{Q}$. The Schubert variety $\mathcal{Q}_w$ is as usual the union $\mathcal{Q}_w = \cup_{w' \le w} L^{>0}G\bar{w}'.\mathbf{1}$ where $\mathbf{1}$ is the class of the trivial (trivialized) G-bundle). The choice of a local coordinate near the marked point trivializes the restriction q_w of q to $\mathcal{Q}_w$ proving that q_w is flat. Each Schubert variety $\mathcal{Q}_w(s)$ over $s \in \mathcal{M}_{g,1}(\mathbf{C})$ is projective, and integral. Moreover, the natural morphism $\mathrm{Pic}(\mathcal{Q}) \to \mathrm{Pic}(\mathcal{Q}_{s,w})$ is an isomorphism. By construction, the restriction of $M = \mathcal{L}^{\deg(L)} \otimes L^{-1}$ to $\mathcal{Q}_w(s)$ is trivial. Because $\mathcal{M}_{g,1}$ is reduced, the base change theorem implies that the direct image $q_{*,w}M_w$ of the restriction M_w to $\mathcal{Q}_w$ is a line bundle $\bar{M}_w$ on $\mathcal{M}_{g,1}$ and that the morphism $q_w^* q_{w,*}M = q_w^*\bar{M}_w \to M_w$ is surjective and therefore an isomorphism. The isomorphisms $(M_w)_{\mathcal{Q}_{w'}} \xrightarrow{\sim} M_{w'}$ for $w' \leq w$ induces isomorphisms $\bar{M}_{w'} \xrightarrow{\sim} \bar{M}_w$; let $\bar{M}$ be the direct limit $\varinjlim \bar{M}_w$ (which is isomorphic to each of the $\bar{M}_w$). By construction, $L \xrightarrow{\sim} \mathcal{L}^{\deg(L)} \otimes q^*\bar{M}$. q.e.d.

Remark 10.2. In particular, the Picard group of $\mathcal{Q}$ is $\mathbf{Z}^3$.

Lemma 10.3. *Let H be a $\mathbf{C}$-group. Let H_1, H_2 be 2 $\mathbf{C}$-subgroups of H and $\psi_2 : H_2 \to G_m$ a character defining a line bundle $\mathcal{L}_2$ on H/H_2. The pull-back $\mathcal{L}_{1,2}$ on $H_1/H_{1,2}$ (where $H_{1,2} = H_1 \cap H_2$) of $\mathcal{L}_2$ is the line bundle associated to the restriction $\psi_{1,2}$ of ψ_2 to $H_{1,2}$.*

Proof. By definition, $\mathcal{L}_2$ is defined by the morphism

$$H/H_2 \to BH_2 \to BG_m,$$

where $H/H_2 \to BH$ is defined by the (H_2-equivariant) morphism $H \times H/H_2$ (H being seen as an H_2-torsor over H/H_2, and $BH_2 \to BG_m$ being $B\psi_2$). The pull-back on $H_1/H_{1,2}$ is defined by the composite

$$H_1/H_{1,2} \to H/H_2 \to BH_2 \to BG_m.$$

The diagram

$$\begin{array}{ccccc} H_1/H_{1,2} \to & H/H_2 & \to & BH_2 \\ \searrow & & \nearrow & \\ & BH_{1,2} & & \end{array}$$

is 2-commutative ($BH_{1,2} \to BH_2$ being the natural morphism deduced from $H_{1,2} \hookrightarrow H_2$). The proposition follows because the composite $BH_{1,2} \to BH_2 \to BG_m$ is $B\psi_{1,2}$. q.e.d.

10.4. Let σ the section of $\mathcal{Q}$ defined by the trivial G-bundle (with its canonical trivialization on the punctured curve) over $X \times \mathcal{M}_{g,1}$. It corresponds to the unit section of $\widehat{LG} \to \mathcal{M}_{g,1}$. The above lemma proves that $\sigma^*\mathcal{L}$ is trivial. We can therefore rewrite Proposition 10 in the following form: for every $L \in \mathrm{Pic}(\mathcal{Q})$, one has the formula

$$L = \mathcal{L}^{\deg(L)} \otimes q^*(\sigma^* L). \tag{10.1}$$

10.5. Let $\rho : G \to \mathbf{S}L_N$ be a linear representation of G, which can be assumed to be nontrivial. Let $\mathcal{E}$ be the universal G-bundle on $\mathcal{Q} \times_{\mathcal{M}_{g,1}} \mathcal{X}$, and L_ρ the line bundle on $\mathcal{Q}$

$$L_\rho = \det(R\Gamma\mathcal{E}(\mathbf{C}^N))^{-1}. \tag{10.2}$$

The degree $\deg(L_\rho)$ is the Dynkin index d_ρ of the representation ρ (see [13]). The formula (10.1) gives therefore an isomorphism of $L_{\mathcal{X}}G$-linearized bundles

$$L_\rho \otimes q^* \det R\Gamma\mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} \mathcal{L}^{\mathrm{d}_\rho} \tag{10.3}$$

well defined up to $H^0(\mathcal{M}_{g,1}, \mathcal{O}^*)$[1].

Remark 10.6. Both sides of (10.3) descends to the universal moduli space. The corresponding projective bundles of global sections

$$\mathbf{P}R\Gamma L_\rho \text{ and } \mathbf{P}R\Gamma\mathcal{L}^{d_\rho}$$

have therefore a Hitchin's connection and are isomorphic (as projective bundles). The construction of Hitchin's connection is certainly functorial and the preceding isomorphism is *flat*.

11. Appendix

For completeness, let me prove a codimension estimate (see [8, Theorem II.6] for similar statements) which is certainly well known to the experts.

Lemma 11.1. *Let $\pi : E \to S$ be a (right) G-bundle over a connected $\mathbf{C}$-scheme S with G reductive. Assume that E has a noncentral automorphism of finite order N. Then E has an L-structure F where L is a proper reductive subgroup of G.*

[1] One can show that this group is in fact $\mathbf{C}^*$, proving that (10.3) is well-defined up to a non-zero scalar.

Proof. Let e be a point of $E(\mathbf{C})$ and g the (unique) point of $G(\mathbf{C})$ such that $\phi(e) = e.g$. Let $T \to S$ be an S-scheme and $F(T)$ be the set

$$F(T) = \{\epsilon \in \mathrm{Hom}_S(T, E) \text{ such that } \phi(\epsilon) = \epsilon g\}.$$

The obvious functor

$$F : \begin{cases} \text{Schemes}^{\text{opp}} & \to \quad \text{Ens}, \\ \quad T & \longmapsto \quad F(T) \end{cases}$$

is a formally principal homogeneous space under the centralizer L of G. This group is reductive (not necessary conneceted) and proper ($g \notin Z(G)$). One has to check that $F(s)$ is nonempty for every $s \in S(\mathbf{C})$.

Let $s \in S(\mathbf{C})$ and $t \in E_s$ over s. There exists a unique $g_t \in G(\mathbf{C})$ such that $\phi(t) = t.g_t$. The conjugacy class of g_t depends only on s. Because g_t is of finite order, it is semisimple and one can define a map $f : S(\mathbf{C}) \to T/W(\mathbf{C})$ which sends s to the conjugacy class of the semi-simple element g_t. Because E is locally trivial, f is algebraic. The functions on T/W are generated by the characters of the fundamental representations. Because g_t is of order N, the eigenvalues of the corresponding matrices are in μ_N and therefore the image of f is finite. Since S is conneceted, this image is a point, the class of g say. This proves the lemma. q.e.d.

Remark 11.2. Suppose that S is a smooth complete curve and that E is semistable. Because L is reductive, the morphism $\mathfrak{g} \longrightarrow\!\!\!\!\rightarrow \mathfrak{g}/\mathrm{Lie}(L)$ has a G-invariant section. The degree-0 vector bundle $\mathrm{Ad}(F)$ is therefore a direct summand of $\mathrm{Ad}(E) = F(\mathfrak{g})$ and is semistable.

11.3. Let X be a smooth complete and projective complex curve, and G a reductive algebraic group.

Definition 11.4. A regularly stable G-bundle on X is a stable bundle with $\mathrm{Aut}(E) = Z(G)$.

Lemma 11.5. *The locus of regularly stable bundles is open in the moduli space of stable G-bundles $M^s_G(X)$.*

Proof. By [17], $M^s_G(X)$ is the GIT quotient of some smooth polarized quasi-projective scheme S by $\mathbf{SL}_N$. Moreover, all points of $S(\mathbf{C})$ are stable (properly stable in the old terminology) for a suitable linearization (induced by some embedding of $S \hookrightarrow \mathbf{P}^{N-1}$). Let $\mathcal{G}$ be the S-group scheme defined as the inverse image of the diagonal by

$$(g, x) \longmapsto (x, gx).$$

The geometric fibers of $\mathcal{G}$ are automorphisms groups of stable bundles and therefore are finite. In particular, $\mathcal{G} \to \mathcal{S}$ is quasi-finite. By Corollary 2.5 of [15], the action $G \times S \to S$ is proper (all the points are assumed to be stable), and hence $\mathcal{G} \to \mathcal{S}$ is finite (proper and quasi finite). By the theorem of formal function, the locus in S where

$$Z(G)_S \hookrightarrow \mathcal{G}$$

is an isomorphism is open. q.e.d.

Proposition 11.6. *The closed subset B of $M_G(X)$ parameterizing semistable bundles E which are not regularly stable is of codimension ≥ 3 for $g \geq 3$ or $g = 2$, and $\mathfrak{g}$ has a factor of type A_1 or C_2.*

Proof. One can assume that G is semisimple (divide by the neutral component of $Z(G)$). Let E be a semistable bundle which is not regularly stable. If E is not stable, there exist a unique standard parabolic subgroup P and a P-structure $\tilde{F}$ of F such that $F = \tilde{F}/\mathrm{rad_u P}$ is stable (as $P/\mathrm{rad_u P}$-bundle). If L is a Levi subgroup of P, this shows that $[E] = [F(G)]$ is in the image of the rational map $M_L(X) \to M_G(X)$ in this case. If now E is assumed stable with $\mathrm{Aut}(E) \neq Z(G)$, let us choose a a noncentral automorphism ϕ, necessarily of finite order. Let F be the L-structure of $gr(E)$ determined by ϕ. Then, F is semistable and E is in the image of the rational morphism $M_L(X) \to M_G(X)$. We have therefore to compare

$$\begin{aligned} \dim M_L(X) =& (g-1)\dim(L) + \dim Z(L) \quad \text{and} \\ (g-1)\dim(G) =& \dim M_G(X). \end{aligned}$$

The function

$$L \longmapsto (g-1)\dim(L) + \dim Z(L)$$

is increasing; one can assume that L is maximal. In this case, the dimension of $Z(L)$ is at most 1 and, except that $\mathfrak{g}$ has a factor of type A_1 or C_2, one has $\dim(G) - \dim(L) \geq 4$ (use exercices VIII.3.2 and VI.4.4 of [2] for instance). q.e.d.

References

[1] A. Beauville, *Conformal blocks, fusions rules and the Verlinde formula*, Israel Math. Conf. Proc. **9** (1996) 75-96.

[2] N. Bourbaki, *Groupes et algèbres de Lie*, Masson Paris, 1990.

[3] A. Beauville & Y. Laszlo, *Conformal blocks and generalized theta functions.* Comm. Math. Phys. **164** (1994) 385-419.

[4] A. Beauville, Y. Laszlo & C. Sorger, *The Picard group of the moduli stack of G-bundles on a curve*, to appear in Compositio Math., preprint alg-geom/9608002.

[5] P. Deligne, *Théorie de Hodge III*, Inst. Hautes Études Sci. Publ. Math. **44** (1974) 5-77.

[6] V. Drinfeld & C. Simpson, *B-structures on G-bundles and local triviality,* Math.-Res.-Lett. **2** (1995) 823–829.

[7] G. Faltings, *A proof for the Verlinde formula,* J. Algebraic Geom. **3** (1994) 347-374.

[8] ———, *Stable G-bundles and projective connections,* J. Algebraic Geom. **2** (1993) 507-568.

[9] N. Hitchin, *Flat Connections and geometric quantization,* Comm. Math. Phys. **131** (1994) 347-380.

[10] V. Kac, *Infinite dimensional Lie algebras,* Progr. Math. **44** Birkhäuser, Boston, 1983.

[11] S. Kumar, *Demazure character formula in arbitrary Kac-Moody setting,* Invent. Math. **89** (1987) 395-423.

[12] S. Kumar, M. S. Narasimhan & A. Ramanathan, *Infinite Grassmannians and moduli spaces of G-bundles*, Math. Ann. **300** (1994) 41-75.

[13] Y. Laszlo & Ch. Sorger, *The line bundles on the moduli of parabolic G-bundles over curves and their sections*, Ann. Sci. École Norm. Sup. 4 Sr. **30** (1997) 499-525.

[14] O. Mathieu, *Formules de caractères pour les algèbres de Kac-Moody générales,* Astérisque **159-160** (1988).

[15] D. Mumford, *Geometric invariant theory*, Springer, Berlin, 1965.

[16] P. Slodowy, *On the geometry of Schubert varieties attached to Kac-Moody Lie algebras*, Canad. Math. Soc. Conf. Proc. **6** (1986) 405-442.

[17] A. Ramanathan, *Moduli for principal bundles over algebraic curves*, I and II, Proc. Indian Acad. Sci. Math. Sci. **106** (1996) 301-328 and 421-449.

[18] C. Sorger, *La formule de Verlinde*, Sém. Bourbaki **794** (1994).

[19] ———, *On moduli of G-bundles overcurves for exceptional G*, Ann. Sci. École Norm. Sup., to appear.

[20] Y. Tsuchimoto, *On the coordinate-free description of the conformal blocks*, J. Math. Kyoto Univ. **33** (1993) 29-49.

[21] A. Tsuchiya, K. Ueno & Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries,* Adv. Stud. Pure Math. **19** (1989) 459-566.

[22] G. Welters, *Polarized abelian varieties and the heat equation,* Compositio Math. **49** (1983) 173-194.

DMI - École Normale Supérieure, Paris