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A C^∞ SCHWARZ REFLECTION PRINCIPLE IN ONE AND SEVERAL COMPLEX VARIABLES

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1. Introduction

The classical Schwarz Reflection Principle of one complex variable is a theorem about boundary behavior of holomorphic mappings. It is easy to state a reasonable analogue of the reflection principle in the C^∞ category, and not too hard to back the statement with a proof. In this paper, we will prove a version of a one variable C^∞ reflection principle in such a way that it generalizes naturally and with very little alteration to a setting in several complex variables.

The one variable C^∞ reflection principle that interests us here is the following.

Theorem 1. *Suppose that γ_1 and γ_2 are C^∞ smooth curves in the complex plane, and suppose there is a point $z_0 \in \gamma_1$ and a disc D centered at z_0 such that $D-\gamma_1$ consists of exactly two simply connected components, which we denote by D_+ and D_-. Suppose that there is a holomorphic function f defined on D_+, which extends continuously to γ_1 such that the extension maps γ_1 to γ_2. Then f extends C^∞ smoothly up to γ_1 near z_0. Furthermore, if f is not a constant function, there is a positive integer n such that $f^{(n)}(z_0) \neq 0$.*

In [5], Čirka proved the regularity part of this theorem and went on to prove more general results about mappings in several variables. In [11], Rosay extended Čirka's results to apply to a class of nonholomorphic mappings; Rosay's proofs, when viewed in the context of Čirka's original result, are simpler and more natural.

The finite order vanishing statement in the theorem was first proved by Alinhac, Baouendi, and Rothschild in [2].

The main result of this paper is an extension of this one variable theorem to a theorem about boundary regularity and uniqueness of holomorphic maps which map a hypersurface into a Levi flat hypersurface. The precise

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statement of the result is made in §3. In §4, the techniques developed for the proof of the main theorem are used to give an alternative proof of the combined results of Čirka, Rosay, Alinhac, Baouendi, and Rothschild.

The paper is organized so that it is a paper on one complex variable until §3, where it becomes a paper on several complex variables. We have done this because the one variable argument is simple and serves well to motivate the several complex variables results which come later. In fact the proof of the one variable results can be understood by a first year graduate student who knows the book of Ahlfors [1].

We would like to thank M. S. Baouendi, D. Barrett, J.-P. Rosay, and X. Saint-Raymond for helpful conversations about this work. It was Baouendi who asked the questions about unique continuation of boundary values of holomorphic mappings which led us to this research. (Baouendi went on to find the answer to his own questions; the results of this paper grew out of his questions and are of independent interest.)

2. Proof of the C^∞ reflection principle

In the real analytic case, the statement about finite order vanishing in Theorem 1 is obvious because holomorphic functions are constant if and only if all their derivatives vanish at a point. In the C^∞ case, this result is not obvious. The proof we give here uses a unique continuation lemma for a $\overline{\partial}$-problem. The spirit of the proof is similar to the proof given by Alinhac, Baouendi, and Rothschild; they use a unique continuation principle for the Laplace operator.

As in the real analytic case, we may assume that γ_1 is the real axis in the complex plane, that $z_0 = 0$, and that D_+ is the upper half disc U_+. (It should be noted, however, that in the C^∞ case, we use the Riemann Mapping Theorem to map D_+ to the upper half plane in such a way that z_0 is mapped to the origin. It is well known that the Riemann Mapping Function is a local C^∞ diffeomorphism up to the boundary near C^∞ smooth boundary points.) We may also assume that $f(0) = 0$. The proof of the theorem rests on the following lemmas. Lemma 1 is a *folk theorem*, and Lemma 2 was proved in various forms in [2], [5], [6], [10]. We prove these lemmas again here for completeness and because we will need to follow our one variable proof as a model for a several variable proof later. (Also, the proofs are very short.) We will use the shorthand notation ϕ_z and $\phi_{\overline{z}}$ to denote derivatives of ϕ with respect to z and $\overline{z}$, respectively. Let U_- denote the lower half of the unit disc.

Lemma 1. *Suppose that* v *is a complex valued* C^∞ *function on* U_+, *which extends continuously to the real axis and which is real valued on the real axis. If* $v_{\bar z}$ *extends* C^∞ *smoothly to the real axis and vanishes to infinite order along the real axis, then the function, which is defined to be equal to* $v(z)$ *for* z *in* $U_+ \cup \mathbf{R}$, *and equal to* $\overline{v(\bar z)}$ *for* z *in* U_-, *is* C^∞ *smooth on the unit disc.*

Let $d_1(z) = \operatorname{Im} z$ denote the distance from a point z in U_+ to $\gamma_1 = \mathbf{R}$, and let $d_2(w)$ denote the distance from a point w to the curve γ_2. Let $U_+(r)$ denote the upper half of the disc of radius r centered at the origin.

Lemma 2. *There is a constant* $C > 0$ *and a radius* $r > 0$ *such that* $d_2(f(z)) \le C d_1(z)$ *for* $z \in U_+(r)$.

The next lemma is a unique continuation theorem and can be proved in a standard way using Carleman estimates. We shall give a very elementary proof below using the Cauchy integral formula.

Lemma 3. *Suppose* v *is a* C^∞ *function on the unit disc such that* $|v_{\bar z}| \le C|v|$ *for some positive constant* C. *If* v *vanishes to infinite order at the origin, then* v *is identically zero.*

Let us now show how the lemmas imply the theorem. Afterwards, we will prove Lemmas 1–3. We require a function Φ which is an almost analytic mapping of γ_2 into the real axis. To be precise, we need Φ to be C^∞ smooth on a neighborhood of $f(z_0) = 0$ such that Φ_z is nonvanishing on γ_2, such that $\Phi_{\bar z}$ vanishes to infinite order along γ_2, and such that Φ is real valued along γ_2. To obtain such a function, we shall use the Riemann Mapping Theorem to map a small one-sided neighborhood of γ_2 near $f(z_0)$ onto the upper half plane in such a way that 0 gets mapped to 0; call this Riemann map F. Now F is C^∞ smooth up to γ_2 near 0 and therefore can be extended to be C^∞ smooth in a neighborhood of 0. It is a classical theorem in conformal mapping that F' cannot vanish along γ_2. Thus, we may take Φ to be equal to the extension of F on a small neighborhood of 0.

Consider the function $v = \Phi \circ f$. It is defined and C^∞ on $U_+(r)$ for some $r > 0$. For convenience, we may assume that $r = 1$. Note that $\Phi \circ f$ extends continuously to the real axis and is real there. We wish to prove that $v_{\bar z} = (\Phi_{\bar z} \circ f)\overline{f'}$ extends C^∞ smoothly to the real axis so that Lemma 1 can be applied. Because f is a bounded holomorphic function on U_+, the classical Cauchy estimates yield that $|f^{(n)}(z)| \le c_n (\operatorname{Im} z)^{-n}$ near the origin. Let D^m denote the differential operator $\partial^m / \partial z^i \partial \bar z^j$ of order $m = i + j$. Since $\Phi_{\bar z}$ vanishes to infinite order on γ_2, we have an estimate of the form $|D^m \Phi_{\bar z}(w)| \le K d_2(w)^N$ near $f(0)$ for each positive integer N, where $K > 0$ is a constant which depends on N and m. A typical term in a derivative of $v_{\bar{z}}$ can be written as $(D^m\Phi_{\bar{z}}) \circ f$ times a product of derivatives of f and $\overline{f}$. Hence, if we choose N in the estimate for $D^m\Phi_{\bar{z}}$ to be sufficiently large, we may use Lemma 2 and the Cauchy estimates for f to deduce that all the derivatives of $(\Phi \circ f)_{\bar{z}}$ tend to zero as $\operatorname{Im} z$ tends to zero. Thus, $(\Phi \circ f)_{\bar{z}}$ extends smoothly to the real axis and vanishes to infinite order there. Lemma 1 yields that $\Phi \circ f$ extends C^∞ smoothly up to the real axis near the origin. Now it follows that f extends C^∞ smoothly up to the real axis near the origin because Φ is a local C^∞ diffeomorphism near the origin.

Finally, we must prove the finite vanishing condition using Lemma 3. Assume that f vanishes to infinite order at the origin. We know that $v = \Phi \circ f$ extends to be a C^∞ function on the unit disc via reflection. We need to see that the extension satisfies the hypotheses of Lemma 3. The finite order vanishing of $\Phi_{\bar{z}}$ along γ_2 allows us to conclude that, for any positive integer N, there is a positive constant C_N such that for z in U_+ near 0,

$$|v_{\bar{z}}(z)| = \Phi_{\bar{z}}(f(z))| \, |f'(z)| \leq C_N d_2(f(z))^N.$$

Now, since $\Phi_z(0) \neq 0$, it follows that $|\Phi(w)| \geq |\operatorname{Im}\Phi(w)| \geq c d_2(w)$ for w in a neighborhood of 0. Thus, $|v(z)| \geq c d_2(f(z)) \geq \text{const}\,|v_{\bar{z}}(z)|$ if z is restricted to be in a small enough neighborhood of the origin. The same inequality holds in the lower half disc by reflection (since $\frac{\partial}{\partial \bar{z}}\overline{v(\bar{z})} = \overline{v_{\bar{z}}(\bar{z})}$). Hence, Lemma 3 implies that v is identically zero near the origin; therefore f maps an open set into γ_2. It follows that f is constant. The proof of Theorem 1 will be complete after we have proved the lemmas.

Proof of Lemma 1. Define a function $\lambda(z)$ on the unit disc to be equal to $v_{\bar{z}}(z)$ for $z \in U_+$, equal to zero for real z, and equal to $\overline{v_{\bar{z}}(\bar{z})}$ for $z \in U_-$. Since $v_{\bar{z}}$ vanishes to infinite order along the real axis, the function λ is seen to be C^∞ on the unit disc U. Now let u be the solution to the $\bar{\partial}$-problem $u_{\bar{z}} = \lambda$ given by

$$u(z) = \frac{1}{2\pi i}\int_U \frac{\lambda(\zeta)}{\zeta - z}\, d\zeta \wedge d\bar{\zeta}$$

(see Hörmander [7, Theorem 1.2.2]). The function u is C^∞ smooth on the unit disc and satisfies the reflection property $u(\bar{z}) = \overline{u(z)}$. Since $u - v$ is holomorphic on U_+ and real on the real axis, the classical reflection principle implies that $u - v$ extends holomorphically to the whole unit disc via reflection. Hence, v extends to be C^∞ smooth on the whole unit disc via reflection.

Proof of Lemma 2. Choose a normal direction ν to the curve γ_2 near $f(0) = 0$. We wish to construct two C^∞ subharmonic functions ρ_+ and ρ_- in a neighborhood of 0, which both vanish on γ_2 such that $\partial\rho_+/\partial\nu > 0$ and $\partial\rho_-/\partial\nu < 0$. To construct ρ_+, let Ω be a small domain with C^∞ smooth boundary whose boundary coincides with γ_2 near the origin such that ν is an outward pointing normal to Ω at 0. Let ϕ be a solution to the Dirichlet problem: $\Delta\phi = 1$ on Ω with $\phi = 0$ on the boundary of Ω. Since ϕ is C^∞ smooth up to the boundary, we may extend ϕ as a C^∞ subharmonic function to a neighborhood of the origin. The classical Hopf Lemma implies that $\frac{\partial\phi}{\partial\nu}(z) > 0$ for $z \in \gamma_2$ near 0. Thus, we may define ρ_+ to be equal to the extension of ϕ restricted to a small neighborhood of 0. To construct ρ_-, we repeat the argument above using a domain Ω whose boundary agrees with γ_2 near 0 such that ν is an *inward* pointing normal to the boundary of Ω near 0. By shrinking the neighborhood of 0 under consideration, we may assume that the functions $\rho_\pm$ are nonzero off of γ_2 in their domain of definition. Now define $\rho = \sup\{\rho_+, \rho_-\}$. This function is subharmonic in a neighborhood of the origin, is zero on γ_2 and positive off of γ_2, and there are positive constants c_1 and c_2 such that $c_1 d_2(w) \le \rho(w) \le c_2 d_2(w)$ for w near 0.

We shall restrict our attention to a small enough half disc $U_+(r)$ so that the composition $\rho \circ f$ is defined. For convenience, we may assume that $r = 1$. Note that $\rho \circ f$ is a nonnegative subharmonic function on U_+, which is continuous up to the real axis and which vanishes there. By composing with a conformal map of the unit disc onto U_+ which maps the boundary point 1 to the origin, we may reduce our task to proving the following proposition (which seems to be well known).

Proposition. *Suppose that λ is a nonnegative subharmonic function on the unit disc which is continuous up to the boundary such that $\lambda(e^{i\theta}) = 0$ for θ in the range $-\delta < \theta < \delta$ for some $\delta > 0$. Then there is a positive constant C such that*

$$\lambda(\zeta) \le C(1 - |\zeta|)$$

for all ζ in the sector $\{|\arg\zeta| < \delta/2\}$.

Proof of the Proposition. Let $P(\zeta, \theta)$ denote the Poisson kernel for the unit disc. Now, we may write

$$\lambda(\zeta) \le \int_{-\pi}^{\pi} P(\zeta, \theta)\lambda(\theta)\, d\theta.$$

But if $|\arg\zeta| < \delta/2$ and $|\theta| > \delta$, we may estimate the Poisson kernel

$$P(\zeta, \theta) \le \frac{2}{\pi} \frac{1 - |\zeta|}{|e^{i\delta/2} - e^{i\delta}|^2}.$$

If we use this inequality in the preceding inequality, we obtain the desired estimate. The proof of Lemma 2 is now complete.

Proof of Lemma 3. Define a function λ on the unit disc to be equal to $v_{\bar{z}}/v$ where v is nonzero, and equal to zero where v is zero. Note that λ is a bounded measurable function. Define a function u on the unit disc U via

$$u(z) = \frac{1}{2\pi i} \int_U \frac{\lambda(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

The function u is continuous on U and has the property that it is C^∞ where v is nonzero and $u_{\bar{z}} = v_{\bar{z}}/v$ there (see Hörmander [7, Theorem 1.2.2]). Now consider the function $h = ve^{-u}$. This function is continuous on U and is holomorphic where it is not zero. Thus, Radó's Theorem implies that h is holomorphic on U. Now it is clear that if v vanishes to infinite order at the origin, then h must be identically zero, and hence, so must v. Lemma 3 is proved.

3. A C^∞ reflection principle in several variables

The most naive generalization of the classical one variable reflection principle to several variables applies to a holomorphic mapping f from the upper half disc in the plane to $\mathbf{C}^n$, which extends continuously up to the real axis and which maps the real axis into the hyperplane $\{\operatorname{Im} z_n = 0\}$. Under these circumstances, one sees that the component f_n extends holomorphically past the real axis. In this section, we will generalize this result to a C^∞ setting in which the hyperplane $\{\operatorname{Im} z_n = 0\}$ is replaced by a Levi flat hypersurface.

The standard example of a holomorphic mapping in several variables, which exhibits bad boundary behavior is given by $(z, w) \mapsto (z, w+h(z))$, where h is a holomorphic function defined on the upper half plane which is continuous, but not C^1, up to the real axis. This mapping is a biholomorphic map of the domain $\{\operatorname{Im} z > 0\}$ in $\mathbf{C}^2$ onto itself which extends continuously to the boundary, but which does not extend smoothly to the boundary. However, the first component of such a mapping must extend smoothly because of the reflection principle mentioned above. This will be seen to be a general phenomenon for the complex normal component of a holomorphic mapping which maps a hypersurface into a Levi flat hypersurface. Before we can state the theorem precisely, we must set up some notation and terminology.

A C^∞ smooth hypersurface M in $\mathbf{C}^n$ is called *Levi flat* if its Levi form vanishes identically. If M is real analytic and Levi flat, then it is possible to make a local biholomorphic change of variables which transforms M into the hypersurface $\{\operatorname{Im} z_n = 0\}$. If M is C^∞, but not real analytic, then the procedure which produces the biholomorphic change of variables in the real analytic case gives only a C^∞ local diffeomorphism Φ which maps M into $\{\operatorname{Im} z_n = 0\}$ and which is almost holomorphic on M in the sense that $\overline{\partial}\Phi_j$ vanishes to infinite order on M for each j. We shall call the component function Φ_n a *flattened complex normal coordinate for* M and we will write $\mathscr{N} = \Phi_n$. We shall call a function *flat* at a point z_0 if there are constants C_N such that $|f(z) - f(z_0)| \le C_N|z - Z_0|^N$ for each positive integer N.

Theorem 2. *Suppose that M_1 and M_2 are C^∞ smooth hypersurfaces in $\mathbf{C}^m$ and $\mathbf{C}^n$, respectively, and suppose there is a point $z_0 \in M_1$ and a ball B centered at z_0 such that $B = M_1$ consists of exactly two connected components, which we denote by B_+ and B_-. Suppose that there is a holomorphic mapping f defined on B_+, which extends continuously to M_1 such that the extension maps M_1 into M_2. If M_2 is Levi flat, then $\mathscr{N} \circ f$ extends C^∞ smoothly up to M_1 near z_0 for any flattened complex normal coordinate $\mathscr{N}$ for M_2. Furthermore, if f does not map B_+ into M_2, then $\mathscr{N} \circ f$ cannot be flat at z_0, and hence, f cannot be flat at z_0.*

Proof. We claim that it is enough to prove the theorem in the simplified case that $m = 1$, M_1 is equal to the real axis, $z_0 = 0$, and B_+ is the upper half disc U_+ in the plane. Indeed, all constants in the estimates that we will prove in this simplified setting can be seen to vary continuously under continuous perturbations of the map f. Hence, in the general setting of Theorem 2, by restricting attention to one-dimensional complex slices which cut the hypersurface M_1 transversally, and by mapping these slices to the upper half disc via the Riemann Mapping Theorem, we may deduce that the function $\mathscr{N} \circ f$ is smooth up to the boundary along any such slice with estimates on derivatives which are uniform under small motions of the slice. Since any partial derivative at a point on M_1 can be written as a finite sum of derivatives along linearly independent slice directions, it is clear that this implies that $\mathscr{N} \circ f$ is smooth up to M_1. The finite vanishing condition also follows by restricting to slices.

Hence, from this point forward, we will be studying a holomorphic map f on the upper half disc U_+ into $\mathbf{C}^n$, which extends continuously to the real axis and which maps the real axis into a Levi flat hypersurface M. The proof of the theorem rests on the following lemmas which together yield the analogue of Lemma 2 in the many variable setting.

Lemma 4. *A C^∞ smooth Levi flat hypersurface M in $\mathbf{C}^n$ is locally the zero set of a C^∞ plurisubharmonic function ρ with the following properties.*

(a) *$d\rho \neq 0$ on M.*

(b) *ρ is strongly plurisubharmonic off M.*

(c) *The sign of the normal derivative of ρ at a point in M can be specified arbitrarily.*

We thank David Barrett for showing us how to prove Lemma 4.

Let $d_1(z) = \operatorname{Im} z$ denote the distance from a point z in U_+ to R, and let $d_2(w)$ denote the distance from a point w in $\mathbf{C}^n$ to the hypersurface M. As before, let $U_+(r)$ denote the upper half of the disc of radius r centered at the origin.

Lemma 5. *There is a constant $C > 0$ and a radius $r > 0$ such that $d_2(f(z)) \leq C\, d_1(z)$ for $z \in U_+(r)$.*

We will now show how Theorem 3 follows from Lemma 5 (using the simplified assumptions made above). Let Φ be an almost analytic mapping which flattens M as in the definition of a flattened complex normal coordinate, and let $v = \Phi_n \circ f$. The Cauchy estimates for the derivatives of the components of the mapping f, the infinite order vanishing of the anti-holomorphic derivatives of Φ, and Lemma 5, together imply that v extends C^∞ smoothly to the real axis and vanishes to infinite order along the real axis. Now Lemma 1 implies that $\Phi_n \circ f$ extends C^∞ smoothly to the real axis. The regularity assertion of Theorem 2 is now proved.

To finish the proof, suppose that $v = \Phi_n \circ f$ is flat at 0. We may consider v to be a C^∞ function defined on the unit disc by the reflection principle of Lemma 1. Now on U_+, we have

$$v_{\bar{z}} = \sum_{j=1}^{n} \left(\frac{\partial \Phi_n}{\partial \overline{w}_j} \circ f \right) \overline{f_j'},$$

and we can combine the Cauchy estimates for the derivatives of each f_j with Lemma 5 and the fact that $\partial\Phi_n/\partial\overline{w}_j$ vanishes to infinite order along M to deduce that, for each positive integer N, there is an $r > 0$ such that $v_{\bar{z}}$ satisfies an estimate of the form

$$|v_{\bar{z}}(z)| \leq C_N\, d_2(f(z))^N$$

for $z \in U_+(r)$. But $|v(z)| = |\Phi_n(f(z))| \geq c\, d_2(f(z))$ for z near 0. Thus $|v_{\bar{z}}| \leq \text{const}\, |v|$ in a sufficiently small upper half disc $U_+(r)$. The same inequality holds on the lower half of that disc by reflection. Therefore, Lemma 3 yields that v is zero in a neighborhood of 0. This implies that f maps an open set into M, and therefore that $\mathcal{N} \circ f$ is identically zero. This completes the proof of Theorem 2. Lemma 5 follows from Lemma 4 exactly as in the proof of Lemma 2. It remains only to prove Lemma 4.

Proof of Lemma 4. We shall use the same notation that we used in the proof of the theorem. The function $\psi = \pm \operatorname{Im} \Phi_n$ is an *almost pluriharmonic* defining function for M in the sense that $\partial^2\psi/\partial w_i \partial \overline{w}_j$ vanishes to infinite order along M for each i and j. Suppose that the origin is contained in M. Let $\rho(w) = \psi(w) + (1 + |w|^2)\psi(w)^2$. This function clearly satisfies conditions (a) and (c) of Lemma 5. We must show that ρ is plurisubharmonic near the origin. We shall write $O(\psi^\infty)$ to indicate that a term in an expression can be dominated by a constant times an arbitrary power of $|\psi|$. For example, since $|\psi(w)|$ is comparable to the distance from w to M and ψ is almost pluriharmonic, it follows that $\psi_{w_i\overline{w}_j}$ is $O(\psi^\infty)$. Now, using the shorthand notation that a subscript i indicates a derivative with respect to w_i and a subscript j indicates a derivative with respect to $\overline{w}_j$, we may compute

$$\rho_{ij} = 2(1 + |w|^2)\psi_i\psi_j + \delta_{ij}\psi^2 + 2\psi\overline{w}_i\psi_j + 2\psi w_j\psi_i + O(\psi^\infty),$$

where δ_{ij} denotes the Kronecker delta function. For $a = (a_1, \cdots, a_n)$, we have

$$\sum_{i,j=1}^n \rho_{ij}a_i\overline{a}_j = 2(1 + |w|^2)\left|\sum a_i\psi_i\right|^2 + |a|^2\psi^2 + 4\psi \operatorname{Re}\left(\sum a_i\psi_i\right)\left(\sum \overline{a}_j w_j\right) + |a|^2 O(\psi^\infty).$$

Now we may use the estimate $2|\operatorname{Re} xy| \le |x|^2 + |y|^2$ to obtain

$$4\psi \operatorname{Re}\left(\sum a_i\psi_i\right)\left(\sum \overline{a}_j w_j\right) \ge -2\left(|\psi \sum \overline{a}_j w_j|^2 + \left|\sum a_i\psi_i\right|^2\right),$$

and hence,

$$\sum_{i,j=1}^n \rho_{ij}a_i\overline{a}_j \ge 2|w|^2\left|\sum a_i\psi_i\right|^2 + \left(|a|^2 - 2\left|\sum \overline{a}_j w_j\right|^2\right)\psi^2 + |a|^2 O(\psi^\infty).$$

It is clear that if w is close enough to the origin, then this expression can be made nonnegative independent of $a \in \mathbf{C}^n$; in fact, the expression can be made strictly positive when $a \neq 0$ and $\psi(w) \neq 0$. Thus, condition (b) of Lemma 5 is verified and the proof is finished.

4. Application to the totally real case

Many authors have studied the boundary behavior of a holomorphic mapping which maps the edge of a wedge into a maximal totally real submanifold of $\mathbf{C}^n$. This is a natural and important object of study because of Webster's version of the reflection principle which uses the edge of the wedge theorem; a strictly pseudoconvex hypersurface is mapped naturally into a totally real manifold in a higher dimensional space. For example, this approach is used by Khasanov and Pinčuk [10] to prove optimal regularity results for biholomorphic mappings.

In this section, we want to indicate how the arguments in the previous section can be modified to study the problem of the boundary behavior of a holomorphic map which maps the edge of a wedge into a totally real submanifold. This problem can be reduced to the study of a holomorphic map defined on a one-dimensional half-disc; in this situation we prove the following theorem.

Theorem 3. *Suppose that Γ is a C^∞ smooth maximal totally real submanifold of $\mathbf{C}^n$. If f is a holomorphic mapping of the upper half disc U_+ in the plane, which extends continuously to the real axis and which maps the real axis into Γ, then f extends C^∞ smoothly up to the real axis. Also, f cannot be flat at a point on the real axis unless f is constant.*

The regularity assertion in the theorem was first proved by Čirka [5] and subsequently generalized by Rosay in [11] and by B. Coupet in his dissertation [6]. (Versions of the regularity theorem with somewhat stronger hypotheses where proved in [4], [8].) The unique continuation clause in the theorem was proved first by Alinhac, Baouendi, and Rothschild and is one of the subjects of their paper [2]. In this section, we will give an alternative proof of the theorem.

In some examples, Theorem 3 can be seen to be a direct consequence of Theorem 2. This is the case whenever Γ can be realized as the transverse intersection of n Levi flat C^∞ hypersurfaces. In §6, we will write down an example of a C^∞ maximal totally real submanifold of $\mathbf{C}^2$ which is not contained in any Levi flat hypersurface. Hence, the proof of Theorem 3 will require some machinery beyond that used in the proof of Theorem 2.

Proof of Theorem 3. Because Γ is maximal totally real, there is a mapping $\Phi\colon \mathbf{C}^n \to \mathbf{C}^n$, which is a local diffeomorphism and almost holomorphic on Γ, and maps Γ into $\{w \in \mathbf{C}^n\colon \operatorname{Im} w_j = 0; j = 1, 2, \cdots, n\}$. It is well known that Γ is locally the zero set of a continuous nonnegative plurisubharmonic function ρ such that $\rho(z)$ is equivalent to the distance $d_\Gamma(z)$ from z to Γ. Using this function, we may argue as in the proof of Lemmas 2 and 5 to see that f satisfies the *distance estimate* $d_\Gamma(f(z)) \le C\,d(z)$. Let $v_k = \Phi_k \circ f$. The boundary distance estimate together with the infinite order vanishing of the $\bar{z}$-derivatives of the components of Φ along Γ allow us to deduce that $(v_k)_{\bar{z}}$ is C^∞ smooth up to the real axis and vanishes to infinite order there. Hence, Lemma 1 implies that v_k extends C^∞ smoothly to the unit disc via reflection. It now follows that each component of f extends C^∞ smoothly to the real axis because Φ is a local diffeomorphism. To finish the proof, we must prove the unique continuation statement. We will require the following generalization of Lemma 3.

Lemma 6. *Suppose $v = (v_1, \cdots, v_n)$ is a vector of C^∞ functions on the unit disc such that $|v_{\bar{z}}| \le C|v|$ for some positive constant C. If v vanishes to infinite order at the origin, then v is identically zero in a neighborhood of the origin.*

Here, the notation $v_{\bar{z}}$ is shorthand for the vector whose k th component is $(v_k)_{\bar{z}}$.

We shall now show how the lemma implies the theorem. Suppose f is flat at $z_0 = 0$ and that $f(0)$ is equal to the origin. Then each v_k vanishes to infinite order at the origin. Now, on U_+, we may calculate

$$(v_k)_{\bar{z}} = \sum_{j=1}^{n} \left(\frac{\partial \Phi_k}{\partial \overline{w}_j} \circ f \right) \overline{f_j'},$$

and hence, it follows (as in the proof of Theorem 1) via the infinite order vanishing of the $\overline{w}$-derivatives of Φ_k along Γ, the distance estimate, and the Cauchy estimates for f_j' that $|(v_k)_{\bar{z}}| \le C\, d_\Gamma(f(z))^N$ for any positive integer N. But $|\Phi(w)| \ge |\operatorname{Im} \Phi(w)| \ge c\, d_\Gamma(w)$ near $w = 0$. Thus, $|v(z)| \ge c\, d_\Gamma(f(z))$ near the origin. Hence, by restricting attention to a small enough neighborhood of the origin in U_+, we deduce that $|v_{\bar{z}}| \le C|v|$ there. The same inequality holds on the lower half disc for the reflection of v. Therefore, Lemma 6 yields that v is zero in a neighborhood of the origin and this implies that f is constant on an open subset of U_+. Thus the proof of the theorem is finished.

Proof of Lemma 6. The following proof is analogous to the proof of Lemma 3 with matrices in place of functions. Let $\varepsilon > 0$ be arbitrary for the moment. By replacing $v(z)$ by $v(tz)$ for a small constant t, we may assume that the constant C in the estimate $|v_{\bar{z}}| \le C|v|$ is less than ε. This implies that there is a matrix A of functions on U which are bounded in modulus by ε such that $v_{\bar{z}} = Av$. To be precise, let us take $A = (a_{ij})$ to be given via

$$a_{ij} = (v_i)_{\bar{z}} \overline{v_j} |v|^{-2}$$

if $v \ne 0$ and $a_{ij} = 0$ otherwise. We now seek to construct a matrix function $B = B(z)$ with bounded, continuous entries such that B is a weak solution to the equation $B_{\bar{z}} = -BA$, and $B(z)$ is invertible at each $z \in U$. For convenience, let us write $B = I + \beta$. We want β to satisfy $-A - \beta A = \beta_{\bar{z}}$ in the sense of distributions. Let T denote the operator which maps a matrix $E = (e_{ij})$ of bounded measurable functions on the unit disc U to a matrix $T(E) = (t_{ij})$ of functions given via

$$t_{ij}(z) = \frac{1}{2\pi i} \int_U \frac{e_{ij}(\zeta)}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}.$$

Note that the operator T maps a matrix of bounded functions to a matrix of bounded *continuous* functions with an L^∞ estimate (see Hörmander [7, Theorem 1.2.2]). Now consider the operator Λ defined by $\Lambda(\beta) = T(-A - \beta A)$. We will be nearly finished if we show that Λ has a fixed point. If ε is sufficiently small, Λ will be a contraction mapping on $L^\infty(U)$, the space of measurable matrix functions on U with bounded entries endowed with the obvious norm. Hence, the Banach Contraction Theorem yields that the sequence of iterates, $\Lambda^k(I)$, converges in L^∞ to the unique fixed point β_0 of Λ viewed as an operator on this space. The entries of β_0 are bounded and continuous on U, and β_0 is a weak solution to the equation $-A - \beta A = \beta_{\bar{z}}$. By choosing ε to be sufficiently small, we may also guarantee that the matrix norm $\|\beta_0\|$ is less than $1/2$ so that $B = I + \beta_0$ is invertible.

To finish the proof of the lemma, consider the vector function Bv. This vector of bounded, continuous functions is a weak solution to $(Bv)_{\bar{z}} = 0$. Hence, the components of Bv are holomorphic functions. Since v is assumed to be flat at the origin, these holomorphic functions must all vanish in a neighborhood of the origin. Because B is invertible, this implies that v is identically zero near the origin. Hence the proof of Lemma 6 is finished.

5. Remarks

It is clear that the proofs of Theorems 1 and 2 can be modified to yield C^k extension results. The papers [2], [5], [6], [10] deal with optimal C^k regularity and extension of mappings.

Theorem 2 is a result in several complex variables, whose proof is essentially a one variable argument. There is another C^∞ version of what most people refer to as *the* reflection principle in several complex variables by Nirenberg, Webster, and Yang [9] with a genuinely multivariable proof.

Their reflection principle applies to mappings between strictly pseudoconvex domains.

6. An example

We conclude this paper by giving an example of a maximal totally real submanifold of $\mathbf{C}^2$ which cannot be realized as a submanifold of a Levi flat hypersurface, as promised in §4.

If a two-dimensional totally real C^∞-manifold $M \subset \mathbf{C}^2$ is contained in a C^2 Levi flat hypersurface L, then L is foliated by Riemann surfaces, and the leaves are nowhere tangent to M. It follows that M and the leaves intersect each other transversely in L, and therefore the intersection of M with a leaf is a (real) curve. Hence, in order to exhibit a totally real M not contained in any Levi flat hypersurface it suffices to construct M such that it intersects no complex curves in a real curve. Then, in fact, no open portion of M will be contained in a Levi flat hypersurface.

Theorem 4. *There is a totally real manifold $M \subset \mathbb{C}^2$ of class C^∞, $\dim M = 2$, such that for any open set $U \subset \mathbf{C}^2$ and any (possibly singular) complex curve $C \subset U$ the set $C \cap M$ is countable.*

Jacobowitz, answering a problem raised by a paper of Baouendi and Treves (see [3]), constructed a totally real M with a somewhat weaker property (unpublished). Jacobowitz's example M is such that for any small ball U centered about a fixed point $p \in M$ and any complex curve $C \subset U$, $p \notin C$, the connected component of $M \setminus C$ containing p is not relatively compact in U. This example was communicated to us by Baouendi.

In order to prove Theorem 4, we shall consider classes of nonanalytic smooth functions. Let Ω be a domain in real euclidean space $\mathbf{R}^d$ (in our case d will be 1 or 2), and let $\{M_{nk}: n, k = 1, 2, \dots\}$ be a double sequence of positive numbers. By $\{M_{nk}\}(\Omega)$ we shall mean the set of those functions $f \in C^\infty(\Omega)$ for which there is a k such that for every $n = 1, 2, \cdots$, we have that

$$\sup_{x\in\Omega}\left|\frac{\partial^\alpha f(x)}{\partial x^\alpha}\right| \le \alpha! M_{nk}$$

for each multi-index α, $|\alpha| = n$. For example, if $M_{nk} = k^n$ and Ω is bounded, then $\{M_{nk}\}(\Omega)$ consists of functions analytic in a neighborhood of $\overline{\Omega}$. We shall always assume that the double sequence M_{nk} has the property that for every k there is an h such that $2M_{nk} \le M_{nh}$. In this case $\{M_{nk}\}(\Omega)$ is a vector space.

Lemma 7. *Let $\{M_{nk}\}$ be a double sequence as above. Then there is another double sequence $\{\widetilde{M}_{nk}\}$ with the following properties:*

(a) *If $f \in \{M_{nk}\}(\Omega)$, then all partials of f are in $\{\widetilde{M}_{nk}\}(\Omega)$.*

(b) *If $f \in \{M_{nk}\}(\Omega)$, and g is a mapping $\Omega' \to \Omega$ whose components are in $\{M_{nk}\}(\Omega')$, then $f \circ g \in \{\widetilde{M}_{nk}\}(\Omega')$.*

(c) *If $\Omega \subset \mathbf{R}^2$ is convex, $f \in \{M_{nk}\}(\Omega)$, $\partial f/\partial x_1 \neq 0$ in Ω, and g is a real function defined on a neighborhood of $0 \in \mathbf{R}$ such that*

$$f(g(t), t) \equiv 0,$$

then $g \in \{\widetilde{M}_{nk}\}(I)$ where I is a neighborhood of $0 \in \mathbf{R}$.

The proof of this lemma is obvious. For example, in (c) one has

$$g'(t) = -\frac{\partial f(g(t), t)/\partial x_2,}{\partial f(g(t), t)/\partial x_1},$$

whence by subsequent derivations one can bound the derivatives of g in terms of those of f.

Next we shall construct smooth functions that are very far from being in some given space $\{M_{nk}\}(\Omega)$.

Lemma 8. *Let c_m be a sequence of positive numbers. Then there is a real valued $\varphi \in C^\infty(\mathbf{R})$ such that for every $x \in \mathbf{R}$ and positive integer m either $|\varphi^{(2m)}(x)| > c_{2m}$ or $|\varphi^{(2m+1)}(x)| > c_{2m+1}$.*

To prove this we shall need the following.

Proposition. *Let n be any positive integer and ε, K positive numbers. There is a real valued 2π-periodic function $\phi \in C^\infty(\mathbf{R})$ such that for any $x \in \mathbf{R}$, the numbers $|\psi(x)|$, $|\psi'(x)|, \cdots, |\psi^{(n-1)}(x)|$ are all bounded by ε, and either $|\psi^{(n)}(x)| > K$ or $|\psi^{(n+1)}(x)| > K$.*

Proof. $\psi(x) = a \sin bx$ with suitably small a and suitably large b will do.

Proof of Lemma 8. The function φ will be the sum of 2π-periodic functions φ_m to be chosen inductively. Let $\varphi_0 \equiv 0$. If 2π-periodic functions $\varphi_0, \cdots, \varphi_{m-1} \in C^\infty(\mathbf{R})$ have already been chosen, put $\varepsilon = 2^{-m}$,

$$K = 1 + c_{2m} + c_{2m+1} + \max_{x\in\mathbf{R}} \sum_{k=0}^{m-1} \left(\left|\varphi_k^{(2m)}(x)\right| + \left|\varphi_k^{(2m+1)}(x)\right| \right),$$

and $n = 2m$. Choose ψ as in the Proposition above, and put $\varphi_m = \psi$.

Having defined the sequence φ_m, let $\varphi = \sum_{m=0}^\infty \varphi_m$. Since

$$|\varphi_m^{(j)}(x)| < 2^{-m} \quad \text{when } 2m > j,$$

$\varphi \in C^\infty(\mathbf{R})$. Furthermore

$$|\varphi^{(2m)}(x)| \geq |\varphi_m^{(2m)}(x)| - \sum_{k=0}^{m-1} |\varphi_k^{(2m)}(x)| - \sum_{k=m+1}^{\infty} 2^{-k},$$

and similarly for $\varphi^{(2m+1)}(x)$. Hence the required property of φ follows.

Proof of Theorem 4. Let us choose a sequence c_n going very fast to infinity, say $c_n = (n!)^2$. Consider the function φ that satisfies Lemma 8. The function constructed there is actually periodic. By adding a linear function we can achieve $\varphi'(x) > 0$ $(x \in \mathbf{R})$. This φ is of course no longer periodic. However, φ', φ'', ... are all bounded functions. Therefore there is a double sequence M_{nk} such that $\varphi \in \{M_{nk}\}(\mathbf{R})$. We can assume $M_{nk} \geq k^n$ so that functions analytic in a neighborhood of a closed interval $\bar{I}$ are in $\{M_{nk}\}(I)$.

The sequence $\{M_{nk}\}$ determines another sequence $\widetilde{M}_{nk} \geq M_{nk}$ according to Lemma 7. We shall iterate this "wave operation" four times to get a sequence $N_{nk} \geq M_{nk}$. Define next a sequence d_n by

$$d_{2m} = N_{2m,n}, \qquad d_{2m+1} = N_{2m+1,m},$$

and using Lemma 8 again, construct a real function $\psi \in C^\infty(\mathbf{R})$ such that for any $x \in \mathbf{R}$ and positive integer m either $|\psi^{(2m)}(x)| > d_{2m}$ or $|\psi^{(2m+1)}(x)| > d_{2m+1}$. We can again assume that $\psi'(x) > 0$.

Define now a surface $M \subset \mathbf{C}^2$ by $M = \{(z, w) \in \mathbf{C}^2 : w = \varphi(\operatorname{Re} z) - i\psi(\operatorname{Im} z)\}$. Since

$$2\frac{\partial}{\partial \bar{z}}\{\varphi(\operatorname{Re} z) - i\psi(\operatorname{Im} z)\} = \varphi'(\operatorname{Re} z) + \psi'(\operatorname{Im} z) > 0,$$

M is indeed totally real. Consider next a complex curve C in some open set $U \subset \mathbf{C}^2$. We shall prove that $C \cap M$ is countable. C has a countable set of singular points; by discarding them from U we may as well assume that C is smooth. Also, it will suffice to prove that any point $p \in C \cap M$ has a neighborhood U' such that $C \cap M \cap U'$ is countable.

Thus pick a point $p \in C \cap M$, $p = (z_0, w_0)$. If T_pC is parallel to the w axis, then C and M intersect transversely in p, hence p is an isolated point of $C \cap M$. Thus we can assume that T_pC is not parallel to the w axis, in which case, near p, C is the graph of a holomorphic function $w = h(z)$. Therefore we have to prove that the equation

$$h(z) = \varphi(\operatorname{Re} z) - i\psi(\operatorname{Im} z) \tag{1}$$

has countably many solutions near z_0 for any holomorphic function h defined on some neighborhood of z_0.

Any solution of (1) satisfies

$$\phi(z) \equiv \operatorname{Re} h(z) - \varphi(\operatorname{Re} z) = 0.$$

The set S characterized by this latter equation is covered by a countable set of curves of the form

$$\operatorname{Re} z = g(\operatorname{Im} z), \tag{2}$$

with $g \in \{\widetilde{\widetilde{M}}_{nk}\}(I)$, I some interval in $\mathbf{R}$. To see this, observe that since $\operatorname{Re} h$ is real analytic while φ has very large derivatives, in every z there is an r such that

$$\partial^r \phi(z)/\partial x^r \neq 0 \qquad (z = x + iy).$$

Hence $S \subset \bigcup_{r=1}^{\infty} S_r$, where

$$S_r = \{z \colon \partial^{r-1}\phi(z)/\partial x^{r-1} = 0,\, \partial^r \phi(z)/\partial x^r \neq 0\}.$$

We can now apply Lemma 8, parts (a) and (c) to conclude first that $\partial^{r-1}\phi/\partial x^{r-1} \in \{\widetilde{M}_{nk}\}(V)$, V a neighborhood of z_0, and then that S_r, locally, is indeed a curve of form (2), $g \in \{\widetilde{\widetilde{M}}_{nk}\}(I)$.

Thus now it would suffice to prove that on any curve of this form, the equation

$$\operatorname{Im} h(z) + \psi(\operatorname{Im} z) = 0$$

has countably many solutions near z_0. In other words

$$\Psi(y) \equiv \operatorname{Im} h(g(y) + iy) + \psi(y) = 0 \tag{3}$$

has countably many solutions y near $y_0 = \operatorname{Im} z_0$. Now this can be shown exactly as the corresponding statement about the equation $\phi(z) = 0$ was shown: One first notices that for any y there is an r such that $\Psi^{(r)}(y) \neq 0$, since the first term in Ψ is in $\{N_{nk}\}(I)$ while the second has much larger derivatives. Secondly, for every r the sets $\{y \colon \Psi^{(r-1)}(y) = 0,\, \Psi^{(r)}(y) \neq 0\}$ are discrete. Since any solution of (3) belongs to one of these sets, it follows that (3) has countably many solutions. This completes the proof of Theorem 4.

Finally we remark that a slight variation of the proof would have shown that the intersection of M with any two-dimensional real analytic set is also countable.

Added in proof. We have learned that Lemma 3 is a special case of a result proved by Lipman Bers many years ago (see pp. 259–263 in the book, *Partial Differential Equations*, by L. Bers, F. John, and M. Schechter, Lectures in Applied Math., Vol. 3A, Amer. Math. Soc., Providence, RI, 1964).

We wish to thank Herb Alexander for pointing out to us that our proof of Theorem 1 actually gives a stronger result with no additional work. The hypothesis that f extend continuously up to the curve γ_1 can be replaced by the weaker condition that *there is a bound* M *such that for every sequence of points* z_n *in* D_+ *such that* $\operatorname{dist}(z_n, \gamma_1) \to 0$ *as* $n \to \infty$, *the sequence* $f(z_n)$ *is bounded in modulus by* M *and* $\operatorname{dist}(f(z_n), \gamma_2) \to 0$ *as* $n \to \infty$.

References

[1] L. Ahlfors, *Complex analysis*, McGraw-Hill, New York, 1979.

[2] S. Alinhac, M. S. Baouendi & L. P. Rothschild, *Unique continuation and regularity for holomorphic functions at the boundary*, preprint.

[3] M. S. Baouendi & F. Treves, *Unique continuation in CR manifolds and in hypoanalytic structures*, Ark. Mat. **26** (1988) 21–40.

[4] E. Bedford & B. Gaveau, *Envelopes of holomorphy of certain 2-spheres in* $\mathbf{C}^n$, Amer. J. Math. **105** (1983) 975–1009.

[5] E. M. Čirka, *Regularity of the boundaries of analytic sets*, Math. USSR Sb. **45** (1983) 291–335.

[6] B. Coupet, *Régularité d'applications holomorphes sur des variétés totalement réelles*, Thesis, Université de Provence, 1987.

[7] L. Hörmander, *An introduction to complex analysis in several variables*, North-Holland, Amsterdam, 1973.

[8] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France **109** (1981) 427–474.

[9] L. Nirenberg, S. Webster & P. Yang, *Local boundary regularity of holomorphic mappings*, Comm. Pure Appl. Math. **33** (1980) 305–338.

[10] S. Pinčuk & S. Khasanov, *Asymptotically holomorphic functions and their applications*, Math. USSR Sb. **62** (1989) 541–550.

[11] J.-P. Rosay, *Une remarque à propos de fonctions holomorphes et du front d'onde* C^∞, Comm. Partial Differential Equations **12** (1987) 1323–1332.

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