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POSITIVE SCALAR CURVATURE AND LOCAL ACTIONS OF NONABELIAN LIE GROUPS

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1. Introduction

Lawson and Yau proved in [7] that if a compact, connected, nonabelian Lie group G acts smoothly and effectively on a compact manifold M, then M admits a riemannian metric of positive scalar curvature. In Theorem A below we show that the same conclusion holds under somewhat weaker assumptions described by the following definition:

1.1. Definition. A local action of nonabelian Lie groups (or $\mathcal{N}$-structure) on a smooth manifold M consists of a finite cover $(U_i)_{i\in I}$ of M by open, connected sets U_i and a family $F_i\colon G_i \times U_i \to U_i$ ($i \in I$) of smooth, effective actions of compact, connected, nonabelian Lie groups G_i such that the following compatibility condition holds:

> for $i, j \in I$ the set $U_{ij} = U_i \cap U_j$ (if nonempty) is both G_i- and G_j-invariant and one of the two groups contains the other if we treat them as subgroups of $\mathrm{Homeo}(U_{ij})$.

Theorem A. *If a compact manifold M admits a local action by nonabelian Lie groups, then it admits a riemannian metric of positive scalar curvature.*

§§4 and 5 contain the main conceptual body of the proof of Theorem A and explain its relation to [7]. The technical core of the proof is deferred to §§9 and 10.

Theorem B (see §2) states that *if M and N are two manifolds with $\mathcal{N}$-structures and* $\dim(M) = \dim(N) \geq 6$, *then the connected sum $M\#N$ also has an $\mathcal{N}$-structure.* This theorem thus provides a method of constructing local actions from global ones and illustrates some flexibility of $\mathcal{N}$-structures, which is not shared by global actions.

Theorem C (see §3) supplies examples of manifolds (with the family $(T^n \times S^2)\#(T^n \times S^2)$, $n \geq 3$, among them) which admit local actions but no global action by a nonabelian group. As those manifolds have metrics

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with positive scalar curvature, they also prove that the converse of the theorem of Lawson and Yau does not hold.

Our interest in local actions of Lie groups comes from the work of Gromov and Cheeger ([3], [4]) who introduced the notion (using different terminology though) and explored the case of abelian groups (tori). Quite naturally the geometric features in the two (abelian and nonabelian) cases differ substantially.

Finally let us mention that some related results were obtained by Phillipe Fullsack.

2. Local actions on connected sums of manifolds

2.1. Theorem B. *If* $\dim(M) = \dim N = n \geq 6$, *and both* M *and* N *admit* $\mathcal{N}$*-structures, then so does the connected sum* $M\#N$.

Proof. We shall successively change the $\mathcal{N}$-structure on M to one for which one of the groups is $\mathrm{SO}(n)$ acting on a neighborhood of a point in the standard way.

Take a group G_1 of maximal dimension acting on its domain $U_1 \subset M$. Let $f\colon M \to [0, +\infty)$ be a smooth function with $\mathrm{supp}(f) \subset U_1$ and $f \not\equiv 0$. Let h be the average of f under the action of G_1. We still have $\mathrm{supp}(h) \subset U_1$ and $h \not\equiv 0$. Let $0 < \varepsilon < \sup(h)$ be a regular value of h. $M_1 = h^{-1}[\varepsilon, +\infty)$ is a nonempty, compact, G_1-invariant submanifold of U_1. We can now define new domains V_i for $i \neq 1$ by putting $V_i = U_i \backslash M_1$. The new domain for G_1 will be the union of $U_1 \backslash M_1$ and a small invariant collar neighborhood of ∂M_1. Let H be a subgroup of G_1 isomorphic to $\mathrm{SO}(3)$ or $\mathrm{SU}(2)$, and let X be a principal orbit of H in $\mathrm{Int}(M_1)$. X has a neighborhood in $\mathrm{Int}(M_1)$ H-equivariantly isomorphic to $D_k(1) \times X$ where $D_k(1)$ is a k-dimensional disc of radius 1 ($k = n - \dim(X)$) and H acts on the second factor.

We enlarge the set of groups of our $\mathcal{N}$-structure as follows. We take H as one of the new groups—its domain will be $\mathrm{Int}(M_1)\backslash\{\overline{D_k(1/2) \times X}\}$. Next we take $\mathrm{SO}(k) \times H$ acting diagonally on $\{D_k(1)\backslash\overline{D_k(1/4)}\} \times X$ and $\mathrm{SO}(k)$ acting on $\{D_k(1/3) \times X\}\backslash\overline{D_n(1/6)}$ where $D_n(r)$ denotes an n-dimensional disc of radius r centered at a fixed point $p = (0, x) \in D_k(1) \in X$, 0 being the center of $D_k(1)$ and x being a point in X. Note that $\mathrm{SO}(k)$ is nonabelian since $k = n - \dim(X) \geq 3$ ($n \geq 6, \dim(X) \leq 3$). Finally we take one group more—$\mathrm{SO}(n)$ acting on $D_n(1/5)$. Now $M^* = M\backslash D_n(1/7)$ is a manifold whose boundary is the $(n-1)$-dimensional sphere with the standard action of $\mathrm{SO}(n)$. Having done the same for N we can glue the boundaries ∂M^* and ∂N^* by a reflection $\phi\colon \partial M^* \to \partial N^*$, $\phi(x_1, \cdots, x_n) = (x_1, \cdots, x_{n-1}, -x_n)$.

The two copies of $\mathrm{SO}(n)$ acting in M and N respectively are identified as transformation groups of a neighborhood of $\partial M^* = \partial N^*$ by an isomorphism $g \mapsto \phi^{-1} \circ g \circ \phi$. This completes the construction of an $\mathscr{N}$-structure on $M \# N$.

2.2. Remark. This argument shows in fact that the initial $\mathscr{N}$-structure on the disjoint union $M \cup N$ can be extended to an $\mathscr{N}$-structure on the standard cobordism between $M \cup N$ and $M \# N$.

2.3. Remark. In view of Theorem A saying that the $\mathscr{N}$-structure implies positive scalar curvature, Theorem B appears to be analogous to the following theorem [5]: *If* $\dim(M) = \dim(N) \geq 3$, *and both M and N admit riemannian metrics of positive scalar curvature, then so does $M \# N$.* In fact Gromov and Lawson proved in [5] that the class of manifolds with positive scalar curvature is closed under surgery in codimension ≥ 3. The same for $\mathscr{N}$-structures would be extremely difficult, if not impossible, although it can be proved in case of trivially attached handles. More precisely, a modification of the above proof of Theorem B shows that if f is a framed embedding of the sphere S^k into M^n extendible to a framed embedding of D^{k+1}, $0 \leq k \leq n-1$, $n \geq 6$, then every $\mathscr{N}$-structure on M extends to an $\mathscr{N}$-structure on $W^{n+1} = M^n \times [0,1] \cup D^{n+1}$, the trace of surgery on M along f.

3. Manifolds with no global nonabelian action

3.1. Theorem C. *Let M_1 and M_2 be closed, oriented, n-dimensional ($n \geq 3$) $K(\pi, 1)$-manifolds (that is $\pi_i(M_k) = 0$ for $i \neq 1$ and $\pi_1(M_k) = G_k$ for $k = 1, 2$). Then the manifold $M = (M_1 \times S^2) \# (M_2 \times S^2)$ has the following properties:*

(i) *M admits a local action of nonabelian groups;*

(ii) *M admits a global action of* $\mathrm{SO}(2)$;

(iii) *M has no global action by a compact, connected, nonabelian Lie group.*

Proof. (i) This follows from the proof of Theorem A since $M_k \times S^2$ admits an action of $\mathrm{SO}(3)$ (on the second factor). We do not need $\dim(M_k \times S^2) \geq 6$ since the orbits are two-dimensional.

(ii) $\mathrm{SO}(2)$ acts on $M_k \times S^2$ as a subgroup of $\mathrm{SO}(3)$. If $x_k \in M_k$ is any point, and $y \in S^2$ is a pole, then $p_k = (x_k, y) \in M_k \times S^2$ is a fixed point of $\mathrm{SO}(2)$, and p_k has a neighborhood $\mathrm{SO}(2)$-equivariantly isomorphic to the product $D^n \times D^2$ with the standard action on the second factor. Hence a small open ball B_k about p_k is SO(2)-invariant and so is $N_k = (M_k \times S^2)\backslash B_k$. In order to form the connected sum $M = (M_1 \times S^2)\#(M_2 \times S^2)$ we have to glue $\partial N_1 \cong S^{n+1}$ and $\partial N_2 \cong S^{n+1}$ by an orientation-reversing linear map $S^{n+1} \to S^{n+1}$. If we change the action of SO(2) on N_2 by the automorphism $g \mapsto g^{-1}$ ($g \in \mathrm{SO}(2)$), then the two actions of SO(2) on N_1 and N_2 agree with the glueing $S^{n+1} \ni (x_1, \cdots, x_{n+2}) \mapsto (x_1, \cdots, x_{n+1}, -x_{n+2}) \in S^{n+1}$, and we get a global action of SO(2) on M.

To prove (iii) we use a theorem of Browder and Hsiang (see [1, p. 412]) which we recall now. Suppose that a compact, connected group G acts on M, and $f\colon M \to K(\pi, 1)$ is a map such that $f_*\colon \pi_1(M) \to \pi_1(K(\pi, 1)) = \pi$ is surjective. Let $p\colon M \to M/G$ be the quotient map, and $i\colon G \to M$ be given by $i(g) = gm$, where m is a base point in M. One proves that $i_*(\pi_1(G))$ is contained in the center of $\pi_1(M)$. Hence $\pi' = \pi/f_*i_*(\pi_1 G)$ is a group. Let $\alpha\colon K(\pi, 1) \to K(\pi', 1)$ be the map induced be the projection $\pi \to \pi'$.

3.2. Theorem (*Browder-Hsiang* [1]). *There exists a map* $\pi\colon H_*(M/G; \mathbf{Q}) \to H_*(K(\pi', 1); \mathbf{Q})$ *such that the following diagram commutes*:

$$\begin{array}{ccc} H_*(M;\mathbf{Q}) & \xrightarrow{f_*} & H_*(K(\pi,1);\mathbf{Q}) \\ \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \alpha_*} \\ H_*(M/G;\mathbf{Q}) & \xrightarrow[\phi]{} & H_*(K(\pi',1);\mathbf{Q}) \end{array}$$

In our application of this theorem we set $\pi = G_1 * G_2$ (the free product of G_1 and G_2). Note that the center of $G_1 * G_2$ is trivial and hence $\pi' = \pi$, $\alpha =$ identity. Obviously $M_1 \vee M_2$ is a $K(\pi, 1)$-space. The map $f\colon M \to M_1 \vee M_2 = K(\pi, 1)$ will be the collapsing of $\partial N_1 = \partial N_2 = S^{n+1}$ to a point: $(M_1 \times S^2)\#(M_2 \times S^2) \to (M_1 \times S^2) \vee (M_2 \times S^2)$ followed by the projection $(M_1 \times S^2) \vee (M_2 \times S^2) \to M_1 \vee M_2$. The induced map $f_*\colon \pi_1(M) \to \pi$ is an isomorphism.

Now $H_n(M_1 \vee M_2; \mathbf{Q}) = H_n(M_1; \mathbf{Q}) \oplus H_n(M_2; \mathbf{Q}) = \mathbf{Q} \oplus \mathbf{Q}$. If z is a point in S^2, then $M_1 \times z$ and $M_2 \times z$ are n-dimensional homology classes in M, and their images in $H_n(M_1 \vee M_2; \mathbf{Q})$ under f_* generate $H_n(M_1 \vee M_2; \mathbf{Q})$ so that f_* is onto.

Suppose now that M admits an effective action by a compact, connected, nonabelian Lie group G. Replacing G by its subgroup $[G, G]$ we can assume that G is semisimple.

To complete the proof of (iii) it is enough to show that $\dim H_n(M/G; \mathbf{Q}) \leq 1$. This is obvious if the dimension of the principal orbits is at least

3 since then $\dim(M/G) \leq \dim(M) - 3 = n - 1$ and $H_n(M/G; \mathbf{Q}) = 0$. G has no homomorphic images of dimension 1 and 2 because there is no semisimple group in these dimensions and the image of a semisimple group is semisimple. It follows that G has no orbit of dimension 1 as the only candidate S^1 would give an epimorphism $G \to \mathrm{SO}(2)$.

We are left with the case when the principal orbits are two-dimensional. Let M^0 denote the union of all the principal orbits. M/G is a pseudomanifold of dimension n, M^0/G is a manifold, and $\dim((M - M^0)/G)$ is strictly less than n. Now M^0/G is connected (see [6, p. 12]) and it follows that $\dim H_n(M/G; \mathbf{Q}) \leq 1$ similarly as for manifolds. Hence $\mathrm{Im}(\phi \circ p_*)$ can be at most 1-dimensional. Since $\dim(\mathrm{Im} f_*) = 2$, we get a contradiction which shows that there is no action of a nonabelian Lie group on M.

3.3. Remark. The class of $K(\pi, 1)$-manifolds includes manifolds of nonpositive curvature (as the torus T^n or hyperbolic manifolds) and also nil-manifolds (as the $(2k+1)$-dimensional Heisenberg nil-manifold—the quotient of Heisenberg group by its standard integral lattice).

It was pointed out by the referee that Theorem C overlaps a theorem of Burghelea and Schultz [2]. In particular they proved that the connected sum of $T^n \times S^2$ with itself has properties 3.1(ii) and (iii).

4. Towards a proof of Theorem A

Let us recall the theorem of Lawson and Yau proved in [7]:

4.1. Theorem [7]. *If a compact, connected, nonabelian Lie group G acts smoothly and effectively on a manifold M, then M admits a riemannian metric of positive scalar curvature.*

The remainder of the present paper is devoted to a proof of the following generalization of Theorem 4.1:

4.2. Theorem A. *If a compact manifold M admits a local action $(G_i \times U_i \to U_i)$ by nonabelian Lie groups G_i (cf. Definition 1.1), then it admits a riemannian metric of positive scalar curvature.*

The action $G_i \times U_i \to U_i$ provides U_i with a metric g_i of positive scalar curvature (obtained for example by means of Theorem 4.1). One could try to build up a metric of positive scalar curvature on M by glueing the metrics g_i together. A natural way (as it is explained in more detail in §5) is to show that the metrics on the intersections $U_{i_1} \cap \cdots \cap U_{i_k}$ obtained from the actions of different groups $G_{i_1}, \cdots, G_{i_k}$ are homotopic. Unfortunately, we are not able to prove this for the metrics obtained from the original construction of Lawson and Yau. In order to display the difficulty which arises here let us recall how the construction of Lawson and

Yau works. For a bi-invariant metric on G the sectional curvature $k(X, Y)$ equals $\frac{1}{4}\|[X, Y]\|^2 \geq 0$, and the submersed metric on a homogeneous space G/K has also nonnegative sectional curvature. For a general global action of G on a manifold N the idea is that shrinking a G-invariant metric on N in the directions tangent to the orbits of G (in a way to be specified) one gets large positive curvature in these directions that predominates over the whole possibly negative curvature coming from the other directions. This does not work in the fixed points N^G where the orbits are zero-dimensional, although the set with possibly negative curvature can be made an arbitrarily small neighborhood of N^G. Lawson and Yau cope with this problem by choosing the initial metric to be a torpedo metric near N^G. This means that small discs transversal to N^G are isometric to a hemisphere of small radius and hence carry large positive curvature. This property guarantees that during the shrinking one has positive scalar curvature on a small but *fixed* neighborhood of N^G. In order to prove the above-mentioned homotopy relations we should be able to shrink a metric along the orbits of all the groups simultaneously, with different weights. The main problem with using torpedo metrics in this situation is that if H is a subgroup of G, then it seems to be impossible to find a G-invariant metric which is torpedo near N^H (since N^H is not G-invariant in general).

Fortunately there is another geometric phenomenon, neglected by Lawson and Yau, which enables us to show that their condition on the initial metric to be a torpedo metric is superfluous. The simplest example of this is the following. Shrinking the orbits of the standard action of S^1 on the flat $\mathbb{R}^2$ we get small curvature on most of $\mathbb{R}^2$ (since S^1 is abelian) *but* large positive curvature near the fixed point. In §9 we show that near the fixed points of any action the situation is very similar. One could observe that the neighborhood of M^G with positive curvature obtained in this way lessens quickly. The aim of §§9 and 10 is to show that the set with positive curvature obtained by shrinking the orbits extends and nears M^G even faster. Thus the two sets eventually cover the manifold and we get positive scalar curvature everywhere.

A rigorous treatment of both sources of positive scalar curvature is based on a modification (presented in §6) of the main construction from [7]. The shrunken metric is obtained as a metric submersed from a riemannian product. Its curvature is described by a formula of O'Neill recalled in §8. In this more technical setting the first source of positive scalar curvature (shrinking the orbits) corresponds to the horizontal curvature while the second source (described conceptually above) makes its appearance as the fundamental tensor A of the riemannian submersion (6.1) and originates in nontrivial representation of isotropy groups acting on the tangent spaces.

5. First step in the proof: local actions reduced to global ones

In the first part of the proof of Theorem A we transform the existence problem on positive scalar curvature metrics in case of local actions into a problem of homotopy of metrics in case of global actions (our aim is to show how Theorem A follows from Proposition 6.2).

With the purpose of illustrating this idea we take a group G_1 of maximal dimension and a compact, G_1-invariant submanifold (with boundary) M_1 of codimension zero in M which covers almost the whole domain U_1 of G_1 (this means that $M_1 \subset U_1$ and $\mathrm{Cl}(M \backslash M_1) \subset \bigcup\{U_i\colon i \neq 1\}$). We fix a collar structure near ∂M_1 (that is, a diffeomorphism of a neighborhood of ∂M_1 in M onto $\partial M_1 \times (-\varepsilon, \varepsilon)$) which is compatible with the action of G_1, and hence with all the actions. Now suppose that we have obtained metrics of positive scalar curvature on $\mathrm{Cl}(M \backslash M_1)$ and M_1 which are product metrics near the boundary ∂M_1 by construction. In order to glue the metrics together it is enough to find a homotopy in $\mathscr{P}(\partial M_1)$ (the space of metrics with positive scalar curvature on ∂M_1) between their restrictions to ∂M_1 since such a homotopy provides a metric of positive scalar curvature on the collar $\partial M_1 \times [0, 1]$, a product near the boundary. Note that the two metrics on ∂M_1 are obtained from local actions: (1) $\{G_i\colon U_i \cap \partial M_1 \neq \emptyset, i \neq 1\}$ and (2) global action by G_1. Note also that all the groups in (1) are contained in G_1 since $\dim(G_1)$ is maximal.

We shall continue this idea and decompose M into pieces with global actions. The metrics of positive scalar curvature provided by these actions can be glued together by one- and multi-parameter homotopies since they are constructed in an almost canonical way: the metric $dm^2_{\xi,t}$ defined by (6.1) depends on a bi-invariant metric on a Lie group, an invariant metric on the manifold, and on a real number t, and the set of these parameters is contractible. Moreover, the metric $dm^2_{\xi,t}$ depends on weights $\xi_i \geq 0$ associated with groups G_i forming a tower $G_n < \cdots < G_1$. Hence there is a standard homotopy joining metrics provided by such a tower and by its subtower—simply by letting some of the weights ξ_i tend to zero.

The decomposition of M which we are going to use is based on the following refinement of the data provided by a local action:

5.1. Proposition. *Let $(F_i\colon G_i \times U_i \to U_i)_{i \in I}$ be a local action of compact, connected (not necessarily nonabelian) Lie groups on a closed manifold M. Then there exist compact submanifolds with boundary M_i of codimension* *zero in* U_i $(i \in I)$ *and a riemannian metric* dm^2 *on* M *satisfying the following conditions*:

(i) *The family* $\{F_i|G_i \times \operatorname{Int} M_i : i \in I\}$ *is a local action on* M.

(ii) ∂M_i *and* ∂M_j *are orthogonal for* $i \neq j$.

Let X_i *be the inward unit normal vector field on* ∂M_i. *For* $J \subset I$ *let* $N_J = \bigcap\{\partial M_j : j \in J\}$.

(iii) *For some* $\varepsilon > 0$ *the exponential map*

$$\exp_J : N_J \times (-\varepsilon, \varepsilon)^J \ni (n, (x_j)_{j \in J}) \mapsto \exp_n \left(\sum x_j X_j(n)\right) \in M$$

is an isometric embedding.

We identify $N_J \times (-\varepsilon, \varepsilon)^J$ *with its image by* $\exp_J$ *in* M.

(iv) *The metric* dm^2 *is* G_i*-invariant on* $M_i \cup (\partial M_i \times (-\varepsilon, \varepsilon))$.

5.2. Remark. It follows immediately that for small ε and $K \subset J \subset I$ $\exp_J$ maps $N_J \times (-\varepsilon, \varepsilon)^K$ into N_{J-K}, and $\bigcap\{\partial M_i \times (-\varepsilon, \varepsilon) : i \in J\} = N_J \times (-\varepsilon, \varepsilon)^J$.

Proof of Proposition 5.1. We can assume that $I = (1, 2, \cdots, n_0\}$ and $i \leq j$ implies $\dim G_i \geq \dim G_j$.

We prove by induction that for $k \in I$ one can find manifolds M_i $(i \leq k)$ and a metric dm^2 with properties (i_k)–(iv_k) analogous to (i)–(iv) above, namely in i_k we require that the family $\{\operatorname{Int}(M_i) : i \leq k,\ U_i : i > k\}$ with the restricted action is a local action and (ii_k)–(iv_k) differ from (ii)–(iv) by the condition $i \leq k$, $j \leq k$, $J \subset \{1, \cdots, k\}$.

Suppose M_i $(i < k)$ and a suitable metric dm^2 are found.

5.3. Lemma. *If* C *is a compact subset of* U_k, *then for small* ε *there is a smooth nonnegative function* h *on* M *such that*:

(a) $\operatorname{supp}(h) \subset U_k$, *$h$ is* G_k*-invariant on* U_k;

(b) *for* $i < k$, *h is* G_i*-invariant on* $M_i \cup (\partial M_i \times [-\varepsilon, \varepsilon])$;

(c) *if* $\varnothing \neq J \subset \{i : i < k\}$ *and* $y \in N_J$, *then h is constant on* $\{y\} \times [-\varepsilon, \varepsilon]^J$;

(d) *h is strictly positive on* C.

Proof of Lemma 5.3. Take ε' slightly greater than ε. For $J \subset \{i : i < k\}$ let $C_J = C \cap (N_J \times [-\varepsilon', \varepsilon']^J)$ and let π_J be the projection from $N_J \times [-\varepsilon', \varepsilon']^J$ onto N_J. If ε and ε' are small, then $B_{\varepsilon'}(\pi_J(C_J)) \times [-\varepsilon', \varepsilon']^J \subset U_k$, where $B_\varepsilon(X)$ is the closed ε-neighborhood of X in M for $\varepsilon > 0$ and $X \subset M$.

For $i < k$ let $P_i^0 = \partial M_i \times [-\varepsilon, \varepsilon]$, $P_i^{+1} = \mathrm{Cl}(M_i \backslash P_i^0)$, and $P_i^{-1} = \mathrm{Cl}(M \backslash P_i^{+1} \backslash P_i^0)$. For $\eta : \{i : i < k) \to \{-1, 0, +1\}$ let $P^\eta = \bigcap\{P_i^{\eta(i)} : i < k\}$. Now $C = \bigcup C^\eta$ where $C^\eta = C \cap P^\eta$. It is relatively easy to construct a function h^η satisfying (a), (b), and (c), strictly positive on C^η. Therefore $h = \sum h^\eta$ meets conditions (a)–(d) as required.

We continue the proof of 5.1 by applying 5.3 to any compact connected subset C of U_k containing $M \backslash \bigcup\{\mathrm{Int}(M_i): i < k\} \backslash \bigcup\{U_i: i > k\}$. Let $c \in (0, \inf h(C))$ be a regular value of h and let M_k be the connected component of $\{p \in M: h(p) \geq c\}$ containing C.

Replace dm^2 by a metric dm_0^2 which is G_k-invariant near M_k, a product near ∂M_k. Then the manifolds M_i $(i \leq k)$, the metric dm_0^2, and a small $\varepsilon > 0$ satisfy (i_k)–(iv_k) and the proof of Proposition 5.1 is complete.

We still assume that $I = \{1, 2, \cdots, n_0\}$ and that $i < j$ implies $\dim(G_i) \geq \dim(G_j)$. Fix a bi-invariant metric dh_i^2 on G_i. Similarly as before let $P_i^0 = \partial M_i \times [-\varepsilon, \varepsilon]$, $P_1^{+1} = \mathrm{Cl}(M_i \backslash P_i^0)$, and $P_i^{-1} = \mathrm{Cl}(M \backslash P_i^{+1} \backslash P_i^0)$ for each $i \in I$. Thus M is decomposed as a union of three manifolds $M = P_i^{+1} \cup P_i^0 \cup P_i^{-1}$ glued along boundaries. For $\eta: I \to \{-1, 0, +1\}$ let again $P^\eta = \bigcap\{P_i^{\eta_i}: i \in I\}$. Now we get the desired decomposition of M into a union $M = \bigcup\{P^\eta\}$ of manifolds with corners.

Let us fix η for a while and describe a riemannian metric of positive scalar curvature on P^η. Let $j_1 < \cdots < j_z$ (resp. $k_1 < \cdots < k_p$) denote all the indices $i \in I$ for which $\eta(i)$ is zero (resp. nonnegative). Now (P^η, dm^2) can be written as a riemannian product $Y \times [-\varepsilon, \varepsilon]^z$ where Y is equal to $\bigcap\{P_i^{\eta(i)}: \eta(i) \neq 0\} \cap \bigcap\{\partial M_i: \eta(i) = 0\}$ and admits a global action by G_{k_1}. Denote by ϕ and ψ two auxiliary smooth increasing functions from $]-\varepsilon, \varepsilon]$ to $[0, 1]$ with $\phi(x) = 0$ for $x \leq 0.1\varepsilon$, $\phi(x) = 1$ near $x = \varepsilon$, and $\psi(x) = 1 - \phi(-x)$. We consider the z-parameter homotopy $\lambda_t: [-\varepsilon, \varepsilon]^z \to \mathscr{M}(Y)$ (where $\mathscr{M}(Y)$ is the space of all riemannian metrics on Y) defined by $\lambda_t(x_{j1}, \cdots, x_{jz}) = dm^2_{\xi, t}$. Here we adopt notation from §6 and use the metric dm^2, the groups $G_{k_1} > \cdots > G_{k_n}$, and the weights ξ given by $\xi_i = 1$ if $\eta(i) = +1$ and $\xi_i = \phi(x_i)$ if $\eta(i) = 0$. The bi-invariant metric used depends on $x \in [-\varepsilon, \varepsilon]^z$ and can be defined as follows. Let $l = \min\{i \in I: \eta(i) = +1\}$ and let $i_1 < \cdots < i_s$ be all the indices preceding l for which $\eta = 0$. Now we choose our bi-invariant metric to be $\psi(x_{i1}) dh_{i1}^2 + [1 - \psi(x_{i1})]\psi(x_{i2}) dh_{i2}^2 + \cdots + [1 - \psi(x_{i1})] \cdots [1 - \psi(x_{is})] dh_l^2$. Note that this metric may not be defined on some groups—but then the corresponding weights are zero.

Proposition 6.2 implies that there is a large value of t, common for all η, such that λ_t maps $[-\varepsilon, \varepsilon]^z$ into $\mathscr{P}(Y)$. Thus for each P^η we get a metric $\lambda_t(x) \oplus f(x)\, dx^2$ of positive scalar curvature, with f, a suitable function elongating the intervals $[-\varepsilon, \varepsilon]$ and dx^2, the standard metric on $[-\varepsilon, \varepsilon]^z$. All these metrics are product metrics with respect to the splittings 5.1(v) and hence give rise to smooth metric of positive scalar curvature on M.

Remark. For a compact, connected, nonabelian Lie group G the commutator group $[G,G]$ is semisimple, and a homomorphism $f\colon H \to G$ induces $\tilde{f}\colon [H,H] \to [G,G]$. Using this we can replace the groups G_i by $[G_i, G_i]$ and work in the sequel with this new $\mathscr{N}$-structure for which the groups are semisimple. We shall exploit this property in the proof of (7.5).

6. Construction of the metric $dm^2_{\xi,t}$

Suppose we are given compact, connected, semisimple Lie groups $G_n < G_{n-1} < \cdots < G_1$ and a smooth, effective action $F_1\colon G_1 \times M \to M$ of G_1 on a compact manifold M. The map

$$\bar{F}\colon G_n \times \cdots \times G_1 \times M \to M, \qquad \bar{F}(g_n, \cdots, g_1, m) = g_n \cdots g_1 m$$

is a principal $(G_n \times \cdots \times G_1)$-bundle with the action Δ given by

$$(g,m)\bar{g} = (g_n\bar{g}_n, \bar{g}_n^{-1} g_{n-1}\bar{g}_{n-1}, \cdots, \bar{g}_1^{-1} g_1 \bar{g}_1, \bar{g}_1^{-1} m).$$

Fix a G_1-invariant metric dm^2 on M and a bi-invariant metric dg_1^2 on G_1. The restriction $dg_i^2 = dg_1^2|_{G_i}$ is a bi-invariant metric on G_i, $i = 1, \cdots, n$. For (ξ, t), where $\xi \in (0,1]^n$ and $t \in [1,\infty)$, let $dm^2_{\xi,t}$ be the only riemannian metric on M such that

$$\bar{F}\colon \xi_n^{-2}\, dg_n^2 \oplus \cdots \oplus \xi_1^{-2}\, dg_1^2 \oplus t^2\, dm^2 \to dm^2_{\xi,t} \tag{6.1}$$

is a riemannian submersion. For $\xi \in [0,1]^n$ let $dm^2_{\xi,t} = dm^2_{\eta,t}$, where η is the sequence ξ with zeros entries omitted. It follows from (7.2) that the map

$$[0,1]^n \times [1,\infty) \to \mathscr{M}(M), \qquad (\xi, t) \mapsto dm^2_{\xi,t}$$

is continuous. Vaguely speaking, $\xi_i \to 0$ means that the group G_i disappears.

6.2. Proposition. *Let $l \leq s \leq n$. There exist $t_0 \in [1,\infty)$ such that for $t \geq t_0$ and any $\xi \in [0,1]^n$ with $\xi_s = 1$ the scalar curvature of $dm^2_{\xi,t}$ is positive. For continuity reasons the same is true if dg_1^2 is allowed to range over a compact set of metrics.*

The proof of 6.2 occupies the succeeding sections from 7 to 10.

7. Some preparatory lemmas

Let $\mathfrak{g}$ (resp. $\tilde{\mathfrak{g}}$) denote the Lie algebra of left (resp. right) invariant fields on a group G. If $F\colon G \times M \to M$ is a left action, then the same letter F

will denote the induced Lie algebra homomorphism $F\colon \tilde{\mathfrak{g}} \to \mathrm{Vect}(M)$. F_i will be the action F_1 restricted to G_i.

We describe now vertical and horizontal fields for the riemannian submersion (6.1).

Lemma. $\Delta\colon \mathfrak{g}_n \oplus \cdots \oplus \mathfrak{g}_1 \to \mathrm{Vect}(G_n \times \cdots \times G_1 \times M)$ *is given by*

$$\Delta(v_n, \cdots, v_1) = (v_n, \tilde{v}_n + v_{n-1}, \cdots, \tilde{v}_2 + v_1, -F_1 v_1), \tag{7.1}$$

where $\mathfrak{g} \ni v \mapsto \tilde{v} \in \tilde{\mathfrak{g}}$ *is the natural Lie algebra isomorphism for which* $\tilde{v}(e) = -v(e)$. *The horizontal lift* $\bar{X}$ *of a field* X *on* M *is equal on* $e \times \cdots \times e \times M$ *to*

$$\bar{X} = (t^2\xi_n^2 F_n^* S^{-1}X, \cdots, t^2\xi_1^2 F_1^* S^{-1}X, S^{-1}X), \tag{7.2}$$

where $S = I + \sum_{i=1}^n t^2\xi_i^2 F_i F_i^* = \sum_{i=0}^n t^2\xi_i^2 F_i F_i^*$, $\xi_0 = 1/t$, $F_0 = I$.

Proof. The formula for Δ is evident. The right-hand side of (7.2) is horizontal since

$$\begin{aligned}
&\langle \sum_{i=0}^n t^2\xi_i^2 F_i^* S^{-1}X, \Delta v\rangle \\
&= \xi_n^{-2}\langle t^2\xi_n^2 F_n^* S^{-1}X, v_n\rangle + \sum_{i=1}^{n-1} \xi_i^{-2}\langle t^2\xi_i^2 F_i^* S^{-1}X, -v_{i+1} + v_i\rangle \\
&\quad - t^2\langle S^{-1}X, F_1 v_1\rangle \\
&= t^2\langle S^{-1}X, F_n v_n\rangle + \sum_{i=1}^{n-1} t^2\langle S^{-1}X, F_i v_i - F_i v_{i+1}\rangle - t^2\langle S^{-1}X, F_1 v_1\rangle = 0.
\end{aligned}$$

Finally

$$F_1\left(\sum_{i=0}^n t^2\xi_i^2 F_i^* S^{-1}X\right) = \left(\sum_{i=0}^n t^2\xi_i^2 F_i F_i^*\right) S^{-1}X = X.$$

In the proof of the next lemma we shall make use of the following blowing-up construction. Let M be a riemannian manifold and N a closed submanifold of M. Let ρ be the injectivity radius for the normal bundle νN and put $r_N(m) = f(d(m,N))$, where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth increasing function such that $f(x) = x$ near zero and $f(x) = 1$ for $x \geq \rho/2$. The function $r_N\colon M \to \mathbb{R}$ is continuous and smooth on $M\backslash N$. We identify the normal disc bundle $D(\nu N, \rho)$ of radius ρ with its image in M by $\exp\colon \nu N \to M$. Let $\tilde{\pi}$ be the linear automorphism of $T(M\backslash N)$, which multiplies by $1/r_N$ the vectors tangent to normal spheres and leaves the vectors normal to the spheres unaltered. Consider the noncomplete riemannian manifold $M\backslash N$ with the metric $\tilde{\pi}^* dm^2$; its completion is a compact riemannian manifold which we shall denote by $\tilde{M} = \tilde{M}(N)$ and call the blowing-up of M along N. The identity on $M\backslash N$ extends to a smooth map $\pi\colon \tilde{M} \to M$ which is covered by the isometry of tangent bundles $\tilde{\pi}$.

Note that topologically $\tilde{M}$ is obtained from M by replacing N by the unit sphere bundle $\partial D(\nu N, 1)$. The metric $\tilde{\pi}^* \, dm^2$ on $\partial D(\nu N, 1)$ coincides with one coming from a standard construction which supplies a riemannian metric on the total space of a vector bundle with a riemannian connection over a riemannian manifold (applied to νN over N here).

Further on we shall need the following estimates.

Lemma. *For some positive constants α and γ and any $i = 1, \cdots, n$ we have*

$$|F_i u| \leq \alpha |u| r_i \quad \textit{and} \quad |F_i^* X| \leq \alpha |X| r_i; \tag{7.3}$$

$$|[F_i^* X, F_i^* Y]| \leq \alpha |X||Y| r_i^2; \tag{7.4}$$

(7.5) *the norm of the tensor $X, Y \mapsto [F_i^* X, F_i^* Y] \in \mathfrak{g}_i$ is greater than γr_i^2,*

where $r_i = r_N$ for $N = M^{G_i}$ and $u \in \mathfrak{g}_i$.

Proof. If $F\colon G \times M \to M$ is an action by isometries and $N = M^G$, then there is an action $\tilde{F}\colon G \times \tilde{M} \to \tilde{M}$ without fixed points for which π is equivariant. It follows that for $F\colon \mathfrak{g} \to TM$ and $\tilde{F}\colon \tilde{\mathfrak{g}} \to T\tilde{M}$ we have $F = T\pi \circ \tilde{F} = r_i \tilde{\pi} \circ \tilde{F}$. $\tilde{F}$ is defined on the compact manifold $\tilde{M}$, and hence $|\tilde{F} u| \leq \alpha |u|$, $|\tilde{F}^* X| \leq \alpha |X|$ and $|[\tilde{F}^* X, \tilde{F}^* Y]| \leq \alpha |X||Y|$ on $\tilde{M}$ for some α. But $\tilde{\pi}$ is an isometry, so (7.3) and (7.4) follow. As regards (7.5) it is enough to show that the tensor $X, Y \mapsto [\tilde{F}^* X, \tilde{F}^* X] \in \mathfrak{g}$ is nonzero at every point of $\tilde{M}$. Now $\operatorname{Im} \tilde{F}^* = (\ker \tilde{F})^\perp$ in $\mathfrak{g}$, and in a semisimple Lie algebra $\mathfrak{g}$ (with a bi-invariant metric) $\mathfrak{k}^\perp$ is abelian (for a subalgebra) $\mathfrak{k} < \mathfrak{g}$) only if $\mathfrak{k} = \mathfrak{g}$. Since $\tilde{F}$ has no fixed points, (7.5) is proved.

8. Curvature of a riemannian submersion

We recall a formula of O'Neill [8] relating horizontal and base sectional curvature of a riemannian submersion $p\colon E \to B$.

Denote by V and H the vertical and horizontal bundles. Let X^v (resp. X^h) be the vertical (resp. horizontal) component of a vector field X on E. For a field Y on B let $\bar{Y}$ be its horizontal lift to E. We define the fundamental tensor $A\colon H \otimes H \to V$ by

$$A(\bar{Y}, \bar{Z}) = \tfrac{1}{2} [\bar{Y}, \bar{Z}]^v. \tag{8.1}$$

The Levi-Civita connection $\bar{\nabla}$ and the curvature tensor $\bar{R}$ on E are related to those on B in the following way.

8.2. Proposition (*O'Neill* [8]).

$$\bar{\nabla}_{\bar{Y}}\bar{Z} = \overline{\nabla_Y Z} + A(\bar{Y},\bar{Z}),$$
$$\langle \bar{R}(\bar{Y},\bar{Z})\bar{Z},\bar{Y}\rangle = \langle R(Y,Z)Z,Y\rangle - 3|A(\bar{Y},\bar{Z})|^2.$$

Define the horizontal scalar curvature at $e \in E$ *as*

$$Scalhor(e) = \sum \langle \bar{R}(\bar{E}_i,\bar{E}_j)\bar{E}_j,\bar{E}_i\rangle,$$

where $\{\bar{E}_i\}$ *is an orthonormal basis for the horizontal space* $H(e)$.

8.3. Corollary. *The horizontal and base scalar curvatures are related to the tensor* A *as follows*:

$$\mathrm{Scal}(p(e)) = \mathrm{Scalhor}(e) + 3\|A\|^2 \quad \textit{for } e \in E.$$

9. The fundamental tensor A

We plan to show that $\mathrm{Scal}(M, dm^2_{\xi,t}) > 0$ using Corollary 8.3. For this we shall calculate the tensor A and the horizontal scalar curvature at $e\times e\times\cdots\times e\times M$.

9.1. Lemma. *There exist small positive constants* ε *and* δ *such that the norm* $\|A\|$ *of the fundamental tensor of the riemannian submersion* (6.1) *is greater than* δ *for any* (ξ,t,m) *satisfying* $r_s(m)\le\varepsilon/t$, $\xi_s=1$.

Proof. Let $v_i=0$ for $n\ge i>s$ and $v_i=w\in\mathfrak{g}_s$ for $s\ge i\ge 1$. Applying (7.3) we see that

$$\begin{aligned}|w|^2 \le |\Delta v|^2_{\xi,t} &\le |w|^2 + t^2|F_1 w|^2 = |w|^2 + t^2|F_s w|^2\\ &\le |w|^2 + t^2\alpha^2 r_s^2|w|^2 \le |w|^2(1+\alpha^2\varepsilon^2),\end{aligned}$$

so that the norm of Δv is uniformly bounded with respect to ξ, t, and to prove the lemma it is enough to bound from below the norm of the tensor

$$(\overline{SX},\overline{SY}) \mapsto \langle A(\overline{SX},\overline{SY}),\Delta v\rangle_{\xi,t}$$

in the metric

$$|\overline{SX}|^2_{\xi,t} = \sum_{i=0}^{n} t^4\xi_i^2|F_i^* X|^2.$$

For $r_s\le\varepsilon/t$ this metric is equivalent to

$$|\overline{SX}|'^2_{\xi,t} = \sum_{i=0}^{s-1} t^4\xi_i^2|F_i^* X|^2, \tag{9.2}$$

because if $i\ge s$ then

$$t^4\xi_i^2|F_i^*X|^2 \le t^4\alpha^2|X|^2 r_i^2 \le t^4\alpha^2|X|^2 r_s^2 \le \alpha^2\varepsilon^2 t^2|X|^2 \le \alpha^2\varepsilon^2|SX|'^2_{\xi,t}$$

(as $r_i \leq r_s$ for $i \geq s$ and small r_s) so that

$$|SX|'^2 \leq |SX|^2 \leq (1 + n\alpha^2\varepsilon^2)|SX|'^2.$$

Now

$$\begin{aligned}
2\langle A(\overline{SX}, \overline{SY}, \Delta v\rangle &= 2\langle \nabla_{\overline{SX}} \overline{SY}, \Delta v\rangle = -2\langle \overline{SY}, \nabla_{\overline{SY}} \Delta v\rangle \\
&= -2t^4 \langle F_s^* Y, \nabla_{F_s^* X} w\rangle - 2\sum_{i=1}^{s-1} t^4 \xi_i^2 \langle F_i^* Y, \nabla_{F_i^* X}(w + \tilde{w})\rangle \\
&\quad - 2t^2 \langle Y, \nabla_X - F_1 w\rangle \\
&= t^4 \langle [F_s^* X, F_s^* Y], w\rangle + 2\sum_{i=1}^{s-1} t^4 \xi_i^2 \langle [F_i^* X, F_i^* Y], w\rangle \\
&\quad + t^2 \langle Y, \nabla_X F_1 w\rangle - t^2 \langle X, \nabla_Y F_1 w\rangle \\
&= \mathbf{a} + \mathbf{b} + \mathbf{c},
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{a} &= t^4 \langle [F_s^* X, F_s^* Y], w\rangle, \\
\mathbf{b} &= \langle \sum_{i=1}^{s-1} t^4 \xi_i^2 F_i F_i^* Y, \nabla_X F_1 w\rangle - \langle \sum_{i=0}^{s-1} t^4 \xi_i^2 F_i F_i^* X, \nabla_Y F_1 w\rangle, \\
\mathbf{c} &= \langle X, \nabla_{F_1 w} \sum_{i=1}^{s-1} t^4 \xi_i^2 F_i F_i^* Y\rangle - \langle Y, \nabla_{F_1 w} \sum_{i=1}^{s-1} t^4 \xi_i^2 F_i F_i^* X\rangle,
\end{aligned}$$

because

$$\begin{aligned}
2\langle [F_i^* X, F_i^* Y], w\rangle &= \langle [F_i^* X, F_i^* Y], w\rangle - \langle [F_i^* Y, F_i^* X], w\rangle \\
&= -\langle F_i^* Y, [F_i^* X, w]\rangle + \langle F_i^* X, [F_i^* Y, w]\rangle \\
&= -\langle Y, F_i [F_i^* X, w]\rangle + \langle X, F_i [F_i^*, Y, w]\rangle \\
&= \langle Y, [F_i F_i^* X, F_i w]\rangle - \langle X, [F_i F_i^* Y, F_i w]\rangle \\
&= \langle Y, \nabla_{F_i F_i^* X} F_i w\rangle - \langle X, \nabla_{F_i F_i^* Y} F_i w\rangle - \langle Y, \nabla_{F_i w} F_i F_i^* X\rangle \\
&\quad + \langle X, \nabla_{F_i w} F_i F_i^* Y\rangle.
\end{aligned}$$

We shall show that the tensors $\mathbf{a}$ and $\mathbf{c}$ have small norms for small ε and that the norm of $\mathbf{b}$ is large enough. By (7.4),

$$\begin{aligned}
|\mathbf{a}(\overline{SX}, \overline{SY}, \Delta v| &= t^4 |\langle [F_s^* X, F_s^* Y], w\rangle| \leq \alpha t^4 |X||Y||w| r_s^2 \\
&\leq \alpha\varepsilon^2 t|X|t|Y||w| \leq \alpha\varepsilon^2 |\overline{SX}||\overline{SY}||\Delta v|.
\end{aligned}$$

Thus $\|\mathbf{a}\| \leq \alpha\varepsilon^2$.

Now let $d = \sup[|\langle X, \nabla_Y F_1 Z\rangle|$ where the supremum is taken over $X, Y \in TM$, $Z \in \mathfrak{g}_1$, $|X| \le 1$, $|Y| \le 1$, and $|Z| \le 1$. Then

$$\begin{aligned}|\langle X, \nabla_{F_1 w} t^4\xi_i^2 F_i F_i^* Y\rangle| &= d|X|\alpha r_s|w|t^4\xi_i^2|F_i^* Y| \\ &\le d\alpha\varepsilon\xi_i t|X|t^2\xi_i|F_i^* Y||w| \le d\alpha\varepsilon\xi_i|\overline{SX}||\overline{SY}||\Delta v|,\end{aligned}$$

so that $\|\mathbf{c}\| \le 2sd\alpha\varepsilon$.

To deal with $\mathbf{b}$ we prove the following algebraic lemma.

Lemma. *Suppose that $D\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a tensor and $B\colon \mathbb{R}^n \to \mathbb{R}^n$ is a selfadjoint, nonnegative linear map. Let $\Phi = t^2(I + B)$ and $C(X, Y) = D(X, \Phi Y) + D(\Phi X, Y)$. Then we have $\|C\|_\varphi \ge 2\|D\|$, where $\|C\|)_\varphi$ is the norm of C taken with respect to the inner product $\langle X, Y\rangle_\varphi = \langle \Phi X, Y\rangle$.*

Proof. Let $\langle e_i, e_j\rangle = \delta_{ij}$ and $Be_i = \lambda_i e_i$. Then $\langle e_i, e_j\rangle_\varphi = t^2(1 + \lambda_i)\delta_{ij}$ and

$$\begin{aligned}\|C\|_\varphi^2 &= \sum_{i,j}\left|C\left(\frac{e_i}{t\sqrt{1+\lambda_i}}, \frac{e_j}{t\sqrt{1+\lambda_j}}\right)\right|^2 \\ &= \sum \frac{|D(e_i, t^2(1+\lambda_j)e_j) + D(t^2(1+\lambda_i)e_i, e_j)|^2}{t^4(1+\lambda_i)(1+\lambda_j)} \\ &= \sum |D(e_i, e_j)|^2[(1+\lambda_i) + (1+\lambda_j)]^2/[(1+\lambda_i)(1+\lambda_j)] \\ &\ge 4\sum |D(e_i, e_j)|^2 = 4\|D\|^2,\end{aligned}$$

since $(a + b)^2/ab \ge 4$.

Applying the lemma to $D(X, Y) = \langle X, \nabla_Y F_1 w\rangle$ and

$$\Phi(X) = \sum_{i=0}^{s-1} t^4\xi_i^2 F_i F_i^* X$$

we get that $\|b\|_{\xi,t}$ is greater than the norm of D in the metric dm^2. But D and dm^2 are independent of ξ, t. Moreover D is nonzero at every point of $M_s = M^{G_s}$. To see this, note that $F_1 w = 0$ on M_s, and hence $\nabla_Y F_1 w$ does not depend on the metric on M. Thus D is determined by the orthogonal representation of G_s in $T_m M$ ($m \in M_s$), which is the same at every point of the connected component of M_s. For $O(n)$ acting on $\mathbb{R}^n$, $\langle X, \nabla_Y Fw\rangle = \langle X, Fw \cdot Y\rangle$ where $Fw \in \mathfrak{o}(n) \subset \mathrm{End}(\mathbb{R}^n)$. As $\|D\| > 0$ on M_s, for some ε, δ we have $\|D\| \ge \delta > 0$ if $r_s(m) \le \varepsilon > 0$. This proves Lemma 9.1.

10. Horizontal curvature

Now we consider the other part of 8.3—the horizontal scalar curvature Scalhor. For orthonormal vectors X, Y in a riemannian product $N_1 \times N_2$

we have

$$\begin{aligned}\operatorname{Sect}(X,Y) = \langle R(X,Y)Y,X\rangle &= \sum_{i=1}^{2}\langle R_i(X_i,Y_i)Y_i,X_i\rangle \\ &= \sum_{i=1}^{2}(|X_i|^2|Y_i|^2 - \langle X_i,Y_i\rangle^2)\operatorname{Sect}(X_i,Y_i),\end{aligned}$$

where X_i, $Y_i \in TN_i$ are the components of X, Y.

In this way the horizontal curvature Scalhor decomposes as the curvature coming from $G_n \times \cdots \times G_1$ and from M:

$$\text{Scalhor} = \text{Scalhor}\, G + \text{Scalhor}\, M \tag{10.1}$$

with $\text{Scalhor}\, G \geq 0$ and

$$|\text{Scalhor}\, M| \leq t^{-2}\dim(M)(\dim(M)-1)\sup|\operatorname{Sect}(M)|$$

so that in order to prove Proposition 6.2 we have to show that for any K the nonnegative terms $3\|A\|^2 + \text{Scalhor}\, G$ in 8.3 are greater than Kt^{-2} for large t. From Lemma 9.1 this is clear for $r_s(m) \leq \varepsilon/t$. To show this for $r_s(m) \geq \varepsilon/t$ we shall use $\text{Scalhor}\, G$.

10.2 Lemma. *For fixed* ξ Scalhor G *is a monotonically increasing function of* t.

Proof. For a tensor C let $C(X,Y)\colon \Phi(X)$ denote $\|C\|^2$ with respect to the inner product for which $\langle X,X\rangle = \Phi(X)$. In this notation

$$\begin{aligned}4\,\text{Scalhor}\, G &= \left(\sum_{i=1}^{n} t^4\xi_i^3[F_i^*X, F_i^*Y]\right) : \left(\sum_{i=0}^{n} t^4\xi_i^2|F_i^*X|^2\right) \\ &= \left(\sum_{i=1}^{n} \xi_i^3[F_i^*X, F_i^*Y]\right) : \left(\sum_{i=1}^{n} \xi_i^2|F_i^*X|^2 + t^{-2}|X|^2\right),\end{aligned}$$

because for a Lie group with a bi-invariant metric $\langle R(X,Y)Y,X\rangle = \frac{1}{4}|[X,Y]|^2$. For $t \to \infty$, $t^{-2}|X|^2$ decreases monotonically, and the other parts of the expression are independent of t.

It follows that we may assume that $t = 1$ for $r_s \geq \varepsilon$ and $t = \varepsilon/r_s(m)$ for $r_s(m) \leq \varepsilon$. In the first case we have a positive lower bound for $\text{Scalhor}\, G$ since by (7.5) the tensor $[F_s^*X, F_s^*Y]$ vanishes only on M_s and so $\text{Scalhor}\, G$ is everywhere strictly positive on $M\backslash M_s$. It remains to consider the case where $0 < r_s \leq \varepsilon$, $t = \varepsilon/r_s$. Here, following (9.2) and then using (7.3)–(7.5) we have

$$\begin{aligned}
4(1+n\alpha^2\varepsilon^2)^2 \operatorname{Scalhor} G &\geq \left(\sum_{i=1}^{n} \xi_i^3 [F_i^* X, F_i^* Y]\right) : \left(\sum_{i=0}^{s-1} \xi_i^2 |F_i^* X|^2\right) \\
&\geq \left(\sum_{i=1}^{s} \xi_i^2 [F_i^* X, F_i^* Y]\right) : \left(\sum_{i=0}^{s-1} \xi_i^2 |F_i^* X|^2\right) \\
&\geq \sum \xi_i^3 [F_i^* X, F_i^* Y] : (\alpha^2 \sum \xi_i^2 r_i^2) |X|^2 \\
&\geq \gamma^2 \alpha^{-4} \sum \xi_i^6 r_i^4 \left(\sum \xi_i^2 r_i^2\right)^{-2} \\
&= \gamma^2 \alpha^{-4} \sum (\xi_i r_i)^4 \xi_i^2 \left(\sum (\xi_i r_i)^2\right)^{-2} =: \gamma^2 \alpha^{-4} B.
\end{aligned}$$

Consider now two cases:

(a) If $\max_{1 \leq i \leq s} t\xi_i r_i = t\xi_k r_k \geq \beta \geq 1$, then

$$\mathbf{B} \geq (\xi_k r_k)^4 \xi_k^2 (s\xi_k^2 r_k^2)^{-2} \geq s^{-2}(\beta / \operatorname{tr}_k)^2 \geq \beta^2 s^{-2} t^{-2},$$

and $\beta^2 s^{-2}$ is large for large β.

(b) If $t\xi_i r_i \leq \beta$ for $1 \leq i \leq s$, then

$$\mathbf{B} \geq (\xi_s r_s)^4 \xi_s^2 t^4 (s\beta^2)^{-2} = \varepsilon^4 s^{-2} \beta^{-4},$$

which is constant and positive. This completes the proof of Proposition 6.2 and simultaneously proves Theorem 4.2.

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