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CURVATURE AT INFINITY OF OPEN NONNEGATIVELY CURVED MANIFOLDS

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1. Introduction

Let M be a complete open manifold of nonnegative sectional curvature. By the structure theory of [6] M is diffeomorphic to the normal bundle of a compact totally convex submanifold Σ of M. Σ is called a soul of M. We fix a soul Σ and define $\kappa(t) = \sup\{K(\sigma) | \operatorname{dist}(\pi(\sigma), \Sigma) \geq t\}$, where $K(\sigma)$ is the sectional curvature of a tangent plane σ at a point $\pi(\sigma)$ with distance $\operatorname{dist}(\pi(\sigma), \Sigma)$ to the soul. We obtain a bound on the decay of the function κ under suitable topological conditions.

Theorem 1. *Let $n = \dim M$ be odd, $k = \dim \Sigma$, and $\alpha = 2 - 2k/(n-1)$. If $\chi(M) \neq 0$ where $\chi(M)$ is the Euler characteristic of M, then* $\limsup \kappa(t) \cdot t^\alpha > 0$ *or M is isometric to a euclidian n-space $\mathbb{R}^n$.*

The theorem implies in particular that Σ is a point and thus M is diffeomorphic to $\mathbb{R}^n$ if $\kappa(t) \cdot t^\beta \to 0$ for an exponent $\beta \geq 2 - 2/(n-1)$. If κ decays quadratically, i.e., $\kappa(t) \cdot t^2 \to 0$, then M is indeed isometric to the euclidean space. The last result was proved first by Greene and Wu [11] under the additional assumption that M has a pole. In general this was recently proved by Kasue [15].

In the case where the codimension of Σ is ≤ 3 we obtain the much stronger result that the function $\kappa(t)$ cannot tend to 0 if the soul is not flat. More precisely we prove

Theorem 2. *If* $\operatorname{codim} \Sigma = n - k \leq 3$ *and* $\kappa(t) \to 0$, *then either M is flat or the universal cover $\tilde{M}$ splits isometrically as $X \times \mathbb{R}^k$, where X is diffeomorphic to $\mathbb{R}^{n-k}$. The fundamental group of M is a Bieberbach group of rank k, and any soul of M is a compact flat k-dimensional manifold.*

The last result leads us to the following conjecture:

Let M be a complete open manifold with nonnegative sectional curvature. If $\kappa(t) \to 0$, then the soul Σ of M is flat.

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In §3, the final section of this paper, we discuss some examples and give generalizations of the above theorems.

2. Proof of the theorems

Proof of Theorem 1. We distinguish the cases $\dim\Sigma \geq 1$ and $\dim\Sigma = 0$.

Case 1. $\dim\Sigma \geq 1$. Since $\chi(M) \neq 0$, the fundamental group $\pi_1(M)$ is finite by [6, Corollary 9.4]. Since the universal covering $\tilde{M} \to M$ is finite, one checks easily that $\tilde{M}$ satisfies the same curvature conditions as M. Thus we assume without loss of generality that M is simply connected.

Let Σ be a soul of M arising from the basic construction of [6, §1] using all rays starting from a fixed point $p_0 \in M$. Then there exists a nonnegative, convex, nonexpanding map $f\colon M \to \mathbb{R}_+$ with $f|_\Sigma = 0$. Up to an additive constant, f is the supremum of the Busemann functions of rays at p_0. The sublevels $C_t = \{p \in M | f(p) \leq t\}$ with $t \geq 0$ are compact and convex. Since f is nonexpanding, we have

$$B_t := \{p \in M | d(p,\Sigma) \leq t\} \subseteq C_t, \qquad d(\partial C_{t_1}, \partial C_{t_2}) \geq |t_1 - t_2|. \tag{1}$$

The convexity of f implies

$$C_t \subset B_{At} \tag{2}$$

for all $t \geq 1$ and $A := \max\{d(p,\Sigma) | p \in C_1\}$.

Let us assume $\limsup \kappa(t) \cdot t^\alpha = 0$. Under this assumption and the condition $\dim\Sigma \geq 1$ we obtain:

Proposition 1. *There exist a sequence $t_i \to \infty$ and a sequence H_i of compact hypersurfaces $H_i \subseteq M$ homeomorphic to the unit normal bundle $N_1(\Sigma)$ with the properties*:

(a) $H_i \subseteq M - B_{t_i}$.

(b) $\|L_i\| \cdot t_i^{\alpha/2} \to 0$ *for the Weingarten map L_i of H_i.*

(c) $\operatorname{vol}(H_i) \leq \text{const} \cdot t_i^{n-1-k}$.

We will prove this proposition below. We remark here that Proposition 1 does not hold in the case $\dim\Sigma = 0$ and $\alpha = 1$. In this case, instead of (b) we only get that $\|L_i\| \cdot t_i$ is bounded.

Note that the hypersurfaces $H_i \cong N_1\Sigma$ are oriented, since M is simply connected. Let G_i be the Chern-Gauss-Bonnet integrand of H_i. Then

$$\chi(H_i) = \chi(N_1\Sigma) = \frac{2}{\omega_{n-1}} \cdot \int_{H_i} G_i \, dV_i, \tag{3}$$

where ω_{n-1} is the volume of the standard $(n-1)$-sphere S^{n-1}. The dimension k of Σ is even since $\chi(\Sigma) = \chi(M) \neq 0$. Thus the fibers of $N_1\Sigma$

have even dimension. So we obtain $\chi(N_1\Sigma) = \chi(\Sigma) \cdot \chi(S^{n-k-1})$ (cf. [3, p. 182]) and in particular $|\chi(N_1\Sigma)| \geq 2$. With $Q_i := G_i - \det L_i$ from (3) we obtain

$$\omega_{n-1} \leq |P_i| + |T_i|,$$

with

$$P_i := \int_{H_i} \det L_i \, dV_i, \qquad T_i = \int_{H_i} Q_i \, dV_i.$$

We will show that the curvature assumption $\limsup \kappa(t) t^{\alpha} = 0$ implies $P_i, T_i \to 0$. This contradiction proves Case 1 of the theorem.

A purely algebraic computation (cf. [11, p. 70]) shows

$$\|Q_i\| \leq \text{const} \cdot \sum_{p=1}^{m} \kappa(t_i)^p (\|L_i\|^2)^{m-p},$$

with $m = \frac{1}{2}(n-1)$.

By property (c) of the proposition we have $\text{vol}(H_i) \leq \text{const} \cdot t_i^{n-1-k} = \text{const} \cdot t_i^{\alpha \cdot m}$. Thus

$$|T_i| \leq \|Q_i\| \text{vol}(H_i) \leq \text{const} \cdot \sum_{p=1}^{m} (\kappa(t_i) t_i^{\alpha})^p (\|L_i\|^2 t_i^{\alpha})^{m-p}.$$

Hence $T_i \to 0$ by the curvature assumption and the boundedness of $\|L_i\| t_i^{\alpha/2}$ by property (b).

For P_i we obtain

$$|P_i| \leq \text{const} \cdot \|L_i\|^{n-1} \cdot t_i^{n-1-k} = \text{const} \cdot (\|L_i\| t_i^{\alpha/2})^{n-1},$$

and hence also $P_i \to 0$ by (b).

It remains to prove Proposition 1. We need some preparations. The first result follows from [6, Theorem 2.5].

Lemma 1. *For $t > 0$ the boundary of the set C_t is homeomorphic to the unit normal bundle $N_1\Sigma$.*

Lemma 2. *Assume that the boundary $S_t = \partial C_t$ is a smooth hypersurface for some $t > 0$. If $0 \leq K \leq \varepsilon^2$ on $M \backslash C_t$, then the distance set $S_t^r = \{p \in M | d(p, C_t) = r\}$ is a smooth hypersurface for $0 \leq r \leq \frac{1}{2}\pi/\varepsilon$, and the Weingarten map L_t^r of S_t^r (with respect to the outer normal vector) satisfies*

$$1/r \geq L_t^r \geq -\varepsilon \tan(\varepsilon r). \tag{4}$$

In particular, the Weingarten map L_t of $H_t = S_t^r$ for $r = \frac{1}{4}\pi/\varepsilon$ satisfies

$$\|L_t\| \leq \left(\frac{4}{\pi}\right) \cdot \varepsilon. \tag{5}$$

Proof. Since S_t is a convex hypersurface, a standard comparison argument implies that the focal points of S_t in $M \setminus C_t$ have distance $\geq \frac{1}{2}\pi/\varepsilon$ from S_t. If the nearest cut point of S_t in $M \setminus C_t$ is closer, then there exists an unbroken geodesic $\gamma\colon [0, 2a] \to M \setminus \text{interior}(C_t)$ with endpoints on S_t. Now $f \circ \gamma(0) = f \circ \gamma(2a) = t$ but $f \circ \gamma(s) > t$ for $s \in (0, 2a)$. This is a contradiction to the convexity of f. Thus there are no cutpoints, and S_t^r is embedded for $0 \leq r \leq \frac{1}{2}\pi/\varepsilon$. The inequality $1/r \geq L_t^r$ follows from $K \geq 0$ and comparison with the Euclidean situation ($K = 0$) while we obtain $L_t \geq -\varepsilon \tan(\varepsilon r)$ by comparison with the sphere of curvature ε^2 (e.g. cf. Proposition 2.3 of [8]).

Smoothing. In general, the hypersurface S_t is not smooth. To remedy this, we use a smoothing process ([9], [10], [14], [1], [7]) for f in a neighborhood of C_t, i.e., we pass to the function

$$\tilde{f}(x) = \int_{T_x M} f(\exp_x(v))\phi(\|v\|)\, dv,$$

where $\phi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a weight function being constant near 0 with support in $[0, \varepsilon]$. Then $\tilde{f}$ is a smooth function whose Hessian is bounded from below by $-\delta$ with $\delta \to 0$ as $\varepsilon \to 0$, and $|\tilde{f} - f| < \varepsilon$. Moreover, for any $p \in S_t$ we have $f(p) = t$ and $d(p, \Sigma) \leq A \cdot t$. Hence for sufficiently small ε, the gradient $\nabla \tilde{f}$ satisfies

$$\langle \nabla \tilde{f}(p), -\gamma'(0) \rangle \geq \tfrac{1}{2} A,$$

where γ is any shortest unit speed geodesic from p to Σ. Thus the principal curvatures of the regular hypersurface $\tilde{S}_t = \{\tilde{f} = t\}$ are bounded from below by $-2A\delta$, and

$$d(\tilde{S}_t, S_t) \leq 2A\varepsilon.$$

Now we may replace S_t with $\tilde{S}_t$ and recover the estimates (4) and (5) of Lemma 1 up to an arbitrary small error.

Lemma 3. *Let D_t, $t \geq t_0$, be a family of compact subsets of M with $D_t \subset D_{t'}$ for $t' > t$ and*

$$d(\partial D_{t'}, \partial D_t) \geq t' - t, \tag{6}$$

such that $H_t = \partial D_t$ is a smooth hypersurface. Let there be $\varepsilon > 0$ such that the sets $H_t^s = \{p \in M | d(p, D_t) = s\}$ are smooth hypersurfaces for all $s \in [0, \varepsilon]$ and all $t \geq t_0$. If there is a continuous function h such that $\operatorname{vol}(H_t^s) \geq h(t)$ *for all $t > t_0$ and $s \in [0, \varepsilon]$, then* $\operatorname{vol}(D_{t_2} - D_{t_1}) \geq \int_{t_1}^{t_2} h(t)\, dt$ *for $t_2 > t_1 \geq t_0$.*

Proof. Let $s_0 < \cdots < s_q$ be any subdivision of $[t_1, t_2]$ with $r_i := s_{i+1} - s_i \leq \varepsilon$. Then (6) implies that

$$\operatorname{vol}(D_{t_2} - D_{t_1}) \geq \sum_{i=0}^{q-1} \int_0^{r_i} \operatorname{vol}(H_{t_i}^{\sigma})\, d\sigma \geq \sum_{i=0}^{q-1} h(t_i)(r_i).$$

Lemma 4. *The function* $t \to \operatorname{vol}(B_t)/t^{n-k}$ *is monotone decreasing.*

The proof of Lemma 4 is analogous to the Bishop-Gromov inequality (cf. [13], [2], [8]). One estimates the dilatation of the normal exponential map of Σ using the comparison theorems of Rauch (cf. [8, Theorem 6.4]). For a different approach compare [15].

Proof of Proposition 1. Since $\limsup \kappa(t) \cdot t^{\alpha} = 0$ and $\alpha < 2$, there exists a monotone decreasing function a with $a(t) \to 0$ as $t \to \infty$, and with $a(t) \geq \max\{\sqrt{\kappa(t)} \cdot t^{\alpha/2}, (1/t)^{1-\alpha/2}\}$. We consider the family $\tilde{C}_t = \{\tilde{f}_t \leq t\}$, where $\tilde{f}_t$ denotes the smoothing of f near C_t as described above. By (1) and the choice of the function $a(t)$ we have $0 \leq K \leq a^2(t) \cdot t^{-\alpha}$ on $M \backslash C_t$. Put

$$D_t := \{p \in M | d(p, \tilde{C}_t) \leq (\pi/4) a(t)^{-1} t^{\alpha/2}\},$$

and let $H_t = \partial D_t$. By Lemma 2, H_t is smooth with Weingarten map L_t satisfying

$$\|L_t\| t^{\alpha/2} \to 0. \tag{7}$$

Furthermore, H_t is homeomorphic to S_t which in turn is homeomorphic to $N_1\Sigma$ by Lemma 1. To finish the proof, we show that there is a sequence $t_i \to \infty$ such that

$$\operatorname{vol}(H_{t_i}) \leq c \cdot t_i^{n-1-k},$$

where $c = 4(n-k)(A + \pi/4)^{n-k} \operatorname{vol}(N_1\Sigma)$, and A is the constant of (2).

Let us assume to the contrary that $\operatorname{vol}(H_t) > c \cdot t^{n-1-k}$ for all $t \geq t_0$. By (5) the second fundamental form of H_t tends to zero. The same is true for the ambient curvature. By Lemma 2, for $0 \leq s \leq 1$, the distance sets

$$H_t^s = \{p \in M | d(p, D_t) = s\}$$

are smooth hypersurfaces, and one checks easily that

$$\operatorname{vol}(H_t^s) \geq \frac{c}{2} \cdot t^{n-1-k}$$

for all $s \in [0, 1]$ and all t sufficiently large. Thus the sets D_t satisfy the conditions of Lemma 3 for t_0 large enough, and $h(t) = c/2 \cdot t^{n-1-k}$. Thus

$$\operatorname{vol}(D_t - D_{t_0}) \geq \frac{c}{2(n-k)} \cdot t^{n-k} - c^*$$

for some constant c^*.

Now $C_t \subset B_{A \cdot t}$ by (2) for $t \geq 1$, and since $a(t)^{-1} \cdot t^{\alpha/2} \leq t$ by the choice of $a(t)$ we have $D_t \subset B_{A' \cdot t}$ with $A' = A + \pi/4$. This leads to a lower bound for the volume growth of distance balls:

$$\begin{aligned}\liminf_{t\to\infty} \frac{\operatorname{vol}(B_t)}{t^{n-k}} &= \liminf_{t\to\infty} \frac{\operatorname{vol}(B_{A'\cdot t})}{(A'\cdot t)^{n-k}} \\ &\geq \frac{c}{2(n-k)\cdot A'^{n-k}} = 2\cdot \operatorname{vol}(N_1\Sigma).\end{aligned}$$

By Lemma 4, $\operatorname{vol}(B_t)/t^{n-k}$ is monotone decreasing, thus

$$\lim_{t\to\infty} \frac{\operatorname{vol}(B_t)}{t^{n-k}} \leq \lim_{t\to 0} \frac{\operatorname{vol}(B_t)}{t^{n-k}} = \operatorname{vol}(N_1\Sigma).$$

This is a contradiction.

Case 2. $\dim \Sigma = 0$. We have to prove that $\lim_{t\to\infty} \kappa(t)\cdot t^2 = 0$ implies the flatness of M. Let therefore $a(t)$ be a positive monotone decreasing function with $\lim_{t\to\infty} a(t) = 0$ and $\kappa(t)\cdot t^2 \leq a^2(t)$. Let $r(t) = \frac{1}{4}\pi/b(t)$ where $b(t) = a(t)/t$. By Lemma 2, we can consider the embedded hypersurfaces S_t^r with $r \leq r(t)$.

We have $1/r \geq L_t^r \geq -b(t)\tan(b(t)r)$. Since $r \leq r(t) = \frac{1}{4}\pi/b(t)$ we have

$$b(t)\tan(b(t)r) \leq b(t)\tan(\pi/4) = b(t) \leq 4b(t)/\pi = 1/r(t) \leq 1/r.$$

Thus $\|L_t^r\| \leq 1/r$ for all $0 \leq r \leq r(t)$. For S_t^r we consider the Gauss-Bonnet integrand G_t^r and $Q_t^r = G_t^r - \det L_t^r$. As in Case 1 we have

$$\begin{aligned}\|Q_t^r\| &\leq \operatorname{const}\sum_{p=1}^{m} \kappa(t+r)^p(\|L_t^r\|^2)^{m-p} \\ &\leq \operatorname{const}\sum_{p=1}^{m} \frac{a^{2p}(t+r)}{(t+r)^{2p}}(\|L_t^r\|^2)^{m-p} \\ &\leq \operatorname{const}\sum_{p=1}^{m} \frac{a^{2p}(t)}{r^{2p}}\left(\frac{1}{r^2}\right)^{m-p} \\ &\leq \operatorname{const}\sum_{p=1}^{m} a(t)\cdot\frac{1}{r^{n-1}} \leq c\cdot a(t)\cdot \frac{1}{r^{n-1}},\end{aligned}$$

if $a(t) \leq 1$, where the constant c depends only on the dimension. We claim for t large enough

$$\operatorname{vol}(S_t^r) \geq \omega_{n-1}r^{n-1}(1 - c\cdot a(t)), \tag{8}$$

where ω_{n-1} is the volume of S^{n-1}. Suppose therefore that $\operatorname{vol}(S_t^r) \leq \omega_{n-1}r^{n-1}$. Then for t large enough such that $a(t) \leq 1$ we have

$$\int_{S_t^r} \|Q_t^r\|\, dV \leq c\cdot a(t)\cdot \frac{1}{r^{n-1}}\cdot \omega_{n-1}r^{n-1} = c\cdot a(t)\cdot \omega_{n-1}.$$

Since

$$\int_{S_t^r} G_t^r \, dV = \frac{\omega_{n-1}}{2}\chi(S_t^r) = \omega_{n-1},$$

we have

$$\int_{S_t^r} |\det L_t^r| \, dV \geq \omega_{n-1} - \int_{S_t^r} \|Q_t^r\| \, dV = \omega_{n-1}(1 - c \cdot a(t)).$$

Since $\|L_t^r\| \leq 1/r$ we have $|\det L_t^r| \leq 1/r^{n-1}$ and this implies (8).

By construction $S_t^r \subset B_{A \cdot t + r(t)}$ for $t \geq 1$, $r \leq r(t)$, and A as in (2). Therefore

$$\begin{aligned}\lim_{t\to\infty} \frac{\operatorname{vol}(B_t)}{t^n} &= \lim_{t\to\infty} \frac{\operatorname{vol}(B_{A\cdot t+r(t)})}{(A\cdot t + r(t))^n} = \lim_{t\to\infty} \frac{\operatorname{vol}(B_{A\cdot t+r(t)})}{r(t)^n} \\ &= \lim_{t\to\infty} \frac{1 - c\cdot a(t)}{r(t)^n} \int_0^{r(t)} \omega_{n-1} s^{n-1} \, ds = \frac{1}{n}\omega_{n-1}.\end{aligned}$$

The second equality follows from the fact that $r(t)/t \to \infty$. Since the volume growth is euclidean, M is isometric to $\mathbb{R}^n$ (cf. e.g. [11, Lemma 1]).

Proof of Theorem 2. We study first the simply connected case and prove:

($*$) Let M be simply connected with $\operatorname{codim}\Sigma \leq 3$. If $\kappa(t) \to 0$, then Σ is a point.

We need a result on the normal holonomy of Σ.

Proposition 2. *Let Σ be a soul of codimension ≤ 3 in the simply connected manifold M. Then one of the following holds*:

(A) *There exists a parallel normal unit vectorfield on Σ.*

(B) *There is a constant $C > 0$ such that for any two unit normal vectors v and w at points $p, q \in \Sigma$ there exists a piecewise differentiable path $c\colon [0,1] \to \Sigma$ of length $\leq C$ from p to q such that the parallel translation of v along c gives w.*

We will prove the proposition at the end of this section.

Consider first case (A) and assume that V is a parallel normal unit vectorfield on Σ. Let $\phi\colon \Sigma \times \mathbb{R} \to M$ be the map $\phi(p,t) = \exp_p tV(p)$. We claim that ϕ is a totally geodesic isometric immersion.

For $|t| < \varepsilon$, where ε is small, $\Sigma_t = \phi(\Sigma, t)$ is an embedded submanifold of M. By Rauch's theorem [5, 1.31] the map $\phi_t\colon \Sigma \to \Sigma_t$, $\phi_t(p) = \phi(p,t)$ is contracting, i.e., $d(\phi_t(p), \phi_t(q)) \leq d(p,q)$. By the work of Sharafutdinov [17], [18], there exists a contracting map $\psi\colon M \to \Sigma$ with $\psi|_\Sigma = \mathrm{id}|_\Sigma$. It follows that $\psi \circ \phi_t\colon \Sigma \to \Sigma$ is a contracting map which is homotopic to the identity. Such a map is an isometry (see e.g. [17, Lemma 1.2]) and hence ϕ_t is an isometry. By the rigidity part of Rauch's theorem, ϕ is an isometric immersion on $\Sigma \times [-\varepsilon, \varepsilon]$. The above argument shows more generally that the set of all $t > 0$ such that ϕ is an isometric immersion on $\Sigma \times [-t, t]$ is open. Since it is clearly closed, it follows that ϕ is a totally geodesic isometric immersion.

Using the structure theory of [6] one checks easily that the image of ϕ does not stay in a compact subset of M. Since we assume that $\kappa \to 0$ and Σ_t is totally geodesic and isometric to Σ, it follows that Σ is flat. Thus Σ is compact, simply connected and flat. Hence Σ is a point.

It remains to consider case (B) of the proposition. We first make the following general comments:

Let $h\colon [0, \infty) \to M$ be a ray parametrized by arc length with $p = h(0) \in \Sigma$. Then [6, 8.22(3)] implies that $\dot{h}(0)$ is perpendicular to Σ. If $g\colon \mathbb{R} \to \Sigma$ is a geodesic in Σ with $g(0) = p$ and $V(s)$ the parallel vectorfield along g with $V(0) = \dot{h}(0) = v$, then [6, 8.22(4)] implies that $\Phi\colon \mathbb{R} \times [0, \infty) \to M$, $\Phi(s, t) = \exp_{g(s)} tV(s)$ is a totally geodesic isometric immersion and the geodesics $h_s(t) = \Phi(s, t)$ are rays for all s.

Let $c\colon [0, 1] \to \Sigma$ be any piecewise differentiable path in Σ with $c(0) = p$ and let $V(s)$ be the parallel vectorfield along c with $V(0) = v$. Using an approximation of c by piecewise geodesics we see that the vectors $V(s)$ are all initial vectors of rays. By (B) every unit normal vector of Σ is in the same orbit of the normal holonomy as v. It follows that all geodesics normal to Σ are rays. Therefore $H_t = \{x \in M | d(x, \Sigma) = t\}$ is a smoothly embedded submanifold. Note that H_t is canonically diffeomorphic to the normal sphere bundle $N_1(\Sigma)$. Since Φ is an isometric immersion, one checks that all rays h_s have the same Busemann function. It follows that the sets H_t are the level sets of the Busemann function of h, and hence H_t is a convex hypersurface. It follows now from Lemma 2 that the norm of the second fundamental tensor L_t of H_t satisfies $\|L_t\| \le 1/t$, in particular $\|L_t\| \to 0$. Since $\kappa(t) \to 0$ also the intrinsic curvature of H_t goes to 0.

We claim that the diameter of H_t is bounded by the constant C of Proposition 2. Let therefore $x, y \in H_t$. Then $x = \exp_p tv$ and $y = \exp_q tw$, where v, w are unit normal vectors at $p, q \in \Sigma$. By (B) there exists a piecewise differentiable curve $c\colon [0, 1] \to \Sigma$ of length $\le C$ such that $V(1) = w$ for the parallel vectorfield V along c with $V(0) = v$. By the above comments the curve $c_t(s) = \exp_{c(s)} tV(s)$ has the same length as c and is contained in H_t.

Thus the metrics on H_t, $t > 0$, define a family of metrics on $N_1(\Sigma)$ with bounded diameter and curvature converging to 0. Thus $N_1(\Sigma)$ is an almost flat manifold in the sense of Gromov [12]. We consider the homotopy sequence of the S^{n-k-1}-bundle $N_1(\Sigma)$ and obtain the exact sequence

$$\pi_1(S^{n-k-1}) \to \pi_1(N_1(\Sigma)) \to \pi_1(\Sigma).$$

Since $\pi_1(\Sigma)$ is trivial by assumption and $\pi_1(S^{n-k-1})$ is either trivial of isomorphic to $\mathbb{Z}$, it follows that $\pi_1(N_1(\Sigma))$ is either trivial or cyclic. By Gromov's theorem [12] (compare also [4]) an almost flat manifold with this fundamental group is the circle S^1. It follows that M is 2-dimensional, and since M is simply connected, M is diffeomorphic to $\mathbb{R}^2$ and Σ is a point.

We now consider the general case. By [6, Corollary 6.2] the universal cover $\tilde{M}$ splits isometrically as $X \times \mathbb{R}^s$, where the isometry group of X is compact. We can assume that M is not flat which implies that X is not trivial. Let $\pi\colon \tilde{M} \to M$ be the covering and $\pi_X\colon \tilde{M} \to X$ the projection. Note that $\pi^{-1}(\Sigma)$ is a totally convex submanifold of $\tilde{M}$ and hence also $\Sigma' = \pi_X(\pi^{-1}(\Sigma))$ is totally convex. We claim that Σ' is a point. Since the isometry group of X is compact and $\pi^{-1}(\Sigma)$ covers the compact set Σ, it follows that Σ' is compact. Therefore Σ' is a totally convex compact submanifold without boundary. By [6, Theorem 2.1] the inclusion $\Sigma' \subset X$ is a homotopy equivalence. Now let Σ_X be a soul of X. Then Σ' and Σ_X have the same homotopy type and in particular $\dim \Sigma_X = \dim \Sigma'$. Since $\operatorname{codim} \Sigma \leq 3$, it follows that the codimension of Σ_X in X is ≤ 3. Let κ_X be the curvature function on X. Since the isometry group of X is compact, the projection $\pi|_{X\times\{0\}}$ is proper. Since $\kappa(t) \to 0$ on M it follows that $\kappa_X(t) \to 0$. By $(*)$, Σ_X (and hence Σ') is a point.

Thus $\pi^{-1}(\Sigma) = \{q\} \times H$, where q is a point in X, and H is an affine subspace of $\mathbb{R}^s$. We claim that $H = \mathbb{R}^s$. Since X is not flat, there exists a tangent plane σ at a point $q' \in X$ with $K(\sigma) > 0$. If $H \neq \mathbb{R}^s$, then there are points $t_i \in \mathbb{R}^s$ with $d(t_i, H) \to \infty$. Thus the points $q_i\colon \pi((q', t_i))$ satisfy $d(q_i, \Sigma) \to \infty$, but the curvature at q_i does not tend to 0. Hence $H = \mathbb{R}^s$, $s = k$, and Σ is a compact quotient of $\mathbb{R}^k$. X has dimension $n-k$, and since Σ_X is a point, it is diffeomorphic to $\mathbb{R}^{n-k}$.

Proof of Proposition 2. Since the statement is easy to prove for codim $\Sigma \leq 2$, let $\operatorname{codim}\Sigma = 3$. We assume that there is no parallel unit normal vectorfield on Σ and prove (B). We fix a point $p \in \Sigma$ and consider closed piecewise differentiable curves on Σ at p. The parallel translation of c defines an element $\phi(c)$ in the normal holonomy group at p. Since Σ is simply connected, $\phi(c)$ is orientation preserving and thus we can consider $\phi(c)$ as an element of SO(3). Since there is no normal parallel vectorfield on Σ, there are closed smooth curves c_0, c_1 at p, such that the elements $\phi(c_0), \phi(c_1) \in \mathrm{SO}(3)$ have different axes. Since Σ is simply connected there is a smooth homotopy c_t between c_0 and c_1. We consider the differentiable map $a(t) = \phi(c_t)$ in SO(3). Since $a(0)$ and $a(1)$ have different axes, $\dot{a}(t)$ is not everywhere a multiple of a given left invariant vectorfield. Thus there are $t_1, t_2 \in (0,1)$ such that the left translation of $\dot{a}(t_1)$ to $a(t_2)$ is linearly independent from $\dot{a}(t_2)$.

Let $h_t^i = c_{t_i}^{-1} \circ c_{t_i+t}$ for $i = 1,2$. The h_t^i are two families of piecewise differentiable curves which are defined for $|t|$ small. By construction the maps $a_i(t) = \phi(h_t^i)$ are smooth such that $\dot{a}_1(0)$ and $\dot{a}_2(0)$ are linearly independent vectors at the identity e of SO(3).

Put $k_t = h_t^1 \circ h_t^2 \circ (h_t^1)^{-1} \circ (h_t^2)^{-1}$ and $h_u^3 = k_{\sqrt{u}}$. Then $a_3(u) = \phi(h_u^3)$ defines a C^1-curve in SO(3) starting at e with

$$\dot{a}_3(0) = [\dot{a}_1(0), \dot{a}_2(0)].$$

Hence $\dot{a}_1(0), \dot{a}_2(0), \dot{a}_3(0)$ are linearly independent. Then $h(t,s,u) = h_t^1 \circ h_s^2 \circ h_u^3$ is a 3-parameter family of piecewise differentiable curves at p, which is defined for $(t,s,u) \in I^3$ where I is a small interval containing 0. Let $\Phi\colon I^3 \to \mathrm{SO}(3)$, $\Phi(t,s,u) = \phi(h(t,s,u))$. By construction, the differential of Φ is nonsingular in 0, and thus $\Phi(I^3)$ is a neighborhood of e in SO(3).

Thus there is a constant $N \in \mathbb{N}$ such that for every element $\alpha \in \mathrm{SO}(3)$ there exists an element $\beta \in \Phi(I^3)$ with $\beta^N = \alpha$. There is a constant C_1 such that every curve $h(t,s,u)$ has length $\leq C_1$. Thus for every $\alpha \in \mathrm{SO}(3)$ there exists a curve c of length $\leq N \cdot C_1$ with $\phi(c) = \alpha$.

Let now v and w be as in the statement of (B). Choose a path c_1 of length $\leq$ diameter(Σ) from p to q. Then there exists a suitable closed curve c_2 at p of length $\leq N \cdot C_1$ such that the parallel translation along $c = c_1 \circ c_2$ maps v onto w. The length of c is bounded by $C = N \cdot C_1 + \mathrm{diameter}(\Sigma)$.

3. Final remarks

1. For the last step of the proof of Theorem 1 it suffices to know that

$$\limsup_{t\to\infty} \frac{\mathrm{vol}(B_t)}{t^{n-k}} < \infty.$$

If $\dim \Sigma = k$ this is a consequence of Lemma 4. Theorem 1 remains valid, if we assume instead of $\dim \Sigma = k$ only that the volume grows of order t^{n-k} where k is any positive real number.

2. The proof of Theorem 2 shows that $\kappa(t)$ cannot tend to 0 if the normal holonomy of the soul Σ satisfies either (A) or (B) of Proposition 2. One of these conditions may also hold in higher codimension. We give two examples:

(a) If M has two different souls Σ_1 and Σ_2, then one can show that on Σ_1 there exists a normal parallel vectorfield pointing towards Σ_2. Thus if $\kappa(t) \to 0$ on M, then the souls of M are flat.

(b) One can show that condition (B) holds if Σ is simply connected and the normal holonomy group acts transitively on $N_1\Sigma$. In special cases the transitivity follows from the topology of the bundle. If $\operatorname{codim}\Sigma = 4$, then there exists a parallel normal vectorfield or the holonomy is transitive or the holonomy group is $SO(2) \times SO(2)$ and $N\Sigma$ splits into two parallel subbundles. If, e.g., $\Sigma = \mathbb{C}P^2$ and $N\Sigma = T\mathbb{C}P^2$ (as a vector-bundle), then from analysing the Stiefel-Whitney classes it follows that the bundle does not split, and thus the holonomy group acts transitively. Hence the conclusion of Theorem 2 holds also for this bundle.

3. Note that the constant C of Proposition 2 gives a universal bound for the diameter of the distance tubes H_t. It follows that there exists a positive lower bound for $\kappa(t)$ which depends only on C and the dimension n. Thus the geometry of $N_1\Sigma$ determines already a lower bound for the curvature function κ.

4. By the O'Neill formula one can compute the function $\kappa(t)$ explicitly for homogeneous vectorbundles $(G \times \mathbb{R}^n)/H$, where G is a compact Lie group, and H is a closed subgroup operating on $\mathbb{R}^n$ by a representation $\mu\colon H \to O(n)$. We consider the special case $(S^3 \times \mathbb{R}^2)/S^1$, where S^3 and $\mathbb{R}^2$ have their standard metrics, S^1 operates by the Hopf-action on S^3 and by rotation on $\mathbb{R}^2$. Then the soul is $S^2 = \mathbb{C}P^1$ with a metric of constant curvature 4. The distance tubes H_t are diffeomorphic to S^3 and carry a Berger metric, where the Hopf-circles are multiplied by a factor $t/(1+t^2)^{1/2}$. For $t \to \infty$, the metric converges to the standard metric on S^3. The maximal value of the curvature is 4 and is assumed only on the soul. Hence $\kappa(0) = 4$ and $\kappa(t) \to 1$ for $t \to \infty$.

5. After finishing this work, we learned that V. B. Marenich [16] has published a proof of our conjecture (cf. §1) using different methods. This would also imply Theorem 2 and the $(k > 0)$-part of Theorem 1. However, from what is written in [16], we were not able to verify that the proof is correct.

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