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STABILITY OF SINGULARITIES OF MINIMIZING HARMONIC MAPS

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1. Introduction

Singularities of energy minimizing harmonic maps may occur with 3-dimensional domains. Perhaps the simplest example is the map $x/|x|$ which has least energy [2] among all finite energy maps from the 3-ball $\mathbf{B}$ to the 2-sphere $\mathbf{S}^2$ having boundary values given by the identity map of $\mathbf{S}^2$. Moreover, in dimension 3, singularities are, by the work of R. Schoen and K. Uhlenbeck [10, Theorem], [11, 2.7], at most isolated. As the boundary data varies the singularities presumably move. In [5] was noted the impossibility of a sequence of minimizing configurations in which a pair of oppositely oriented singularities come together and cancel, leaving a singularity-free configuration. This followed from the strong convergence of minimizers and the basic small energy regularity theorem [10, 2.6]. These arguments left open the possibility of three singularities, two oppositely oriented, merging and leaving a single singularity. This is not precluded by either topological degree considerations or by the monotonicity of energy [11, 2.4]. However, the estimates of the present paper, in particular, rule out any such cancellation. Our results are based on the following:

Perturbation Lemma. *There exist positive constants δ_0, c_0, and α so that if $\varphi \in \operatorname{Lip}(\mathbf{S}^2, \mathbf{S}^2)$, $\delta = \|\varphi - \mathrm{id}_{\mathbf{S}^2}\|_{\mathrm{Lip}} \le \delta_0$, and $u \in H^1(\mathbf{B}, \mathbf{S}^2)$ is energy minimizing with $u|\mathbf{S}^2 \equiv \varphi$, then u has only one singular point a,*

$$|a| \le c_0 \delta^{1/2} \quad \textit{and} \quad \left\| u - \theta\left(\frac{x-a}{|x-a|}\right)\right\|_{C^\alpha} \le c_0 \delta^{1/4}$$

for some orthogonal rotation θ of $\mathbf{R}^3$ with $\|\theta - \mathrm{id}_{\mathbf{R}^3}\| \le c_0 \delta^{1/4}$.

This leads to the following general

Stability Theorem. *Suppose Ω is a smooth bounded domain in $\mathbf{R}^3$, $\psi \in \operatorname{Lip}(\partial\Omega, \mathbf{S}^2)$, and v is the unique energy-minimizing map from Ω to $\mathbf{S}^2$ with $v|\partial\Omega \equiv \psi$. There exists a positive number β and, for any positive ε, a*

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positive number δ so that for any $\varphi \in C^{1,\alpha}(\partial\Omega, \mathbf{S}^2)$ with $\|\varphi - \psi\|_{\mathrm{Lip}} \leq \delta$ and for any energy-minimizing $u \in H^1(\Omega, \mathbf{S}^2)$ with $u|\partial\Omega \equiv \varphi$, one has

$$\|u - v \circ \eta\|_{C^\beta} \leq \varepsilon$$

for some bi-lipschitz transformation η of Ω with $\|\eta - \mathrm{id}_\Omega\|_{\mathrm{Lip}} \leq \varepsilon$. In particular, η maps the singularities of u onto the singularities of v.

The uniqueness hypothesis may be obtained by restricting a minimizer to a proper subdomain. For examples of nonuniqueness, see [7], [1], and §5 below.

Our proof of these perturbation estimates is based on the regularity theory of energy minimizing maps ([10], [11], §2), the classification of minimizing tangent maps into $\mathbf{S}^2$ by H. Brezis, J. M. Coron, and E. Lieb [2] and asymptotic estimates of L. Simon [12], [13]. The latter works provide a description of the behavior of a minimizer in a punctured ball based only on information concerning its behavior on an annular region and its energy density at the center.

The Stability Lemma implies an interior estimate on the distance between singularities. Along with the uniform boundary estimate (2.1), it leads readily to global estimates on the number of singularities in terms of the Lipschitz norm of the boundary data. F. J. Almgren, Jr. and E. Lieb have in [1] independently obtained estimates in terms of the energy of the boundary data.

In our proof of the Perturbation Lemma, decay in an annulus is controlled by the boundary values. The proof easily generalizes to minimizers of other functionals as considered in [12]. In §5 we remark how the example of [8] leads to the existence of a smooth function $\varphi : \mathbf{S}^2 \to \mathbf{S}^2$ which serves as boundary values for two distinct energy-minimizing maps, one having singularities and one completely free of singularities.

2. Preliminary lemmas

We will use $c_0, c_1, \ldots$ to denote universal constants, and let

$$\mathbf{B}_r(a) = \{x \in \mathbf{R}^3 : |x - a| < r\}, \quad \mathbf{B}_r = \mathbf{B}_r(0), \quad \mathbf{B} = \mathbf{B}_1(0), \quad \mathbf{S}^2 = \partial\mathbf{B}.$$

As in [9] we define, for any C^1 function $g : \mathbf{R}^2 \to \mathbf{R}$ with $g(0) = 0 = |\nabla g(0)|$ and $\mathrm{Lip}\, g \leq 1$, the domain

$$\Omega_g = \{(x_1, x_2, x_3) \in \mathbf{B} : x_3 < g(x_1, x_2)\}.$$

Lemma (*Uniform boundary regularity*). *There exists positive numbers ρ_0 and σ_0 so that if g is as above with $\mathrm{Lip}\, g \leq \sigma_0$ and if $u \in H^1(\Omega_g, \mathbf{S}^2)$ is energy* *minimizing with* $\mathrm{Lip}(u|\mathbf{B}\cap\partial\Omega_g)\le 1$, *then* $u|(\mathbf{B}_{\rho_0}\cap\overline{\Omega}_g)$ *is Hölder continuous with*

$$\|u|(\mathbf{B}_{\rho_0}\cap\overline{\Omega}_g)\|_{C^{1/2}}\le 1.$$

Proof. Recalling [4, 5.4, 5.5], it suffices to find ρ_0 so that

$$\rho_0^{-1}\int_{\mathbf{B}_{2\rho_0}\cap\Omega_g}|\nabla u|^2\,dx\le\alpha_0 \tag{1}$$

for some suitable positive constant α_0. Before doing this we will establish an absolute bound

$$\int_{\mathbf{B}_{1/2}\cap\Omega_g}|\nabla u|^2\,dx\le c_1. \tag{2}$$

This is analogous to the interior energy bound [6, 3.1].

First choose, as in [9, 5.2], uniformly bi-lipschitz maps $\Upsilon_\sigma:\mathbf{B}_\sigma\cap\Omega_g\to\mathbf{B}_\sigma$ for $\sigma\in[\frac{1}{2},1]$. Then select, as in [4, 2.3], an extension $\omega_\sigma\in H^1(\mathbf{B}_\sigma,\mathbf{S}^2)$ of $u\circ\Upsilon_\sigma^{-1}$ satisfying the estimate

$$\int_{\mathbf{B}_\sigma}|\nabla\omega_\sigma|^2\,dx\le c_2\left(\int_{\partial\mathbf{B}_\sigma}|\nabla_{\tan}(u\circ\Upsilon_\sigma^{-1})|^2\,ds\right)^{1/2}.$$

Transforming with Υ_σ and Υ_σ^{-1} and abbreviating

$$\mathbf{D}(\sigma)=\int_{\mathbf{B}_\sigma\cap\Omega_g}|\nabla u|^2\,dx,$$

we conclude from the minimality of u that, for almost all $\sigma\in[\frac{1}{2},1]$,

$$\begin{aligned}\mathbf{D}(\sigma)&\le\int_{\mathbf{B}_\sigma\cap\Omega_g}|\nabla(\omega_\sigma\circ\Upsilon_\sigma)|^2\,dx\\&\le c_3\left(\int_{\mathbf{B}_\sigma\cap\partial\Omega_g}|\nabla_{\tan}u|^2\,dS+\int_{\partial\mathbf{B}_\sigma\cap\Omega_g}|\nabla_{\tan}u|^2\,dS\right)^{1/2}\\&\le c_3(\mathbf{D}'(\sigma)+\sigma^2)^{1/2}\end{aligned}$$

for some $c_3\ge 1$. This implies that $\mathbf{D}(\frac{1}{2})\le 4c_3^2$. In fact otherwise, for $\sigma\in[\frac{1}{2},1]$,

$$\mathbf{D}(\sigma)\ge\mathbf{D}(\tfrac{1}{2})>4c^2>2c_3,$$
$$\mathbf{D}'(\sigma)\ge c_3^{-2}\mathbf{D}(\sigma)^2-\sigma^{-2}\ge c_3^{-2}\mathbf{D}(\sigma)^2-4\ge\tfrac{1}{2}c_3^{-2}\mathbf{D}(\sigma)^2,$$

and we could integrate $-\mathbf{D}'/\mathbf{D}^2$ from $\frac{1}{2}$ to 1 to find the contradiction

$$-\mathbf{D}(\tfrac{1}{2})^{-1}\le\mathbf{D}(1)^{-1}-\mathbf{D}(\tfrac{1}{2})^{-1}\le-\tfrac{1}{4}c_3^{-2}\quad\text{and}\quad\mathbf{D}(\tfrac{1}{2})\le 4c_3^2.$$

Having established the bound (2), we now argue by compactness. If the lemma were false, then, by (1), there would exist sequences $\rho_i\to 0$, $\sigma_i\to 0$

and minimizers $u_i \in H^1(\Omega_{g_i}, \mathbf{S}^2)$ where $\operatorname{Lip} g_i \le \sigma_i$, $\operatorname{Lip}(u_i|\mathbf{B} \cap \partial\Omega_{g_i}) \le 1$, and

$$\liminf_{i\to\infty} \rho_i^{-1} \int_{\mathbf{B}_{2\rho_i}\cap\Omega_{g_i}} |\nabla u_i|^2\,dx \ge \alpha_0. \tag{3}$$

By (2) (applied with $g(x)$ replaced by $g_i(2\rho_i x)$) the sequence of scaled functions $v_i(x) = u_i(2\rho_i x)$ have bounded energy on $\mathbf{B}$. A subsequence of the v_i converge weakly in H^1 to a function $v \in H^1(\mathbf{B} \cap \{x_3 > 0\}, \mathbf{S}^2\}$ with $v|\mathbf{B} \cap \{x_3 = 0\} \equiv$ constant $(= \lim_{i\to\infty} u_i(0))$. By [9, 6.4, 5.7] and [11, 2.6], v must be a constant and the energies

$$(2\rho_i)^{-1} \int_{\mathbf{B}_{2\rho_i}\cap\Omega_{g_i}} |\nabla u_i|^2\,dx = \int_{\mathbf{B}\cap(2\rho_i)^{-1}\Omega_{g_i}} |\nabla u_i|^2\,dx$$

converge to 0, contradicting (3).

Lemma 2.2. *There exist a positive constant δ_0 so that if $\varphi \in \operatorname{Lip}(\mathbf{S}^2, \mathbf{S}^2)$, $\delta = \|\varphi - \mathrm{id}_{\mathbf{S}^2}\|_{\mathrm{Lip}} \le \delta_0$, and $u \in H^1(\mathbf{B}, \mathbf{S}^2)$ is energy minimizing with $u|\mathbf{S}^2 \equiv \varphi$, then u has only one singular point a and $|a| \le c_4\delta^{1/2}$.*

Proof. First note that the minimality of such a u implies the estimate

$$\begin{aligned} \int_{\mathbf{B}} |\nabla u|^2\,dx &\le \int_{\mathbf{B}} |\nabla\varphi\left(\frac{x}{|x|}\right)|^2\,dx = \int_{\mathbf{S}^2} |\nabla_{\tan}\varphi|^2\,ds \\ &\le \int_{\mathbf{B}} |\nabla\left(\frac{x}{|x|}\right)|^2\,dx + c_5\|\varphi - \mathrm{id}\,\|_{\mathrm{Lip}}^2 \\ &= 8\pi + c_5\|\varphi - \mathrm{id}\,\|_{\mathrm{Lip}}^2. \end{aligned} \tag{4}$$

Assuming now that the lemma is false, we would find a sequence of energy minimizing maps $u_i : \mathbf{B} \to \mathbf{S}^2$ which do not satisfy the conclusion but which have Lipschitz boundary values $\varphi_i = u_i|\mathbf{S}^2$ so that

$$\delta_i = \|\varphi_i - \mathrm{id}_{\mathbf{S}^2}\|_{\mathrm{Lip}} \to 0 \quad \text{as } i \to \infty.$$

By (4), any subsequence of the u_i contains a subsequence that is weakly convergent in $H^1(\mathbf{B}, \mathbf{S}^2)$. Any limit function, having energy at most 8π and having boundary values equalling $\mathrm{id}_{\mathbf{S}^2}$, must be $x/|x|$ [2]. Thus, our original sequence u_i converges to $x/|x|$ weakly, and, in fact, by [10, 4.6], strongly in H^1.

By Lemma 2.1, the functions are all uniformly Hölder continuous with uniformly bounded energies on some neighborhood of $\partial(\mathbf{B} \sim \mathbf{B}_{1/2})$. For i sufficiently large, φ_i has degree one, and so $\operatorname{Sing}(u_i)$, the singular set of u_i, must be nonempty. Arguing as above using the interior regularity theory [10] and the Hölder continuity of $x/|x|$ away from the origin, we now find

$$\sup\{|a| : a \in \operatorname{Sing}(u_i)\} \to 0 \quad \text{as } i \to \infty.$$

Choose a point $a_i \in \mathrm{Sing}(u_i)$, let

$$v_i \colon \mathbf{B}_{1-|a_i|} \to \mathbf{S}^2, \qquad v_i(x) = u_i(x + a_i),$$

and note, by the interior regularity theory, that

$$\begin{aligned} &\left\| v_i - \frac{x}{|x|} \right\|_{C^2(\mathbf{B}_{2/3}(a_i)\sim\mathbf{B}_{1/3}(a))} \\ &\quad \le \left\| u_i(x) - \frac{x}{|x|} \right\|_{C^2(\mathbf{B}_{3/4}\sim\mathbf{B}_{1/4})} + \left\| \frac{x}{|x|} - \frac{x - a_i}{|x - a_i|} \right\|_{C^2(\mathbf{B}_{2/3}\sim\mathbf{B}_{1/3})} \to 0 \end{aligned}$$

as $i \to \infty$. As in the discussion in [12, §8], Theorem 1 of [12] is applicable to v_i. It implies that, for i sufficiently large, 0 is the only singularity of v_i, hence a_i is the only singularity of u_i.

Finally we need to estimate $|a_i|$. For this, we define the function $w_i \in H^1(\mathbf{B}, \mathbf{S}^2)$,

$$w_i(x) = u_i(2x) \quad \text{for } x \in \mathbf{B}_{1/2}, \qquad w_i(x) = z_i(x)/|z_i(x)| \text{ for } x \in \overline{\mathbf{B}} \sim \mathbf{B}_{1/2},$$

where

$$z_i(x) = (2 - 2|x|)\varphi_i\left(\frac{x}{|x|}\right) + (2|x| - 1)\frac{x}{|x|}.$$

Noting that $\|\varphi_i - \mathrm{id}_{\mathbf{S}^2}\|_{L^\infty} \le c_6\delta_i$, we readily compute that

$$\begin{aligned} \int_{\mathbf{B}} |\nabla w_i|^2\,dx - c_7\delta_i &\le \int_{\mathbf{B}_{1/2}} |\nabla u_i|^2\,dx + \int_{\mathbf{B}\sim\mathbf{B}_{1/2}} \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 dx \\ &= \int_{\mathbf{B}_{1/2}} |\nabla u_i|^2\,dx + 4\pi = \int_{\mathbf{B}_{1/2}} |\nabla u_i|^2\,dx + 2\int_{1/2}^1 \mathrm{Area}\, u_i(\partial\mathbf{B}_r)\,dr \\ &= \int_{\mathbf{B}_{1/2}} |\nabla u_i|^2\,dx + \int_{1/2}^1 \int_{\partial\mathbf{B}_r} |\nabla_{\tan} u_i|^2\,dS\,dr \le \int_{\mathbf{B}} |\nabla u_i|^2\,dx \end{aligned} \tag{5}$$

because $\mathrm{degree}(u_i|\partial\mathbf{B}_r) = 1$ for $\frac{1}{2} \le r \le 1$ and i large enough to guarantee that $\mathrm{degree}(\varphi_i) = 1$ and $a_i \in \mathbf{B}_{1/2}$. Since $w_i|\mathbf{S}^2 \equiv \mathrm{id}_{\mathbf{S}^2}$ and w_i has a single degree one singularity at a_i, we may also infer, using [2], the energy lower bound

$$\int_{\mathbf{B}} |\nabla w_i|^2\,dx \ge e\int_{\mathbf{S}^2} |\omega - a_i|\,dS\omega \ge 8\pi + c_8|a_i|^2.$$

Combining this inequality with (4) and (5), we obtain the estimate

$$|a_i| \leq c_4 \delta_i^{1/2},$$

which contradicts the original choice of u_i.

3. Proof of the Perturbation Lemma

By Lemma 2.2, u has, for δ sufficiently small, a single singularity a with $|a| \leq c_4 \delta^{1/2}$. From [2] we infer that

$$\limsup_{r \downarrow 0} r^{-1} \int_{\mathbf{B}_r(a)} |\nabla u|^2 \, dx = 8\pi,$$

and that any tangent map [10] of u at a must be in the form $\theta(x/|x|)$ for some rotation θ of $\mathbf{R}^3$. Theorem 1 of [12] implies that θ is unique and that

$$A(r) = (\|(\partial/\partial r)v(r\cdot)\|_{C^1(\mathbf{S}^2)} + \|v(r\cdot) - \theta\|_{C^2(\mathbf{S}^2)}) \to 0 \quad \text{as } r \to 0,$$

where $v(x) = u(x+a)$. Moreover, by the integrability of the Jacobi fields of $x/|x|$ [3], [12, Theorem 1] may here be replaced by the more elementary alternative argument of [13, §6]. The latter implies, for some positive $\beta < 1$ and δ sufficiently small, the estimates

$$\|\theta - \mathrm{id}_{\mathbf{R}^3}\| \leq c_q \varepsilon \quad \text{and} \quad A(r) \leq c_q \varepsilon r^\beta, \tag{6}$$

where $\varepsilon = \|v(x) - x/|x|\|_{C^2(\mathbf{B}_{2/3} \sim \mathbf{B}_{1/3})}$. In particular, for any positive $\alpha < \beta$,

$$\left\| u - \theta\left(\frac{x-a}{|x-a|}\right) \right\|_{C^\alpha(\mathbf{B}_{1/2})} \leq c_{10} \varepsilon. \tag{7}$$

Since, by (6) and the definitions of δ and ε,

$$\begin{aligned} &\left\| u - \theta\left(\frac{x-a}{|x-a|}\right) \right\|_{\mathrm{Lip}\, \partial(\mathbf{B} \sim \mathbf{B}_{1/2})} \\ &\qquad \leq c_{11}(\varepsilon + |a|) + \delta + \left\| u - \theta\left(\frac{x-a}{|x-a|}\right) \right\|_{\mathrm{Lip}\, \partial(\mathbf{B}_{1/2})} \leq c_{12}(\varepsilon + \delta^{1/2}), \end{aligned}$$

standard interior and boundary estimates also imply that

$$\left\| u - \theta\left(\frac{x-a}{|x-a|}\right)\right\|_{C^{1/2}(\mathbf{B}\sim\mathbf{B}_{1/2})} \le c_{13}(\varepsilon + \delta^{1/2}) \tag{8}$$

for δ sufficiently small. By (7) and (8) it only remains to estimate ε.

For $1/8 \le |x| \le 1$, we find by integration radially and Schwarz's inequality that

$$\begin{aligned}\left|u(x) - \frac{x}{|x|}\right| &\le \left|\varphi\left(\frac{x}{|x|}\right) - \frac{x}{|x|}\right| + \left|\int_{|x|}^1 \left(\frac{\partial u}{\partial \rho}\right)\left(\rho\frac{x}{|x|}\right) d\rho\right| \\ &\le \|\varphi - \mathrm{id}\,\|_{L^\infty} + \left(\int_{1/8}^1 \left(\frac{\partial u}{\partial \rho}\right)^2 \left(\rho\frac{x}{|x|}\right) d\rho\right)^{1/2}. \\ &\le \|\varphi - \mathrm{id}\,\|_{L^\infty} + \left(64\int_{1/8}^1 \left(\frac{\partial u}{\partial \rho}\right)^2 \left(\rho\frac{x}{|x|}\right) \rho^2\, d\rho\right)^{1/2}.\end{aligned}$$

Squaring and integrating give

(9)
$$\begin{aligned}\int_{\mathbf{B}\sim\mathbf{B}_{1/8}} \left|u(x) - \frac{x}{|x|}\right|^2 dx &\le 8\pi\|\varphi - \mathrm{id}\,\|_{L^\infty}^2 + 512\pi\int_{\mathbf{B}\sim\mathbf{B}_{1/8}} \left(\frac{\partial u}{\partial \rho}\right)^2 dx \\ &\le 8\pi^3\|\varphi - \mathrm{id}\,\|_{\mathrm{Lip}}^2 + 512\pi\int_{\mathbf{B}\sim\mathbf{B}_{1/8}} \left(\frac{\partial u}{\partial \rho}\right)^2 dx,\end{aligned}$$

because $\varphi(\omega) = \omega$ for some $\omega \in \mathbf{S}^2$. To estimate the latter term we will use the monotonicity equality [10, 2.4], [9, 4.1] on the annulus $\mathbf{B} \sim \mathbf{B}_{1/8}$ and on the ball $\mathbf{B}_{1/8-|a|}(a) \subset \mathbf{B}_{1/8}$ to find that

$$\begin{aligned}\int_{\mathbf{B}\sim\mathbf{B}_{1/8}} \left(\frac{\partial u}{\partial \rho}\right)^2 dx &\le \int_{1/8}^1 \left(2\rho^{-1}\int_{\partial\mathbf{B}_\rho} \left(\frac{\partial u}{\partial \rho}\right)^2 dS\right) d\rho \\ &= \int_{\mathbf{B}} |\nabla u|^2\, dx - 8\int_{\mathbf{B}_{1/8}} |\nabla u|^2\, dx \\ &\le \int_{\mathbf{B}} |\nabla u|^2\, dx - (1-8|a|)\left[\frac{8}{1-8|a|}\right]\int_{\mathbf{B}_{1/8-|a|}(a)} |\nabla u|^2\, dx \\ &\le \int_{\mathbf{B}} |\nabla u|^2\, dx - (1-8|a|)\lim_{\rho\to 0}\rho^{-1}\int_{\mathbf{B}_\rho(a)} |\nabla u|^2\, dx \\ &\le \int_{\mathbf{B}} |\nabla u|^2\, dx - (1-8|a|)8\pi.\end{aligned}$$

From this inequality, (9), (4), and 2.2 we now deduce that

$$\int_{\mathbf{B}\sim\mathbf{B}_{1/8}} \left|u(x) - \frac{x}{|x|}\right|^2 dx \le 8\pi^3\delta^2 + c_5\delta^2 + 64\pi c_4\delta^{1/2}.$$

Hence, by standard interior estimates [12, §1],

$$\left\| u(x) - \frac{x}{|x|} \right\|_{C^2(\mathbf{B}_{3/4} \sim \mathbf{B}_{1/4})} \leq c_{14}\delta^{1/4}$$

for δ sufficiently small. Thus,

$$\begin{aligned} \varepsilon &= \left\| v(x) - \frac{x}{|x|} \right\|_{C^2(\mathbf{B}_{2/3} \sim \mathbf{B}_{1/3})} \\ &\leq \left\| u(x) - \frac{x}{|x|} \right\|_{C^2(\mathbf{B}_{3/4} \sim \mathbf{B}_{1/4})} + \left\| \frac{x}{|x|} - \frac{x-a}{|x-a|} \right\|_{C^2(\mathbf{B}_{2/3} \sim \mathbf{B}_{1/3})} \\ &\leq c_{15}\delta^{1/4} + c_{16}|a| \leq c_{17}\delta^{1/4}, \end{aligned}$$

which, along with (7) and (8), completes the proof.

3. Proof of the Stability Theorem

Assume that u_i is a sequence of energy minimizing maps from Ω to $\mathbf{S}^2$ which have Lipschitz boundary values $\varphi_i = u_i|\partial\Omega$ with

$$\|\varphi_i - \psi\|_{\mathrm{Lip}} \rightarrow \quad \text{as } i \rightarrow \infty.$$

First we note that the energies $\int_\Omega |\nabla u_i|^2\,dx$ are bounded. In fact the harmonic vectors $h_i : \overline{\Omega} \rightarrow \mathbf{R}^3$ with $h_i|\partial\Omega = \varphi_i$ clearly have bounded energies. The same holds for the maps w_i obtained by suitably projecting, as in [9, 6.2], the h_i onto $\mathbf{S}^2$. Moreover,

$$\int_\Omega |\nabla u_i|^2\,dx \leq \int_\Omega |\nabla w_i|^2\,dx$$

by minimality.

Next we reason as in the proof of the Stability Lemma and use the uniqueness of v to deduce that the functions u_i converge strongly in H^1 to v. By [10, Theorem II], [11, 2.7] the set of singularities of v is a finite (possibly empty) subset $A = \{a_1, a_2, \cdots, a_k\}$ of Ω.

For each singularity a_j, we infer from [10], [13], and [2] that there is a unique tangent map of v at a_j in the form $\theta_j(\frac{x}{|x|})$ for some rotation θ_j of $\mathbf{R}^3$. As in the proof of the Perturbation Lemma, we may by [12, Theorem 1] or [3], [13, §6] find a fixed positive constants τ and γ (depending on v) so that

$$\tau < \frac{1}{2} \min_j \operatorname{dist}(a_j, (A \sim \{a_j\}) \cup \partial\Omega\},$$

$$\left\| v - \theta_j \left(\frac{x - a_j}{|x - a_j|} \right) \right\|_{C^2(\mathbf{B}_{3\tau/2}(a_j) \sim \mathbf{B}_{\tau/2}(a_j))} \leq \frac{1}{2}\delta_0, \tag{10}$$

$$\left\| v - \theta \left(\frac{x - a_j}{|x - a_j|} \right) \right\|_{C^\alpha(\mathbf{B}_\rho(a_j))} \leq \gamma \rho^{1/2\alpha}. \tag{11}$$

The uniqueness of v and a compactness argument now show that

$$\delta_i = \|u_i - v\|_{\mathrm{Lip}(\partial \mathbf{B}_\tau(a_j))} \to 0 \quad \text{as } i \to \infty. \tag{12}$$

As in the proof of the Stability Lemma, the uniform boundary regularity Lemma 2.1, and the interior regularity theory now gives the estimate

$$\|u_i - v\|_{C^\alpha(\Omega \sim \bigcup_{j=1}^k \mathbf{B}_\tau(a_j))} \leq c_{18}\delta_i \quad \text{for } \delta_i \leq c_{18}^{-1}. \tag{13}$$

By (10) and (12) we may, for i sufficiently large, translate,rotate, and scale to apply the Perturbation Lemma. We obtain single points $a_{ji} \in \mathbf{B}_\tau(a_j)$ and rotations θ_{ji} of $\mathbf{R}^3$ so that

$$|a_{ji} - a_j| + \|\theta_{ji} - \theta_j\| + \left\| u_i - \theta_{ji} \left(\frac{x - a_{ji}}{|x - a_{ji}|} \right) \right\|_{C^\alpha(\mathbf{B}_\tau(a_j))} \leq c_{19}\delta_i^{1/4}. \tag{14}$$

We now let

$$\tau_i = \max_j |a_{ji} - a_j|^{1/2} \leq c_{20}^{1/2}\delta_i^{1/8}, \tag{15}$$

and define $\eta_i : \Omega \to \Omega$ so that

$$\begin{aligned} &\eta_i(x) = x \quad \text{on } \Omega \sim \bigcup_{j=1}^k \mathbf{B}_{\tau_i}(a_j), \\ &\eta_i(x) = \lambda_{ji}(x)\xi_{ji}(x) + [1 - \lambda_{ji}(x)]x \quad \text{on } \mathbf{B}_{\tau_i}(a_j), \end{aligned}$$

where

$$\begin{aligned} &\xi_{ji}(x) = \theta_j^{-1}\theta_{ji}(x - a_{ji}) + a_j, \\ &\lambda_{ji} \in C^\infty(\Omega, [0,1]), \qquad |\nabla \lambda_{ji}| \leq \tau_i/3, \\ &\lambda_{ji} \equiv 1 \quad \text{on } \mathbf{B}_{\tau_i/2}(a_j), \quad \text{and} \quad \lambda_{ji} \equiv 0 \quad \text{on } \Omega \sim \mathbf{B}_{\tau_i}(a_j). \end{aligned}$$

Using (11), (13), (14), and (15), we now conclude that

$$\|\eta_i - \mathrm{id}_\Omega\|_{\mathrm{Lip}} \leq c_{20}\delta_i^{1/8} \quad \text{and} \quad \|u_i - v \circ \eta_i\|_{C^\beta} \leq c_{20}\delta_i^{1/4}$$

for any positive $\beta < \frac{1}{2}\alpha$ and i sufficiently large.

4. Remark

An easy consequence of the Stability Theorem is that

the number of singularities of an energy-minimizing map $u \in H^1(\Omega, \mathbf{S}^2)$ is bounded by a constant C, depending only on Ω and $\|u|\partial\Omega\|_{\mathrm{Lip}}$.

To see this one infers by scaling and 2.1 that there is, for each positive L, a positive number $\delta = \delta(\Omega, L)$ so that any energy-minimizing map $u \in H^1(\Omega, \mathbf{S}^2)$ with $\|u|\partial\Omega\|_{\mathrm{Lip}} \leq L$ has no singularities on $\Omega \sim \Omega_\delta$ where $\Omega_\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}$. On the other hand, any sequence of energy-minimizing maps $u_i \in H^1(\Omega, \mathbf{S}^2)$ has, by the universal interior energy bound of [6, 3.1], a subsequence which converges in H^1_{loc} to a map $u_0 \in H^1_{\mathrm{loc}}(\Omega, \mathbf{S}^2)$ which minimizes energy on each compact subset of Ω. In particular u_0 has only a finite number of singularities in Ω_δ. By the Stability Theorem, u_i has, for i sufficiently large, precisely the same number of singularities in Ω_δ.

5. An example of nonuniqueness

There exists a smooth function $\varphi : \mathbf{S}^2 \to \mathbf{S}^2$ which serves as boundary data for two energy minimizing maps from $\mathbf{B}^3$ to $\mathbf{S}^2$, one having no singularities and one having at least two singularities.

Here one may choose as in [8] a smooth function $\psi : \mathbf{S}^2 \to \mathbf{S}^2$ of degree 0 so that any energy-minimizing map with boundary data ψ must have at least two singularities. There is a smooth family of smooth functions $\varphi_t : \mathbf{S}^2 \to \mathbf{S}^2$ for $0 \leq t \leq 1$, so that $\varphi_1 \equiv \psi$ and φ_0 is a constant function. For all sufficiently small positive t, any energy-minimizer with boundary data φ_t must, by the regularity theory [10], [11], be free of singularities. Let

$$\tau = \sup\{t\colon \text{ every energy-minimizer with boundary data } \varphi_t \text{ has no singularities}\}.$$

Then $0 < \tau < 1$. Choose a sequence $s_i \uparrow \tau$ and singularity-free, energy-minimizing maps $f_i \in H^1(\mathbf{B}, \mathbf{S}^2)$ with $f_i|\mathbf{S}^2 = \varphi_{t_i}$. Also choose a sequence $t_i \downarrow \tau$ (possibly all $t_i = \tau$) and singular energy-minimizing maps $g_i \in H^1(\mathbf{B}, \mathbf{S}^2)$ with $g_i|\mathbf{S}^2 = \varphi_{t_i}$. Passing to subsequences, without changing notation, f_i and g_i converge in $H^1(\mathbf{B}, \mathbf{S}^2)$ to energy-minimizing maps f and g with $f|\mathbf{S}^2 = \varphi_\tau = g|\mathbf{S}^2$.

The map f has no singularities. In fact, any possible singularity a must, by [10], [11], occur on the interior. For a small positive ρ, $f|\partial\mathbf{B}_\rho(a)$ is smooth and of nonzero degree. The same would hold for f_i, for i sufficiently large, by the interior regularity theory. But this would contradict the continuity of $f_i|\overline{\mathbf{B}_\rho(a)}$.

Finally the map g must have (interior) singularities because otherwise the regularity theory would imply the smoothness of g_i, for i sufficiently large.

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