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LOWER BOUNDS FOR λ_1 ON A FINITE-VOLUME HYPERBOLIC MANIFOLD

JOZEF DODZIUK & BURTON RANDOL

1. Introduction

Suppose M is an n-dimensional hyperbolic manifold having finite volume V, and denote by λ_1 the first positive element in the discrete spectrum for the problem $\Delta f + \lambda f = 0$ on M. If M is compact and $n = 2$, it is known [13] that there exists a constant $c > 0$, depending only on the genus of M, such that $\lambda_1 \geqslant cl$, where l is the total length of a smallest (in the sense of total length) collection of simple closed geodesics separating M. If M is compact and $n \geqslant 3$, it is known [12] that there exists a constant $c > 0$, depending only on n, and such that $\lambda_1 \geqslant cV^{-2}$. The 2-dimensional infinite volume case is discussed in [10].

In this paper we will discuss lower bounds for λ_1 in the finite volume case. Our results agree with those of [13] and [12] in cases covered by those papers. Our main purpose, however, is to illustrate a simple and general approach to this question, which depends on lower bounds for the first Dirichlet eigenvalues of some of the basic building blocks for hyperbolic manifolds.

In more detail, the Margulis lemma [2], [3], [6], [7], [9], [11], [14], implies that there exists $\varepsilon(n) > 0$, such that M is the union of two not necessarily disjoint subsets A and B, a thick and a thin part, for which:

1. A is not empty, and the injectivity radius at each point of A is greater than $\varepsilon(n) > 0$, so $V > v_1(n) > 0$. For $n \geqslant 3$, A is connected.

2. B is either empty or is a disjoint union of pieces, each of which is either:

(a) A closed embedded tubular neighborhood N of a simple closed geodesic γ, for which we may assume, by taking $\varepsilon(n)$ small enough, that the radius of N, i.e., the distance of any point on ∂N to γ, is greater than 1, or

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(b) of the form $[a, \infty) \times F$, where F is a compact flat $(n-1)$-dimensional manifold. The metric on such a piece is $y^{-2}((dx_1)^2 + \cdots + (dx_{n-1})^2 + (dy)^2)$, where $(x_1, \cdots, x_{n-1}) \in F$ and $y \in [a, \infty)$.

In both cases (a) and (b), we may assume that the volume of a piece is greater than $v_2(n) > 0$.

Remark. In the class of finite-volume M, the subset B contains pieces of the type described in (b) if and only if M is not compact.

Definition. We will call the pieces of B, if present, the thin components of M.

Definition. Suppose T is a thin component of M of the compact type, based on the geodesic γ. Denote by ρ the radial variable of T, i.e., the distance of a point of T from γ, and suppose the radius of T is R. We will call the subset of T for which $R - 1 \leqslant \rho \leqslant R$ the shell of T. If T is of the noncompact type, we will call the subset of T given by $[a, a+1] \times F$ the shell of T. Note that if T is of the compact type and $n = 2$, the shell consists of two components, both of which are topologically cylinders.

The $\varepsilon(n)$ of the Margulis lemma can be chosen so that the shell of each component of B is also contained in A, and we will henceforth suppose that this is the case. Additionally, in what follows, all functions will be real-valued. We begin with three lemmas which deal with various aspects of the Dirichlet problem for thin components.

Lemma 1. *Suppose T is a thin component of M. Then $\lambda_1(T) > ((n-1)/2)^2$, where $\lambda_1(T)$ is the first Dirichlet eigenvalue for T (the set of test functions over which the Rayleigh quotient is minimized consists of continuous piecewise C^1 functions which vanish on ∂T and whose L^2 norm over T is* 1).

Proof ([4], [8]). Assume initially that T is of the compact type, and is based on the geodesic γ. Introduce Fermi coordinates for T based on γ. A point $x = (t, \rho, \sigma)$ of T is then specified by its position t on γ, its distance ρ from γ, and a point $\sigma \in S^{n-2}$. The metric in these coordinates is given by

$$(ds)^2 = (\cosh^2 \rho)(dt)^2 + (d\rho)^2 + (\sinh^2 \rho)(d\sigma)^2,$$

and the volume element is $(\sinh^{n-2} \rho \cosh \rho)\, dt\, d\rho\, d\sigma$ ([2], [4]). Suppose $f \not\equiv 0$ is a function which vanishes on the boundary of T.

Now $(\int_T f^2)^2 = (\int_{S^{n-2}} d\sigma \int_0^l dt \int_0^R f^2 \sinh^{n-2} \rho \cosh \rho\, d\rho)^2$, where $l =$ length(γ) and R is the radius of T.

Integrating the inner integral of the last expression by parts, and using the fact that f vanishes on ∂T, we find that

$$\left| \int_0^R f^2 \sinh^{n-2} \rho \cosh \rho\, d\rho \right| = \left| (2/(n-1)) \int_0^R f f' \sinh^{n-1} \rho\, d\rho \right|,$$

where the derivative is taken with respect to ρ. Since $\sinh\rho < \cosh\rho$, the last quantity is less than $(2/(n-1))\int_0^R |ff'|\sinh^{n-2}\rho\cosh\rho\, d\rho$. I.e.,

$$\left(\int_T f^2\right)^2 < (2/(n-1))^2\left(\int_{S^{n-2}} d\sigma \int_0^l dt \int_0^R |ff'|\sinh^{n-2}\rho\cosh\rho\, d\rho\right)^2$$
$$= (2/(n-1))^2\left(\int_T |ff'|\right)^2.$$

By Schwarz's inequality, $(\int_T |ff'|)^2 \leqslant (\int_T f^2)(\int_T (f')^2)$, so since $|f'|^2 \leqslant |\nabla f|^2$, we conclude that $((n-1)/2)^2(\int_T f^2) < \int_T |\nabla f|^2$, which proves the result for the case of compact T.

If T is of the noncompact type, we use the fact that λ_1 is the infimum of the Rayleigh quotient taken over all C^∞ functions with compact support in the interior of T, and then invoke McKean's argument [8], integrating by parts in the y-direction, to show that any such Rayleigh quotient is greater than $((n-1)/2)^2$.

Lemma 2 (*approximate version of Lemma* 1). *For each n, there exists a constant $\delta > 0$ such that if T is a thin component of M with shell S, and if f is a function defined on T and satisfying*:

(a) $\int_T |f|^2 = c > 0$,

(b) $\int_S |\nabla f|^2 < \delta c$,

(c) $\int_S |f(x)|^2 < \delta c$,

then $\int_T |\nabla f|^2 > (c/2)((n-1)/2)^2$.

Proof. Assume initially that T is of the compact type, and let $R > 1$ denote the radius of T. Consider the function F on T which is defined in Fermi coordinates by

$$\begin{aligned} F(t,\rho,\sigma) &= f(t,\rho,\sigma) && (\rho \leqslant R-1) \\ &= (R-\rho)f(t,\rho,\sigma) && (R-1 \leqslant \rho \leqslant R). \end{aligned}$$

By (c), if δ is small enough, most of the contribution to the integral in (a) comes from $T - S$. Thus, for small δ, $\int_T F^2 > (3/4)c$, so it follows from Lemma 1 that $\int_T |\nabla F|^2 > (3/4)((n-1)/2)^2 c$.

On the other hand,

$$\int_S |\nabla F|^2 = \int_S |(R-\rho)\nabla f + f\nabla(R-\rho)|^2$$
$$\leqslant \left[\left(\int_S |\nabla f|^2\right)^{1/2} + \left(\int_S |f|^2\right)^{1/2}\right]^2$$
$$= O(\delta c) \quad \text{by (b) and (c)}.$$

Thus, for sufficiently small δ,

$$\int_{T-S} |\nabla F|^2 = \int_{T-S} |\nabla f|^2 > (c/2)((n-1)/2)^2,$$

which clearly implies the lemma.

If T is of the noncompact type, an essentially identical argument leads to the desired conclusion.

Lemma 3. *Suppose $n = 2$ and T is a thin component of the compact type having radius R, and with associated closed simple geodesic γ of length l. T is then topologically a cylinder. Denote by Γ_1 and Γ_2 the two boundary components of T, which are topologically circles. Suppose f is a function on T for which*

$$\min_{(x,x^*)\in\Gamma_1\times\Gamma_2}|f(x)-f(x^*)| \geqslant c > 0,$$

where x^* is the reflection of x through γ. Then $\int_T |\nabla f|^2 \geqslant c^2 l/4$.

Proof. In the disk model of H^2, with cartesian coordinates x and y, lift γ to a segment $\bar\gamma = \{(x, y)|0 \leqslant x \leqslant d;\ y = 0\}$, which projects to γ in a 1-1 manner except at the endpoints. T then lifts to a four-sided figure $\overline{T}$, whose intersection with the domain defined by $0 \leqslant x \leqslant d$ we will denote by Q. Q is a quadrilateral figure bounded on its sides by two vertical Euclidean segments, and on its top and bottom by arcs of the locus of points at hyperbolic distance R from $\bar\gamma$. Denoting by F the lift of f to $\overline{T}$, we have

$$\int_T |\nabla f|^2 = \int_{\overline{T}} |\nabla F|^2 \geqslant \int_Q |\nabla F|^2,$$

where the integrals are taken with respect to the hyperbolic volume element. Now in two dimensions, these integrals are conformal invariants and hence can be computed with respect to the Euclidean volume element. Since $d \sim \frac{1}{2}l$, the lemma now follows from Fubini's theorem, the fact that $(\partial F/\partial x)^2 + (\partial F/\partial y)^2 \geqslant (\partial F/\partial y)^2$, and the fact that if $-1 < a < b < 1$ and $|F(a) - F(b)| \geqslant c$, then $\int_a^b |F'(x)|^2\,dx > c^2/2$, since

$$c^2 \leqslant \left(\int_a^b F'(x)\,dx\right)^2 \leqslant \left(\int_a^b 1^2 dx\right)\left(\int_a^b |F'(x)|^2\,dx\right).$$

2. Lower bounds for λ_1

Case 1: $n \geqslant 3$. The constants which arise in our discussion of this case depend only on n. Now if $n \geqslant 3$, the thick part A of M is connected. Denote by φ a normalized eigenfunction corresponding to λ_1. I.e., $\Delta\varphi + \lambda_1\varphi = 0$, $\int_M |\varphi|^2 = 1$, and $\int_M \varphi = 0$.

Suppose B is an embedded ball in M of a given radius, and suppose $\int_B |\nabla\varphi|^2 = \int_B |d\varphi|^2 < \varepsilon$, where d is the exterior derivative operator. Then since $\Delta^k(d\varphi) = d(\Delta^k\varphi) = (-\lambda_1)^k d\varphi$, where Δ is the Laplacian on forms, it follows that $\sum_{k=0}^N \|\Delta^k(d\varphi)\|_2 \leqslant c_1\varepsilon^{1/2}$ for a fixed N, where the L^2 norms are taken over B. It thus follows from the Sobolev lemma that $|d\varphi| = |\nabla\varphi| \leqslant c_2\varepsilon^{1/2}$ for all points of $B/2$, the ball concentric with B of half the radius of B. Now it is easy to see that any two points of A can be connected by a chain $B_1/2,\cdots, B_k/2$ of overlapping balls such that $k \leqslant c_3V$, with the B_j's embedded balls of identical radius, depending on n, for which each B_j intersects at most a finite number β of other B_j's. It follows that

$$\beta\lambda_1 = \beta\int_M |\nabla\varphi|^2 \geqslant \sum_j \int_{B_j} |\nabla\varphi|^2.$$

But

$$k^{1/2}\left(\sum_j \int_{B_j} |\nabla\varphi|^2\right)^{1/2} \geqslant \sum_j\left(\int_{B_j} |\nabla\varphi|^2\right)^{1/2} \geqslant (c_2)^{-1}\sum_j \|\nabla\varphi\|_{\infty, B_j/2},$$

so since $k \leqslant c_3V$, it follows that if $\alpha > 0$ is a small constant, and if $\lambda_1 \leqslant \alpha V^{-2}$, then the oscillation of φ on A is $\leqslant c_4(\alpha/V)^{1/2}$. Thus, if $\sup_{x\in A}|\varphi(x)| > c_4(\alpha/V)^{1/2}$, then φ must be of one sign on A, which we may take to be positive. If the thin part B is empty, this is an immediate contradiction, since $\int_M \varphi = 0$. On the other hand, if B is not empty, then it follows that $\varphi(x) < 0$ for some $x \in B$. Denote by T the component of B containing x. Since $\varphi > 0$ on ∂T, we conclude that φ is a Dirichlet eigenfunction for a subset of T, and it then follows from Lemma 1 and the domain monotonicity of Dirichlet eigenvalues that $\lambda_1 > ((n-1)/2)^2$.

If $\sup_{x\in A}|\varphi(x)| \leqslant c_4(\alpha/V)^{1/2}$, most of $\int_M \varphi^2 = 1$ comes from the thin components, and it follows easily that $\int_T \varphi^2 > c_5\,\mathrm{vol}(T)/V$ for some thin component T. Thus, if we take $c = \mathrm{vol}(T)/V$ and $\delta = c_6\alpha$ in Lemma 2, we find that $\int_T |\nabla\varphi|^2 \geqslant \frac{1}{2}\mathrm{vol}(T)/V \geqslant c_7V^{-1}$, which for small α contradicts the assumption that $\lambda_1 \leqslant \alpha V^{-2}$. We are thus led to the following theorem:

Theorem 1 (*cf.* [12]). *If M is as above and $n \geqslant 3$, then there exists $c(n)$ such that $\lambda_1 \geqslant c(n)V^{-2}$.*

Case 2: $n = 2$. In our discussion of this case, we will assume that the volume V of M is fixed, and the constants which arise will be understood to depend on V.

Now from our point of view the only essential difference between $n = 2$ and $n \geqslant 3$ is that the thick part A may not be connected. If A is connected, the argument of Case 1 leads to the estimate of Theorem 1. Assume that A is not connected. Then the argument of Case 1 shows that if λ_1 is small, the oscillation of a normalized eigenfunction φ over the set A cannot be too small or we will arrive at a contradiction. I.e., if λ_1 is small, the oscillation of φ over A must be $\geqslant c_8 > 0$. At the same time, the argument used in Case 1 also shows that the oscillation of φ over each component of A must be small if λ_1 is small. It follows easily from these facts that there exists $c_9 > 0$ such that if $\lambda_1 \leqslant c_9$, then there is a collection $T_1, \cdots, T_k$ of thin components of the compact type, with associated closed simple geodesics $\gamma_1, \cdots, \gamma_k$ separating M, and for which the oscillation in the sense of Lemma 3 of φ across each T_j is $\geqslant c_{10} > 0$. Assuming $\lambda_1 \leqslant c_9$, it then follows from Lemma 3, adding up the $\int_{T_j} |\nabla\varphi|^2$'s, that $\lambda_1 \geqslant \frac{1}{4}c_{10}^2(l_1 + \cdots + l_k)$, where $l = \text{length}(\gamma_j)$. We are therefore led to the following theorem:

Theorem 2 (*cf.* [13], [10]). *There exists $c = c(V) > 0$ such that if M is as above and $n = 2$, then $\lambda_1 \geqslant cL$, where L is the minimal length of a separating chain of simple closed geodesics. (If A is connected, then L cannot be small. Also, L can be bounded above in terms of V (cf.* [1]).)

We conclude by showing that in the 2-dimensional case the methods of this paper can also be applied to derive counterparts of the estimates for higher eigenvalues discussed in [13] for the compact 2-dimensional case.

Let $n = 2$ and let M be as before. Then M is topologically a compact surface of genus g with p punctures, and $V = 2\pi(2g + p - 2)$. For $k \geqslant 1$, denote by L_k the minimal total length of a chain C of disjoint closed simple geodesics separating M into $k + 1$ components. If no such chain exists, set $L_k = 1$. (It is well known that no such chain exists for $k > 2g + p - 3$.)

Theorem 3 (*cf.* [13]). *There exists $c > 0$ such that $\lambda_k \geqslant cL_k$.*

By the minimax principle [5, pp. 409–410], Theorem 3 is a direct consequence of the following lemma:

Lemma 4. *Assume $k \leqslant 2g + p - 3$. Let P be any one of the pieces $M_1, \cdots, M_{k+1}$ into which C separates M, and let μ_1 be the eigenvalue for the problem $\Delta\Psi + \mu\Psi = 0$ on P, subject to $\int_P \Psi = 0$ and zero Neumann data on the boundary geodesics of P. Then there exists $c > 0$ such that:*

1. *If there exists a chain of disjoint closed simple geodesics separating P, then $\mu_1 \geqslant cL_{k+1}$.*

2. *If there does not exist such a chain, then $\mu_1 \geqslant c$. (Note that such a chain cannot exist for $k = 2g + p - 3$.)*

Proof of Lemma 4. *Case* 1. Let L be the length of a chain of the above type which separates P. Then it is easy to see that $L \geqslant m^{-1}L_k$, where m is the number of geodesics in the chain C corresponding to L_k, since otherwise L would be smaller than one of the geodesics γ in C, and replacement of γ by L

would result in a smaller value of L_k. In particular, $(m + 1)L \geqslant L_k + L \geqslant L_{k+1}$, or $L \geqslant c_{11}L_{k+1}$, since it is well known that $m \leqslant 3g + p - 3$. Now let $\bar{P}$ be the double of P along its boundary geodesics, and extend the Neumann eigenfunction Ψ on P to $\bar{P}$ by reflection. Using the methods and notation of the proof of Theorem 2, applied to Ψ and $\bar{P}$, it is then evident that we can ignore any cylinder T_j associated to a boundary geodesic of P, since the extended Ψ has no oscillation across such a T_j. The proof of Theorem 2 then shows that there exists $c' > 0$ such that $\mu_1 \geqslant c'L \geqslant cL_{k+1}$.

Case 2. If no separating chain of the required type exists, then the proof of Theorem 2, coupled with the fact that we can ignore any T_j associated to a boundary goedesic of P, completes the proof of the lemma.

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