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GENERIC SUBMANIFOLDS OF AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

U-HANG KI & JIN SUK PAK

Dedicated to Professor Kentaro Yano on his 70th birthday

0. Introduction

Recently several authors have studied generic submanifolds (anti-holomorphic submanifolds) immersed in Kaehlerian manifolds by using the method of Riemannian fibre bundles ([3], [4] and [8] etc.).

The purpose of the present paper is to characterize generic submanifolds of an even-dimensional Euclidean space.

In §1, we recall fundamental properties and structure equations for generic submanifolds immersed in an even-dimensional Euclidean space.

In §2, we prove some lemmas under the assumption that the f-structure induced on the submanifold and the second fundamental tensors commute.

In §3, we characterize generic submanifolds of an even-dimensional Euclidean space under certain conditions.

In 1971 Yano and Ishihara [6] proved the following.

Theorem A. *Let M be a complete submanifold of dimension n immersed in a Euclidean space E^m of dimension m $(1 < n < m)$ with nonnegative sectional curvature. Suppose that the normal connection of M is flat and the mean curvature vector of M is parallel in the normal bundle. If the length of the second fundamental form of M is constant in M, then M is a sphere $S^n(r)$ of dimension n, an n-dimensional plane $E^n(\subset E^m)$, a pythagorean product of the form*

$$(1) \quad S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N), \quad p_1 + \cdots + p_N = n, \ 1 < N \leqslant m - n,$$

or a pythagorean product of the form

$$(2) \quad \begin{aligned} &S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \times E^p, \\ &\qquad p_1 + \cdots + p_N + p = n, \ 1 < N \leqslant m - n, \end{aligned}$$

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where $S^p(r)$ is a p-sphere with radius r, and E^p ($\subset E^m$) a p-dimensional plane. If M is a pythagorean product of the form (1) *or* (2), *then M is of essential codimension N.*

Using a method quite similar to the one used in Lemma 1.2 of Yano and Kon [8] we can prove that the sectional curvature of an n-dimensional submanifold immersed in E^m with flat normal connection is always non-negaive if the second fundamental tensor of the submanifold is parallel. By means of Theorem A, we have

Theorem B. *Let M be a complete submanifold of dimension n immersed in a Euclidean space E^m of dimension m $(1 < n < m)$ with flat normal connection. If the second fundamental tensor of M is parallel, then M is of the same type as stated in Theorem A.*

To characterize the submanifolds we shall use Theorem B.

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1. Structure equations of generic submanifolds

Let E^{2m} be a $2m$-dimensional Euclidean space, and 0 the origin of a Cartesian coordinate system in E^{2m}, and denote by X the position vector representing a point of E^{2m} with respect to the origin. Since E^{2m} is even-dimensional, E^{2m} can be regarded as a flat Hermitian manifold, and hence there exists a tensor field F of type (1,1) with constant components such that

$$F^2 = -I, \quad (FX)\cdot(FY) = X \cdot Y \tag{1.1}$$

for any vectors X and Y, where I denotes the identity transformation, and the dot the inner product in the Euclidean space E^{2m}.

Let M be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and immersed isometrically in E^{2m} by the immersion i: $M \to E^{2m}$. Throughout this paper the indices $h, i, j, k, \cdots, t$ run over the range $\{1, 2, \cdots, n\}$, and the summation convention is used with respect to this system of indices. We identify $i(M)$ with M itself.

Put

$$X_i = \partial_i X, \quad \partial_i = \partial/\partial x^i. \tag{1.2}$$

Then X_i are n linearly independent vector fields tangent to the submanifold M. Denoting by g_{ji} the components of the induced metric tensor of M, we have

$$g_{ji} = X_j \cdot X_i, \tag{1.3}$$

since the immersion is isometric.

Denote by C_x $2m - n$ mutually orthogonal unit normals to M. Throughout this paper the indices u, v, w, x, y and z run over the range $\{n + 1, \cdots, 2m\}$, and the summation convention is used with respect to this system of indices. Therefore denoting by ∇_j the operator of the van der Waerden-Bortolotti covariant differentiation with respect to the Christoffel symbols $\{_j{}^k{}_i\}$ formed with g_{ji}, we have the equations of Gauss and Weingarten for M

$$(1.4) \qquad \nabla_j X_i = h_{ji}{}^x C_x,$$

$$(1.5) \qquad \nabla_j C_x = -h_j{}^k{}_x X_i$$

respectively, where $h_{ji}{}^x$ are the second fundamental tensors with respect to the normals C_x and $h_j{}^i{}_x = h_{jh}{}^y g^{ih} g_{yx}$, g_{yx} being the metric tensor of the normal bundle of M given by $g_{yx} = C_y \cdot C_x$, and $(g^{ji}) = (g_{ji})^{-1}$.

Since the ambient manifold E^{2m} is Euclidean, the equations of Gauss, Codazzi and Ricci for M are respectively given by

$$(1.6) \qquad K_{kji}{}^h = h_k{}^h{}_x h_{ji}{}^x - h_j{}^h{}_x h_{ki}^x,$$

$$(1.7) \qquad \nabla_k h_{ji}{}^x - \nabla_j h_{ki}{}^x = 0,$$

$$(1.8) \qquad K_{jiy}{}^x = h_{jt}{}^x h_i{}^t{}_y - h_{it}{}^x h_j{}^t{}_y,$$

where $K_{kji}{}^h$ and $K_{jiy}{}^x$ are the curvature tensors of M and the connection induced in the normal bundle respectively.

Now we consider the submanifold M of E^{2m} which satisfies

$$N_P(M) \perp F(N_P(M))$$

at each point $P \in M$, where $N_P(M)$ denotes the normal space at P. Such a submanifold M is called a generic submanifold (an anti-holomorphic submanifold), [4], [7]. From now on we consider generic submanifolds immersed in an even-dimensional Euclidean space E^{2m}. Then we can put in each coordinate neighborhood

$$(1.9) \qquad FX_j = f_j{}^i X_i - f_j{}^x C_x,$$

$$(1.10) \qquad FC_x = f_x{}^i X_i,$$

where $f_j{}^i$ is a tensor field of type (1,1) defined on M, $f_j{}^x$ a local 1-form for each fixed index x, and $f_x{}^i = f_j{}^y g^{ji} g_{yx}$.

Applying F to (1.9) and (1.10) respectively, and using (1.1) and those equations, we can easily find

$$(1.11) \qquad f_j{}^t f_t{}^h = -\delta_j^h + f_j{}^x f_x{}^h,$$

$$(1.12) \qquad f_j{}^t f_t{}^x = 0, \quad f_t{}^x f_j{}^t = 0,$$

(1.13) $$f_t{}^x f_y{}^t = \delta_y^x.$$

Moreover, (1.11) and (1.12) imply

$$f_j{}^h f_h{}^t f_t{}^i + f_j{}^i = 0,$$

and consequently M admits the so-called f-structure satisfying $f^3 + f = 0$ (see [2], [3]).

Substituting (1.9) into $(FX_j)\cdot(FX_i) = X_j \cdot X_i$ gives

(1.14) $$f_j{}^h f_i{}^k g_{hk} = g_{ji} - f_j{}^x f_i{}^y g_{xy}.$$

By putting $f_{ji} = f_j{}^t g_{ti}, f_{jx} = f_j{}^y g_{yx}$, we easily see that

(1.15) $$f_{ji} = -f_{ij}, \quad f_{jx} = f_{xj}.$$

If we apply the operator ∇_j of the covariant differentiation to (1.9) and take account of $\nabla_j F = 0$, then we obtain

$$F\nabla_j X_i = (\nabla_j f_i{}^h)X_h - f_i{}^h \nabla_j X_h - (\nabla_j f_i{}^x)C_x - f_i{}^x \nabla_j C_x.$$

Substituting (1.4) and (1.5) into the above equation yields

(1.16) $$\nabla_j f_i{}^h = h_{ji}{}^x f_x{}^h - h_j{}^h{}_x f_i{}^x,$$

(1.17) $$\nabla_j f_i{}^x = h_{jt}{}^x f_i{}^t.$$

In the same way, from (1.10) we can also obtain

(1.18) $$\nabla_j f_x{}^h = h_{jtx} f^{ht},$$

(1.19) $$f_x{}^t h_{jt}{}^y = h_j{}^t{}_x f_t{}^y,$$

where $h_{jtx} = h_j{}^i{}_x g_{it}$ and $f^{ht} = f_j{}^t g^{jh}$ because of (1.4) and (1.5).

We now consider a tensor field S of type (1,2) whose local components are given by

$$S_{ji}{}^h = [f, f]_{ji}{}^h + (\nabla_j f_i{}^x - \nabla_i f_j{}^x) f_x{}^h,$$

where

$$[f, f]_{ji}{}^h = f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h$$

is the Nijenhuis tensor formed with $f_i{}^h$. When the tensor field S vanishes identically, the f-structure induced on M is said to be *normal* (see Nakagawa [2]). But, for generic submanifolds of a Euclidean space, substituting (1.16) and (1.17) into the above equation, we find

$$S_{ji}{}^h = (h_j{}^t{}_x f_t{}^h - f_j{}^t h_t{}^h{}_x) f_i{}^x - (h_i{}^t{}_x f_t{}^h - f_i{}^t h_t{}^h{}_x) f_j{}^x.$$

Hence if $S_{ji}{}^h$ vanishes identically, we have

(1.20) $$(h_{itx} f_h{}^t + h_{htx} f_i{}^t) f_j{}^x - (h_{jtx} f_h{}^t + h_{htx} f_j{}^t) f_i{}^x = 0,$$

because f_{ji} is skew-symmetric.

Transvecting (1.20) with $f_y{}^j$ and taking account of (1.12) and (1.13), we find

$$h_{ity} f_h{}^t + h_{hty} f_i{}^t - \left(h_{jtx} f_h{}^t f_y{}^j\right) f_i{}^x = 0. \tag{1.21}$$

Taking the skew-symmetric part with respect to the indices i and h in (1.21) yields

$$\left(h_{jtx} f_h{}^t f_y{}^j\right) f_i{}^x - \left(h_{jtx} f_i{}^t f_y{}^j\right) f_h{}^x = 0,$$

which, transvected with $f_z{}^i$, gives $h_{jtz} f_h{}^t f_y{}^j = 0$ because of (1.12) and (1.13). Consequently (1.21) becomes $h_{ity} f_h{}^t + h_{hty} f_i{}^t = 0$. Thus we have

Lemma 1.1. *Let M be an n-dimensional generic submanifold of an even-dimensional Euclidean space E^{2m}. Then the f-structure induced on M is normal if and only if*

$$h_j{}^t{}_x f_t{}^i = f_j{}^t h_t{}^i{}_x. \tag{1.22}$$

Here we first notice that the condition (1.22) does not depend on the choice of mutually orthogonal unit normal vectors C_x. In fact, if we take another set of mutually orthogonal unit normals $'C_x$, then we have

$$'C_x = \sigma_x{}^y C_y, \tag{1.23}$$

where $(\sigma_x{}^y)$ is a special orthogonal matrix of degree $2m - n$. Defining the second fundamental tensor $'h_{ji}{}^x$ with respect to $'C_x$ by $\nabla_j X_i = {}'h_{ji}{}^x{}'C_x$, we have,

$$'h_{ji}{}^x = \sigma_y{}^x h_{ji}{}^y,$$

which implies our assertion.

In this point of view we shall investigate some properties concerning the f-structure induced on M satisfying (1.22) for later uses.

2. Lemmas concerning $h_j{}^t{}_x f_t{}^i = f_j{}^t h_t{}^i{}_x$.

In this section, we assume throughout that the f-structure induced on M satisfies (1.22), and the normal connection of M is flat. Then from (1.22) we have

$$h_{jt}{}^x f_i{}^t + h_{it}{}^x f_j{}^t = 0, \tag{2.1}$$

$$h_{jt}{}^x h_i{}^t{}_y - h_{it}{}^x h_j{}^t{}_y = 0, \tag{2.2}$$

which is a direct consequence of the equation (1.8) of Ricci.

Transvecting (2.1) with $f_k{}^i$ and taking account of (1.11), we obtain

$$h_{jk}{}^x - \left(h_{jt}{}^x f_y{}^t\right) f_k{}^y + h_{st}{}^x f_j{}^t f_k{}^s = 0.$$

Taking the skew-symmetric part with respect to j and k in the above equation gives

$$\left(h_{jt}{}^x f_y{}^t\right) f_k{}^y - \left(h_{kt}{}^x f_y{}^t\right) f_j{}^y = 0.$$

Transvecting this equation with $f_z^{\ h}$ we find

$$h_{ji}{}^x f_y{}^t = P_{yz}{}^x f_j{}^z, \tag{2.3}$$

where we have put

$$p_{yz}{}^x = h_{ji}{}^x f_y{}^j f_z{}^i. \tag{2.4}$$

Let $P_{yzx} = g_{wx} P_{yz}{}^w$. Then P_{yzx} is symmetric for all indices because of (1.19) and (2.3).

Next, transvecting (2.2) with $f_z{}^j$ and using (2.3), we can get

$$P_{zu}{}^x P_{yw}{}^u f_i{}^w = P_{zy}{}^u P_{uw}{}^x f_i{}^w,$$

which together with (1.13) gives

$$P_{uz}{}^x P_{yw}{}^u = P_{uw}{}^x P_{yz}{}^u, \tag{2.5}$$

because P_{yzx} is symmetric for all indices. From (2.5) it follows that

$$P_{uz}{}^x P_{yx}{}^u = P_x P_{yz}{}^x, \tag{2.6}$$

where we have put

$$P^x = g^{yz} P_{yz}{}^x. \tag{2.7}$$

Lemma 2.1. *Let M be a generic submanifold of an even-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M satisfies* (1.22), *then we have*

$$h_{jt}{}^x h_i{}^t{}_y = P_{yz}{}^x h_{ji}{}^z. \tag{2.8}$$

Proof. Differentiating (2.3) covariantly along M and using (1.17), we find

$$(\nabla_k h_{jt}{}^x) f_y{}^t + h_j{}^{tx} h_{ksy} f_t{}^s = (\nabla_k P_{yz}{}^x) f_j{}^z + P_{yz}{}^x h_{kt}{}^z f_j{}^t.$$

Taking the skew-symmetric part in the above equation and using (1.7) and (2.1), we obtain

$$2h^{stx} h_{ksy} f_{jt} = (\nabla_k P_{yz}{}^x) f_j{}^z - (\nabla_j P_{yz}{}^x) f_k{}^z + 2P_{yz}{}^x h_{kt}{}^z f_j{}^t. \tag{2.9}$$

Transvecting (2.9) with $f_w{}^j$ gives

$$\nabla_k P_{yw}{}^x = (\nabla_t P_{yz}{}^x) f_w{}^t f_k{}^z, \tag{2.10}$$

which implies

$$(\nabla_k P_{yz}{}^x) f_j{}^z = f_y{}^t (\nabla_t P_{zw}{}^x) f_k{}^w f_j{}^z,$$

since $P_{yz}{}^x = P_{zy}{}^x$. Therefore (2.9) reduces to

$$h_t{}^{sx} h_{ksy} f_j{}^t = P_{yz}{}^x h_{kt}{}^z f_j{}^t,$$

Transvecting the above equation with $f_i{}^j$ and taking account of (1.11), we obtain

$$h_i{}^{sx} h_{ksy} + h_t{}^{sx} h_{ksy} f_i{}^w f_w{}^t = P_{yz}{}^x h_{ki}{}^z + P_{yz}{}^x h_{kt}{}^z f_i{}^w f_w{}^t,$$

which together with (2.3) implies

$$h_i^{sx}h_{ksy} + P_{wz}{}^xP_{uy}{}^zf_k{}^uf_i{}^w = P_{yz}{}^xh_{ki}{}^z + P_{yz}{}^xP_{wu}{}^zf_k{}^uf_i{}^w.$$

Thus (2.8) is verified with the help of (2.5), and consequently the proof of the lemma is completed.

Lemma 2.2. *Under the same assumptions as those stated in Lemma 2.1, we have*

$$\nabla_j h^x = \nabla_j P^x, \tag{2.11}$$

where $h^x = g^{ji}h_{ji}{}^x$.

Proof. Differentiating (2.1) covariantly and using (1.16), we find

$$(\nabla_k h_{jt}{}^x)f_i{}^t + h_{jt}{}^x(h_{ki}{}^yf_y{}^t - h_k{}^t{}_yf_i{}^y) + (\nabla_k h_{it}{}^x)f_j{}^t + h_{it}{}^x(h_{kj}{}^yf_y{}^t - h_k{}^t{}_yf_j{}^y) = 0,$$

which together with (2.3) and (2.8) implies

$$(\nabla_k h_{jt}{}^x)f_i{}^t + (\nabla_k h_{it}{}^x)f_j{}^t = 0.$$

By taking the skew-symmetric part of the above equation with respect to the indices k and j, we see that

$$(\nabla_k h_{it}{}^x)f_j{}^t - (\nabla_j h_{it}{}^x)f_k{}^t = 0.$$

The last two equations together with (1.7) give $(\nabla_k h_{it}{}^x)f_j{}^t = 0$. Transvecting this equation with $f_l{}^j$ and using (1.11) we obtain

$$\nabla_k h_{il}{}^x = (\nabla_k h_{it}{}^x)f_l{}^yf_y{}^t,$$

which transvected with g^{il} thus yields

$$\nabla_k h^x = (\nabla_k h_{it}{}^x)f^{iy}f_y{}^t. \tag{2.12}$$

On the other hand, from (2.4) and (2.7) we have

$$P^x = h_{st}{}^xf^{sy}f_y{}^t.$$

If we differentiate the above equation covariantly and take account of (2.12), then we have

$$\nabla_j P^x = \nabla_j h^x + h_{st}{}^x(\nabla_j f^{sy})f_y{}^t + h_{st}{}^xf^{sy}(\nabla_j f_y{}^t).$$

Substituting (1.18) into the above equation and using (1.12), we arrive at (2.11). Hence Lemma 2.2 is proved.

3. Some characterizations of generic submanifolds

We first prove

Lemma 3.1. *Let M be a generic submanifold of an even-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on*

M satisfies (1.22), *then we have*

$$\frac{1}{2}\Delta\left(h_{ji}{}^{x}h^{ji}{}_{x}\right) = (\nabla_j\nabla_i h^x)h^{ji}{}_x + \|\nabla_k h_{ji}{}^x\|^2, \tag{3.1}$$

where $\Delta = g^{ji}\nabla_j\nabla_i$.

Proof. From the Ricci identity and (1.8) and $K_{jiy}{}^{x} = 0$:

$$\nabla_k\nabla_j h_{ih}{}^x - \nabla_j\nabla_k h_{ih}{}^x = -K_{kji}{}^t h_{th}{}^x - K_{kjh}{}^t h_{it}{}^x,$$

we obtain, in consequence of (1.7),

$$\nabla^k\nabla_k h_{ji}{}^x - \nabla_j\nabla_h h^x = K_{jt}h_i^{tx} - K_{kjih}h^{khx} \tag{3.2}$$

where K_{ji} is the Ricci tensor of M given by

$$K_{ji} = h^x h_{jix} - h_{jt}{}^x h_i{}^t{}_x. \tag{3.3}$$

Transvecting (3.2) with $h^{ji}{}_x$ and making use of (1.6), (2.8), (3.3), (2.2) and (2.7), we get

$$\left(\nabla^k\nabla_k h_{ji}{}^x\right)h^{ji}{}_x - (\nabla_j\nabla_h h^x)h^{ji}{}_x = \left(P_{yxz}P_w{}^{yz}P_u{}^{xw} - P^y P_{yxw}P_u{}^{xw}\right)h^u. \tag{3.4}$$

Consequently (3.4) reduces to

$$\left(\nabla^k\nabla_k h_{ji}{}^x\right)h^{ji}{}_x = (\nabla_j\nabla_i h^x)h^{ji}{}_x$$

because of (2.6).

On the other hand, we have by definition

$$\frac{1}{2}\Delta\left(h_{ji}{}^x h^{ji}{}_x\right) = \left(\nabla^k\nabla_k h_{ji}{}^x\right)h^{ji}{}_x + \|\nabla_k h_{ji}{}^x\|^2.$$

Thus the last two equations give (3.1). This completes the proof of the lemma.

The mean curvature vector

$$H = \frac{1}{n}h^x C_x,$$

which is globally defined on M, is said to be parallel in the normal bundle if $\nabla_j h^x = 0$. In this case we have $\nabla_j P^x = 0$ by means of (2.11). Since $h_{ji}{}^x h^{ji}{}_x = P_x h^x$, the function $h_{ji}{}^x h^{ji}{}_x$ is constant on M. Hence (3.1) implies $\nabla_k h_{ji}{}^x = 0$, and consequently by means of Theorem B in §0 we have

Theorem 3.2. *Let M be an n-dimensional complete generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M satisfies* (1.22), *and the mean curvature vector is parallel in the normal bundle, then M is an n-sphere $S^n(r)$, an n-dimensional plane E^n ($\subset E^{2m}$), a pythagorean product of the form*

(1) $S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N)$,

$$p_1, \cdots, p_N \geqslant 1, p_1 + \cdots + p_N = n, 1 < N \leqslant 2m - n,$$

or a pythagorean product of the form

(2) $S^{p_1}(r_1) \times \cdots \times S^{p_N}(r_N) \times E^p$,

$$p_1, \cdots, p_N, p \geqslant 1, p_1 + \cdots + p_N + p = n, 1 < N \leqslant 2m - n,$$

where $S^p(r)$ is a p-sphere with radius $r > 0$ and E^p a p-dimensional plane. If M is a pythagorean product of the form (1) *or* (2), *then M is of essential codimension N.*

Combining Lemma 1.1 and Theorem 3.2 we conclude

Theorem 3.3. *Let M be an n-dimensional complete generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M is normal, and the mean curvature vector is parallel in the normal bunde, then M is of the same type as stated in Theorem* 3.2.

We next prove

Lemma 3.4. *Under the same assumptions as those stated in Lemma* 3.1, *the scalar curvature of M is constant.*

Proof. From (2.10) we have, in consequence of (2.7),

$$\nabla_i P_x = (f_x{}^t \nabla_t P_z) f_i^z \tag{3.5}$$

which implies

$$f_j^t \nabla_t P_x = 0. \tag{3.6}$$

Differentiating (3.5) covariantly and using (1.17) we find

$$\nabla_j \nabla_i P_x = \nabla_j (f_x{}^t \nabla_t P_z) f_i^z + (f_x{}^t \nabla_t P_z) h_{js}{}^z f_i^s.$$

Taking the skew-symmetric part with respect to j and i in the above equation and using (2.1) and (2.2), we obtain

$$\nabla_j (f_x{}^t \nabla_t P_z) f_i^z - \nabla_i (f_x{}^t \nabla_t P_z) f_j^z + 2(f_x{}^t \nabla_t P_z) h_{js}{}^z f_i^s = 0.$$

Transvecting the above equation with f^{ji} and using (1.11) and (1.12) give

$$(f_x{}^t \nabla_t P_z) h_{js}{}^z (-g^{sj} + f^{sy} f_y{}^j) = 0,$$

which together with (2.4) and (2.7) implies

$$(f_x{}^t \nabla_t P_z)(h^z - P^z) = 0.$$

Transvecting the above equation with f_j^x and using (1.11) and (3.6), we have $(\nabla_j P_x)(h^x - P^x) = 0$. Thus from (2.11) it follows that

$$(\nabla_j h_x)(h^x - P^x) = 0. \tag{3.7}$$

On the other hand, the scalar curvature K of M is given by

$$K = (h^x - P^x) h_x \tag{3.8}$$

because of (2.8) and (3.3). Differentiating (3.8) covariantly and taking account of (2.11) and (3.7), we can see that K is constant on M. Thus Lemma 3.4 is proved.

Finally we prove

Theorem 3.5. *Let M be an n-dimensional compact generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M satisfies* (1.22), *then M is locally symmetric.*

Proof. From (2.8) and (3.3), we have

$$K_{ji} = (h^x - P^x)h_{jix}. \tag{3.9}$$

Differentiating (3.9) covariantly and taking account of (2.11), we find

$$\nabla^k\nabla_k K_{ji} = (h^x - P^x)\nabla^k h_{jix}. \tag{3.10}$$

Substituting (1.6) and (3.9) into (3.2) and using (2.8), we obtain

$$\nabla^k\nabla_k h_{ji}{}^x - \nabla_j\nabla_i h^x = 0.$$

Thus (3.10) becomes

$$\nabla^k\nabla_k K_{ji} = (h^x - P^x)\nabla_j\nabla_i h_x = \nabla_j\nabla_i K$$

because of (2.11) and (3.8). From Lemma 3.4 it follows that $\nabla^k\nabla_k K_{ji} = 0$. Since M is compact, the identity

$$\frac{1}{2}\Delta\left(K_{ji}K^{ji}\right) = \left(\nabla^k\nabla_k K_{ji}\right)K^{ji} + \|\nabla_k K_{ji}\|^2$$

gives

$$\nabla_k K_{ji} = 0. \tag{3.11}$$

On the other hand, if we substitute (1.6) into the right-hand side of the Ricci identity:

$$\nabla_l\nabla_m K_{kjih} - \nabla_m\nabla_l K_{kjih} = K_{mlk}{}^t K_{tjih} + K_{mlj}{}^t K_{ktih} + K_{mli}{}^t K_{kjth} + K_{mlh}{}^t K_{kjit}$$

and use (2.8), then we get

$$\nabla_l\nabla_m K_{kjih} = \nabla_m\nabla_l K_{kjih},$$

which implies that

$$\nabla^l\nabla_m K_{ljih} = \nabla_m\nabla^l K_{ljih}, \tag{3.12}$$

$$\nabla^l\nabla_m K_{klih} = -\nabla_m\nabla^l K_{lkih}. \tag{3.13}$$

By means of (3.11) and the second Bianchi identity:

$$\nabla_l K_{kjih} + \nabla_k K_{jlih} + \nabla_j K_{lkih} = 0, \tag{3.14}$$

we have $\nabla^l K_{ljih} = 0$. Thus (3.12) and (3.13) reduce respectively to

$$\nabla^l\nabla_m K_{ljih} = 0, \quad \nabla^l\nabla_m K_{klih} = 0,$$

which together with (3.14) imply that

$$\nabla^l\nabla_l K_{kjih} = 0. \tag{3.15}$$

Since M is compact, from the identity:

$$\frac{1}{2}\Delta\left(K_{kjih}K^{kjih}\right) = \left(\nabla^l\nabla_l K_{kjih}\right)K^{kjih} + \|\nabla_l K_{kjih}\|^2,$$

it follows that $\nabla_k K_{ljih} = 0$ because of (3.15). This gives the proof of the theorem.

Combining Lemma 1.1 and Theorem 3.5 we have

Theorem 3.6. *Let M be an n-dimensional compact generic submanifold of a $2m$-dimensional Euclidean space E^{2m} with flat normal connection. If the f-structure induced on M is normal, then M is locally symmetric.*

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KYUNGPOOK UNIVERSITY, TAEGU, KOREA