J. DIFFERENTIAL GEOMETRY
12 (1977) 203-208

EXISTENCE OF GENERALIZED SYMMETRIC RIEMANNIAN SPACES OF ARBITRARY ORDER

OLDŘICH KOWALSKI

A Riemannian symmetric space is a Riemmanian manifold (M, g) with the following properties: for each $x \in M$ there is a (unique) isometry J_x on M such that

(a) x is an isolated fixed point of J_x,

(b) $(J_x)^2 =$ identity.

It is also easy to show the following property: for every two points $x, y \in M$ we have

(c) $J_x \circ J_y = J_z \circ J_x$, where $z = J_x(y)$.

The following is a direct generalization of the previous situation.

Definition. *A Riemannian k-symmetric space* ($k \geq 2$) is a Riemannian manifold (M, g) on which a family $\{s_x\}_{x \in M}$ of isometries exists with the following properties:

(a) Each $x \in M$ is an isolated fixed point of the corresponding s_x,

(b) $(s_x)^k =$ identity for all $x \in M$, and k is the minimum number of this property,

(c) for every $x, y \in M$, $s_x \circ s_y = s_z \circ s_x$, where $z = s_x(y)$.

In fact, Ledger and Obata [3] have proved that for every $k > 2$ there is a k-symmetric Riemannian space which is not symmetric. The purpose of this paper is to strengthen the previous result in the following sense: *for every $k > 2$ there is a k-symmetric Riemannian space which is not l-symmetric for $l = 2, \cdots, k - 1$*. (Such a Riemannian space is said to be *generalized symmetric of order k*; see [2]). In our further considerations we shall make full use of the original construction by Ledger and Obata.

1. Let $M = G/H$ be a homogeneous Riemannian space. As usual, we suppose G acting effectively on the coset space G/H. Thus the Lie group G can be considered as a group of isometries on M. Let $\pi: G \to M$ denote the canonical prejection.

Proposition 1. *Let G admit an automorphism σ such that*

(i) *$H = G^\sigma =$ the fixed point set of σ,*

(ii) *$\sigma^k =$ identity,*

(iii) *the transformation s of M determined by $\pi \circ \sigma = s \circ \pi$ is an isometry. Then M is a Riemannian k-symmetric space.*

Communicated by W. P. A. Klingenberg, May 15, 1975.

Proof. For $x \in M$ define a transformation s_x of M by the formula $s_x = g \circ s \circ g^{-1}$, where $g \in \pi^{-1}(x)$. Then s_x is independent of the choice of g. In fact, for each $h \in H$ we have $L_h \circ \sigma \circ L_{h^{-1}} = \sigma$ and $\pi \circ L_h = h \circ \pi$. Hence $(h \circ s \circ h^{-1}) \circ \pi = h \circ (s \circ \pi) \circ L_{h^{-1}} = h \circ (\pi \circ \sigma) \circ L_{h^{-1}} = \pi \circ (L_h \circ \sigma \circ L_{h^{-1}}) = \pi \circ \sigma$, and consequently, $h \circ s \circ h^{-1} = s$. Thus for $g' = gh$ we obtain $g' \circ s \circ g'^{-1} = g \circ s \circ g^{-1}$.

It is clear that $(s_x)^k =$ identity for each $x \in M$. We have to prove that x is an isolated fixed point of s_x. For, it is sufficient to show that the initial point $o \in M$, $o = \pi(H)$, is an isolated fixed point of s. Condition (iii) implies that $s_{*0} \circ \pi_{*e} = \pi_{*e} \circ \sigma_{*e}$ on the tangent space G_e. Let $X \in M_0$ be such that $s_{*0}(X) = X$, and let $\tilde{X} \in G_e$ be a lift of X. Then $\pi_{*e}(\sigma_{*e}(\tilde{X})) = \pi_{*e}(\tilde{X})$, and hence $\sigma_{*e}(\tilde{X}) = \tilde{X} + \tilde{Z}$, where $\tilde{Z} \in H_e$. Now $\sigma_{*e}(\tilde{Z}) = \tilde{Z}$, and $(\sigma_{*e})^k(\tilde{X}) = \tilde{X} + k\tilde{Z} = \tilde{X}$ because $(\sigma_{*e})^k =$ identity. Thus $\tilde{Z} = 0$ and $\tilde{X}$ is a fixed vector of σ_{*e}. We deduce $\tilde{X} \in H_e$ and $X = 0$. Because s_{*0} has no nonzero fixed vectors and s is an isometry of M, we conclude that o is an isolated fixed doint of s.

Finally, we have to prove the formula $s_x \circ s_y = s_z \circ s_x$, $z = s_x(y)$. For this purpose we shall identify the elements of G with the corresponding transformations of M. Then we deduce $s \circ g \circ s^{-1} = \sigma(g)$. Put $s_x = g \circ s \circ g^{-1}$, $s_y = g' \circ s \circ (g')^{-1}$, where $x = g(o)$ and $y = g'(o)$. Then $(g \circ s \circ g^{-1} \circ g' \circ s^{-1})(o) = s_x(g'(o)) = s_x(y)$. On the other hand, $g \circ s \circ g^{-1} \circ g' \circ s^{-1} = g \circ \sigma(g^{-1}g') = g''$ belongs to G. Consequently, $s_x \circ s_y = g \circ s \circ g^{-1} \circ g' \circ s \circ (g')^{-1} = g'' \circ s \circ (g'')^{-1} \circ g \circ s \circ g^{-1} = s_{s_x(y)} \circ s_x$.

2. We shall recall here a class of Riemannian manifolds constructed by Ledger and Obata (see [3]). Let G be a compact connected nonabelian Lie group, G^{k+1} the direct product of G with itself $(k+1)$-times, and ΔG^{k+1} the diagonal of G^{k+1}. Consider the action of G^{k+1} on G^k given by

$$(x_1, \cdots, x_{k+1})(y_1, \cdots, y_k) = (x_1 y_1 x_{k+1}^{-1}, \cdots, x_k y_k x_{k+1}^{-1}) .$$

Then G^{k+1} acts on G^k transitively and effectively, and ΔG^{k+1} is the isotropy group at the identity $o = (e, \cdots, e)$ of G^k. We get a diffeomorphism between G^k and the coset space $G^{k+1}/\Delta G^{k+1}$. Each tangent vector at the identity of G^k can be written in a unique way in the form $(X_1, \cdots, X_k)$, where $X_1, \cdots, X_k \in G_e$.

Now let Φ be an Ad (G)-invariant inner product on G_e, and let $\Phi^{[k]}$ be the Ad (ΔG^{k+1})-invariant inner product on $(G^k)_0$ defined by

$$\Phi^{[k]}((X_1, \cdots, X_k), (X_1, \cdots, X_k)) = \sum_{i=1}^{k} \Phi(X_i, X_i) + \sum_{i<j} \Phi(X_i - X_j, X_i - X_j) .$$

An alternative definition of $\Phi^{[k]}$ is the following: for $i = 1, \cdots, k$ and $X \in G_e$ let $X^{(i)}$ denote the vector $(X_1, \cdots, X_k) \in (G^k)_e$ such that $X_i = X$ and $X_j = 0$ for $j \neq i$. Then $\Phi^{[k]}(X^{(i)}, Y^{(i)}) = k\Phi(X, Y)$, and $\Phi^{[k]}(X^{(i)}, Y^{(j)}) = -\Phi(X, Y)$ for $i \neq j$.

The inner product $\Phi^{[k]}$ can be extended, by the left translations of G^{k+1}, to a Riemannian metric on G^k denoted also by $\Phi^{[k]}$. Then $G^{k+1}/\Delta G^{k+1}$ becomes a homogeneous Riemannian manifold $(G^k, \Phi^{[k]})$.

Let σ be an automorphism of G^{k+1} defined by the rule $\sigma(x_1, \cdots, x_{k+1}) = (x_{k+1}, x_1, \cdots, x_k)$. Then σ satisfies all the conditions of Proposition 1, where we write $k+1, G^{k+1}, \Delta G^{k+1}, G^k$ instead of k, G, H, M respectively. In particular, condition (iii) can be verified as follows: consider the transformation s of G^k determined by $\pi \circ \sigma = s \circ \pi$. Then for any $X \in G_e$ we deduce easily $s_{*0}(X^{(i)}) = X^{(i+1)}$ for $i = 1, \cdots, k-1$, $s_{*0}(X^{(k)}) = -(X^{(1)} + \cdots + X^{(k)})$, and $\Phi^{[k]}(s_{*0}X^{(i)}, s_{*0}Y^{(j)}) = \Phi^{[k]}(X^{(i)}, Y^{(j)})$ for $i, j = 1, \cdots, k$. Thus the Riemannian manifold $(G^k, \Phi^{[k]})$ is $(k+1)$-symmetric.

3. In the remainder of this paper we shall specialize the class of manifolds $(G^k, \Phi^{[k]})$ in a proper way.

Proposition 2. *Consider a homogeneous Riemannian manifold* $(G^k, \Phi^{[k]})$ *such that*

(a) *G is simple,*

(b) *G^{k+1} is the component of unity of the full isometry group $I(G^k, \Phi^{[k]})$. Then $(G^k, \Phi^{[k]})$ is not l-symmetric for any $l < k+1$.*

Proof. Let r be an isometry of $(G^k, \Phi^{[k]})$ with the isolated fixed point $o = (e, \cdots, e)$ such that $r^l =$ identity. Define an automorphism $\tilde{\rho}$ of the group $I(G^k, \Phi^{[k]})$ by the formula $\tilde{\rho}(g) = r \circ g \circ r^{-1}$. Then the restriction of $\tilde{\rho}$ to G^{k+1} is an automorphism ρ of G^{k+1}. We can easily see that $\pi \circ \rho = r \circ \pi$.

Now G^{k+1} is a direct product of simple subgroups $G^{*(i)}$, $i = 1, \cdots, k+1$, all of them being canonically isomorphic to the group G. Then the automorphism $\rho\colon G^{k+1} \to G^{k+1}$ induces a permutation ν of the indices $1, \cdots, k+1$ such that $\rho(G^{*\nu(i)}) = G^{*(i)}$, $i = 1, \cdots, k+1$. Denoting by φ_i the restriction of ρ to $G^{*\nu(i)}$, we get $\rho(g_1, \cdots, g_{k+1}) = (\varphi_1(g_{\nu(1)}), \cdots, \varphi_{k+1}(g_{\nu(k+1)}))$. In particular, $\rho(g, \cdots, g) = (\varphi_1(g), \cdots, \varphi_{k+1}(g))$. Because $\rho(\Delta G^{k+1}) \subset \Delta G^{k+1}$, we obtain $\varphi_1 = \varphi_2 = \cdots = \varphi_{k+1}$ under the canonical identification $G^{*(1)} = \cdots = G^{*(k+1)} = G$, and therefore a unique automorphism $\varphi\colon G \to G$ such that $\rho(g_1, \cdots, g_{k+1}) = (\varphi(g_{\nu(1)}), \cdots, \varphi(g_{\nu(k+1)}))$. Denote by $d\rho$ (respectively, $d\varphi$) the induced automorphism of the Lie algebra $\mathfrak{g}^{k+1}$ (respectively, $\mathfrak{g}$). Then $d\rho(X_1, \cdots, X_{k+1}) = (d\varphi(X_{\nu(1)}), \cdots, d\varphi(X_{\nu(k+1)}))$, $X_1, \cdots, X_{k+1} \in \mathfrak{g}$.

Now let us recall the following result by Borel and Mostow, [1].

Lemma. *A semi-simple automorphism A of a nonsolvable Lie algebra $\mathfrak{g}$ leaves fixed an element X such that* $\operatorname{ad} X$ *is not nilpotent.*

$d\varphi$ is a semi-simple automorphism of $\mathfrak{g}$ because $(d\varphi)^l =$ identity. Let $X \neq 0$ be a fixed vector of $d\varphi$ and suppose $l < k+1$. Then the permutation ν contains a cycle $(i_1, \cdots, i_m)$ of length $m < k+1$. Consider the vector $Z = (X_1, \cdots, X_{k+1}) \in \mathfrak{g}^{k+1}$ such that $X_i = X$ for $i = i_1, \cdots, i_m$ and $X_i = -X$ otherwise. Clearly, $d\rho(Z) = Z$. Now we can identify $\mathfrak{g}^{k+1}$ with the tangent space $(G^{k+1})_e$ and $d\rho$ with the tangent map ρ_{*e}. We have $\pi_{*e} \circ \rho_{*e} = r_{*0} \circ \pi_{*e}$, and thus the projection $\pi_{*e}(Z) \in (G^k)_0$ is a fixed vector with respect to r_{*0}.

Moreover, $Z \in (G^{k+1})_e$ is not tangent to the submanifold ΔG^{k+1} and hence $\pi_{*e}(Z) \neq 0$, a contradiction. This completes the proof.

Proposition 3. *For* $G = SO(3)$ *and* $\Phi(X, Y) = -\frac{1}{2}\operatorname{tr}(\operatorname{ad} X \circ \operatorname{ad} Y)$ *the conditions of Proposition* 2 *are satisfied.*

Proof. In the following, the elements of $\mathfrak{g}$ (respectively, $\mathfrak{g}^k$) are considered as left invariant vector fields on G (respectively, G^k). First of all, there is a basis $\{X_1, X_2, X_3\}$ of $\mathfrak{g}$ such that $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$. We have $\Phi(X_\alpha, X_\beta) = \delta_{\alpha\beta}$ for $\alpha, \beta = 1, 2, 3$, and the vectors $X_\alpha^{(i)}$, $\alpha = 1, 2, 3$, $i = 1, \cdots, k$, form a basis of $\mathfrak{g}^k$. Now recall formulas (14) of [3]: for $X, Y \in \mathfrak{g}$

$$\nabla_{X^{(i)}} Y^{(j)} = \frac{1}{2(k+1)}\{[X, Y]^{(j)} - [X, Y]^{(i)}\} \qquad \text{for } i \neq j ,$$
$$\nabla_{X^{(i)}} Y^{(i)} = \tfrac{1}{2}[X, Y]^{(i)} .$$

A routine calculation shows the following properties of the curvature tensor R of $\Phi^{[k]}$:

$$R(X_\alpha^{(i)}, X_\beta^{(j)})X_\gamma^{(k)} = 0 \text{ whenever } \alpha \neq \beta \neq \gamma \text{ or } \alpha = \beta = \gamma,$$
$$(1) \qquad R(X_\alpha^{(i)}, X_\beta^{(j)})X_\alpha^{(k)} \text{ and } R(X_\alpha^{(i)}, X_\alpha^{(j)})X_\beta^{(k)} \text{ belong to the subspace}$$
$$\text{generated by } X_\beta^{(i)}, X_\beta^{(j)}, X_\beta^{(k)} .$$

Let H_0 be the component of the unity of the isotropy group of $I(G^k, \Phi^{[k]})$ at the origin o, and denote the corresponding Lie algebra by $\mathfrak{h}_0$. Then $\mathfrak{h}_0$ has a faithful isotropy representation by endomorphisms of $\mathfrak{g}^k = (G^k)_0$. Clearly, the necessary condition for $A \in \mathfrak{h}_0$ is that $A(\Phi^{[k]}) = A(R) = 0$, where A acts as a derivation on the tensor algebra of $\mathfrak{g}^k$.

Let $A \in \mathfrak{h}_0$ and set

$$(2) \qquad AX_\alpha^{(i)} = \sum_{\beta=1}^{3} \sum_{j=1}^{k} a_{(j)\alpha}^{(i)\beta} X_\beta^{(j)} , \qquad i = 1, \cdots, k, \ \alpha = 1, 2, 3 .$$

The relation $(A\Phi^{[k]})(X_\alpha^{(i)}, X_\beta^{(j)}) = 0$ implies

$$(3) \qquad k(a_{(i)\alpha}^{(i)\beta} + a_{(j)\beta}^{(j)\alpha}) - \sum_{l \neq i} a_{(l)\alpha}^{(i)\beta} - \sum_{l \neq j} a_{(l)\beta}^{(j)\alpha} = 0 .$$

Further, we can calculate easily

$$R(X_\alpha^{(i)}, X_\beta^{(i)})X_\alpha^{(i)} = -\tfrac{1}{4}X_\beta^{(i)} \qquad \text{for } \alpha \neq \beta .$$

Consider the relation $(AR)(X_\alpha^{(i)}, X_\beta^{(i)})X_\alpha^{(i)} = 0$, i.e.,

$$(4) \qquad \begin{aligned} -\tfrac{1}{4}AX_\beta^{(i)} = R(AX_\alpha^{(i)}, X_\beta^{(i)})X_\alpha^{(i)} + R(X_\alpha^{(i)}, AX_\beta^{(i)})X_\alpha^{(i)} \\ + R(X_\alpha^{(i)}, X_\beta^{(i)})AX_\alpha^{(i)} . \end{aligned}$$

Let us substitute (2) in (4) and consider a vector $X_\gamma^{(j)}$, where $\gamma \neq \alpha, \beta$ and

$j \neq i$. This vector enters into the left-hand side with the coefficient $-\frac{1}{4}a^{(i)\gamma}_{(j)\beta}$. According to (1), there is only one term on the right-hand side the evaluation of which can involve $X^{(j)}_\gamma$, namely, the term $R(X^{(i)}_\alpha, a^{(i)\gamma}_{(j)\beta}X^{(j)}_\gamma)X^{(i)}_\alpha$. Now

$$R(X^{(i)}_\alpha, a^{(i)\gamma}_{(j)\beta}X^{(j)}_\gamma)X^{(i)}_\alpha = a^{(i)\gamma}_{(j)\beta}[(k+2)X^{(i)}_\gamma - X^{(j)}_\gamma]/[4(k+1)^2] .$$

Comparing the coefficients at $X^{(j)}_\gamma$ we finally get $a^{(i)\gamma}_{(j)\beta} = 0$. Thus we have proved

$$(5) \qquad a^{(i)\alpha}_{(j)\beta} = 0 \qquad \text{for } i \neq j,\ \alpha \neq \beta .$$

Substituting in (3) we get

$$(6) \qquad a^{(i)\beta}_{(i)\alpha} + a^{(j)\alpha}_{(j)\beta} = 0 \qquad \text{for } \alpha \neq \beta .$$

In particular, for $i = j$ we obtain

$$(7) \qquad a^{(i)\beta}_{(i)\alpha} + a^{(i)\alpha}_{(i)\beta} = 0 ,$$

and hence

$$(8) \qquad a^{(1)\beta}_{(1)\alpha} = a^{(2)\beta}_{(2)\alpha} = \cdots = a^{(k)\beta}_{(k)\alpha} \qquad \text{for } \alpha \neq \beta .$$

Now let us compare the coefficients at $X^{(j)}_\beta$, $j \neq i$, in the relation (4). $X^{(j)}_\beta$ enters into the left-hand side with the coefficient $-\frac{1}{4}a^{(i)\beta}_{(j)\beta}$. As for the right-hand side, $X^{(j)}_\beta$ can be involved only in the evaluations of the terms $R(a^{(i)\alpha}_{(j)\alpha}X^{(j)}_\alpha, X^{(i)}_\beta)X^{(i)}_\alpha$, $R(X^{(i)}_\alpha, a^{(i)\beta}_{(j)\beta}X^{(j)}_\beta)X^{(i)}_\alpha$, $R(X^{(i)}_\alpha, X^{(i)}_\beta)(a^{(i)\alpha}_{(j)\alpha}X^{(j)}_\alpha)$. After routine calculations we obtain

$$(9) \qquad (3k+2)a^{(i)\alpha}_{(j)\alpha} + (k^2+2k)a^{(i)\beta}_{(j)\beta} = 0 .$$

Writing these relations for $(\alpha, \beta) = (1, 2), (2, 3), (3, 1)$ respectively, we obtain finally

$$(10) \qquad a^{(i)\alpha}_{(j)\alpha} = 0 \qquad \text{for } i \neq j,\ \alpha = 1, 2, 3 .$$

Having $i = j$ and $\alpha = \beta$ in (3), we deduce from (10)

$$(11) \qquad a^{(i)\alpha}_{(i)\alpha} = 0 , \qquad \alpha = 1, 2, 3,\ i = 1, \cdots, k .$$

If we summarize (5), (10) and then (7), (8), (11), we can see that $\mathfrak{h}_0 \subset \Delta([\mathfrak{so}(3)]^k)$. On the other hand, the group $G^{k+1} = SO(3)^{k+1}$ is contained in $I(G^k, \Phi^{[k]})$ so that $\Delta(SO(3)^{k+1})$ is contained in H_0. Thus $H_0 = \Delta(SO(3)^{k+1})$, and consequently $SO(3)^{k+1}$ is the component of the unity of $I(G^k, \Phi^{[k]})$ as required. Hence we can conclude our paper with

Theorem. *For each integer $k \geq 2$ there exists a compact generalized symmetric Riemannian space (M, g) of order k such that the component of the unity of the full isometry group $I(M, g)$ is semi-simple.*

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CHARLES UNIVERSITY, PRAGUE