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TOPOLOGY OF THE COMPLEX VARIETIES $A_s^{(n)}$

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1. Introduction

Define, for $s \leq [n/2]$,

$\tilde{V}_{n,2s}$: manifold of ordered $2s$-tuplets of linearly independent vectors in Euclidean n-space R^n,

$\tilde{A}_s^{(n)}$: space of 2-forms in R^n of rank $2s$,

$\tilde{f}_s^{(n)} : \tilde{V}_{n,2s} \to \tilde{A}_s^{(n)}$: map given by

$$\tilde{f}_s^{(n)}(y_1, \cdots, y_{2s}) = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s} ,$$

$V_{n,2s}$: Stiefel manifold of orthonormal $2s$-frames in R^n,

$A_s^{(n)} = \tilde{f}_s^{(n)}(V_{n,2s})$: subspace of $\tilde{A}_s^{(n)}$ of "normalized" 2-forms in R^n of rank $2s$,

$f_s^{(n)} : V_{n,2s} \to A_s^{(n)}$: the restriction of $\tilde{f}_s^{(n)}$ to $V_{n,2s}$.

It was proved in [4] that the maps $\tilde{f}_s^{(n)}$ and $f_s^{(n)}$ induce the principal $Sp(s; R)$- and $U(s)$-bundles respectively, and that $A_s^{(n)}$ is a strong deformation retract of $\tilde{A}_s^{(n)}$.

One may, equivalently, define $A_s^{(n)}$ as the space of normalized complex s-substructures of R^n, i.e., pairs (p, J) where p is a $2s$-plane in R^n and J is a normalized complex structure on p ($J \in O(p)$, $J^2 = -1$).

To see the equivalence, let $w \in A_s^{(n)}$. Then $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ for an orthonormal $2s$-frame $y = (y_1, \cdots, y_{2s})$. Let p be the $2s$-plane spanned by y. For $x \in p$, let $d_x : p \to \Lambda^2 p$ be forming wedge products with x, i.e., $d_x(z) = x \wedge z$, and $\delta_x : \Lambda^2 p \to p$ be its "adjoint". Define a linear transformation J on p by $J(x) = \delta_x(w)$, $x \in p$. Then $J(y_i) = y_{i+s}$ and $J(y_{i+s}) = -y_i$, $1 \leq i \leq s$. Thus $J \in O(p)$, $J^2 = -1$. Conversely, a normalized complex s-substructure J, $J \in O(p)$, $J^2 = -1$, can be represented by the matrix $\begin{bmatrix} 0 & -I_s \\ I_s & 0 \end{bmatrix}$ relative to some orthonormal $2s$-frame $y = (y_1, \cdots, y_{2s})$ on p. Hence J corresponds to $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ in $A_s^{(n)}$.

It follows from either definition that $A_s^{(n)} = SO(n)/U(s) \times SO(n-2s)$ for $s < n/2$, $A_s^{(2s)} = O(2s)/U(s) = I_s \cup I'_s$ where $I_s = SO(2s)/U(s)$, $A_1^{(n)} = \tilde{G}_{n,2} = Q_{n-2}(C)$ where $\tilde{G}_{n,2}$ is the oriented 2-planes in R^n, and $Q_{n-2}(C)$ is the complex quadric of dimension $n-2$.

The spaces $A_s^{(n)}$ appear as "fibres" in global obstruction problems involving

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2-forms of constant rank, and the foremost among these problems are the existence and decomposability of such forms.

1. The existence of a 2-form of constant rank $2s$ on an R^n-bundle E (or, a complex s-substructure on E) is equivalent to cross-sectioning the associated bundle $A_s(E)$ to E with fiber $A_s^{(n)}$.

2. Globally decomposing a given 2-form w of constant rank $2s$ on E as a sum $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ of products of 1-forms (y_i) on E is equivalent to the lifting of the diagram

$$\begin{tikzcd} & V_{2s}(E) \arrow[d, "f_s^{(n)}"] \\ B \arrow[r, "w"] \arrow[ur, dashed, "?"] & A_s(E) \end{tikzcd}$$

where B is the base manifold, $V_{2s}(E)$ the associated bundle to E with fiber the Stiefel manifold $V_{n,2s}$, and w is represented with respect to a suitable metric on E as a "normalized" 2-form on E of constant rank $2s$, i.e., as a map w: $B \to A_s(E)$. (Refer to [4].)

2a. In the special case when E is a trivial (product) bundle (e.g., the tangent bundles of Lie groups), the diagram reduces to

$$\begin{tikzcd} & V_{n,2s} \arrow[d, "f_s^{(n)}"] \\ B \arrow[r, "w_1"] \arrow[ur, dashed, "?"] & A_s^{(n)} \end{tikzcd}$$

and the primary obstructions to lifting w_1 are the pull-back $w_1^*(c_i) \in H^{2i}(B; Z)$ by w_1 of the Chern classes $c_i \in H^{2i}(A_s^{(n)}; Z)$ of the principal $U(s)$-bundle $V_{n,2s}(A_s^{(n)}; U(s))$.

2b. In the general case (i.e., when the total bundle E is not necessarily trivial) a necessary condition for globally decomposing w is that the $2s$-dimensional subbundle S_w of E defined by w is trivial. Using the triviality of S_w (and a suitable metric on it) w is represented as a map w_1: $B \to I_s$, and then decomposability of w is equivalent to the lifting of the diagram:

$$\begin{tikzcd} & SO(2s) \arrow[d] \\ B \arrow[r, "w_1"] \arrow[ur, dashed, "?"] & I_s \end{tikzcd}$$

(which is the special case of diagram 2a for $n = 2s$) and again the primary obstructions to decomposing w are the pull-back $w_1^*(c_i) \in H^{2i}(B; Z)$ by w_1 of the Chern classes $c_i \in H^{2i}(I_s; Z)$ of $SO(2s)(I_s; U(s))$. (Refer to [4] for details.)

In this paper we make a start on these obstruction problems by studying the topology of the manifolds $A_s^{(n)}$. We represent $A_s^{(n)}$ as the subvariety of the complex Grassmann variety $G_{n,s}^C$ of projective $[s-1]$-planes lying on the complex quadric $Q_{n-2}(C)$. In perfect analogy with the classical Schubert calculus on Grassmann varieties, we define the Schubert cell $\Omega_{a_0a_1\cdots a_{s-1}}$, $0 \leq a_0 < a_1 < \cdots < a_{s-1} \leq n-2$. Then the main result of this paper, the *CW*-structure theorem, states that $A_s^{(n)}$ is a cell complex on the class of Schubert cells

$$(\Omega_{a_0a_1\cdots a_{s-1}} \mid a_i + a_j \neq n-2 \text{ for } 0 \leq i < j \leq n-2) .$$

As a corollary we obtain the additive homology and cohomology of $A_s^{(n)}$. We then develop a duality theory for $A_s^{(n)}$, and using this and the inclusion map $j\colon A_s^{(n)} \to G_{n,s}^c$ we compute the Chern classes $c_i \in H^{2i}(A_s^{(n)}; Z)$. Thus given w we can explicitly determine the primary obstructions $w^*(c_i)$ to decompose w.

The paper, as a whole, is self contained. The arguments are based on elementary projective geometry.

2. Universality of $A_s^{(\infty)}$

For fixed s we have a sequence of principal $U(s)$-bundles:

$$\begin{array}{ccccccccccc} V_{2s,2s} & \subset & V_{2s+1,2s} & \subset \cdots \subset & V_{n,2s} & \subset & V_{n+1,2s} & \subset \cdots & V_{\infty,2s} \\ \downarrow{\scriptstyle f_s^{(2s)}} & & \downarrow{\scriptstyle f_s^{(2s+1)}} & & \downarrow{\scriptstyle f_s^{(n)}} & & \downarrow{\scriptstyle f_s^{(n+1)}} & & \downarrow{\scriptstyle f_s^{(\infty)}} \\ A_s^{(2s)} & \subset & A_s^{(2s+1)} & \subset \cdots \subset & A_s^{(n)} & \subset & A_s^{(n+1)} & \subset \cdots & A_s^{(\infty)} . \end{array}$$

Thus $A_s^{(\infty)} = \operatorname{dir\,lim}_{n\to\infty} A_s^{(n)}$ forms a classifying space for $U(s)$. Let $W_{n,s}$ be the Stiefel manifold of complex orthonormal s-frames in C^n, and define $r_s^{(n)}\colon W_{n,s} \to V_{2n,2s}$ by $r_s^{(n)}(z_1, \cdots, z_s) = (z_1, \cdots, z_s, iz_1, \cdots, iz_s)$, and $w_s^{(n)}\colon V_{n,2s} \to W_{n,s}$ by $w_s^{(n)}(x_1, \cdots, x_{2s}) = ((1/\sqrt{2})(x_1 - ix_{s+1}), \cdots, (1/\sqrt{2})(x_s - ix_{2s}))$ where $i = \sqrt{-1}$. $r_s^{(n)}$ and $w_s^{(n)}$ are $U(s)$-maps, and thus induce imbeddings $\bar{r}_s^{(n)}\colon G_{n,s}^c \to A_s^{(2n)}$ and $\overline{w}_s^{(n)}\colon A_s^{(n)} \to G_{n,s}^c$ on the quotient spaces. $\bar{r}_s^{(n)} \circ \overline{w}_s^{(n)}$ and $\overline{w}_s^{(2n)} \circ \bar{r}_s^{(n)}$ are homotopic to inclusion maps $A_s^{(n)} \subset A_s^{(2n)}$ and $G_{n,s}^c \subset G_{2n,s}^c$ respectively. Hence $\bar{r}_s^{(\infty)}$ and $\overline{w}_s^{(\infty)}$ are the desired homotopy equivalences of $A_s^{(\infty)}$ with the standard classifying space $G_{\infty,s}^c$ of $U(s)$.

Let $Q^c(z_1, \cdots, z_n) = z_1^2 + z_2^2 + \cdots + z_n^2$ be the nonsingular billinear form on C^n. Then it can be easily verified from the definition that

$$\text{Image } w_s^{(n)} = (\pi \in G_{n,s}^c \mid Q^c \text{ vanishes on } \pi) .$$

Let $Q_{n-2}(C)$ be the quadric of the form Q^c in $P_{n-1}(C)$. We can now identify $A_s^{(n)}$ with its image in $G_{n,s}^c$, and write this as a

Representation theorem. *$A_s^{(n)}$ is represented as the complex analytic variety of linear projective $[s-1]$-planes on $Q_{n-2}(C)$.*

3. Preliminaries

We now list the preliminaries to be needed in the sequel, and for details we refer the reader to [6]. In what follows, $\perp_f$ and $\perp_m$ will denote orthogonal complements with respect to the form Q^c and the Hermitian metric on C^n respectively. $\vee$ will denote join, $\cup$ union and $\cap$ intersection.

3.1. The conjugation map $c: C^{n+2} \to C^{n+2}$ given by $c(z_0, z_1, \cdots, z_{n+1}) = (\bar{z}_0, \bar{z}_1, \cdots, \bar{z}_{n+1})$ has the following properties:

(i) $Q^c(z; w) = \langle z | c(w) \rangle$, and thus $z^{\perp_f} = c(z)^{\perp_m}$.

(ii) $Q^c(c(z)) = \overline{Q^c(z)}$, and thus c maps $Q_n(C)$ onto itself.

(iii) The image under c of a projective $[s]$-plane q lying on $Q_n(C)$ is another projective $[s]$-plane q', which also lies on $Q_n(C)$ and is m-orthogonal to q. Thus c induces an involution on $A_{s+1}^{(n+2)}$.

(iv) $Q^c(z; c(z)) \neq 0$ for $z \neq 0$. Thus, if an $[s]$-plane q is $[k]$-degenerate with degeneracy q_0 (i.e., $q_0 = q \cap q^{\perp_f}$), then Q^c is nonsingular on the join $q \vee c(q_0)$.

3.2. Suppose that a projective $[s-1]$-plane q lies on $Q_n(C)$, and that P is a point not on q. Then the join $q \vee P$ lies on $Q_n(C)$ if and only if $P \in Q_n(C) \cap q^{\perp_f}$.

3.3. $Q_n(C)$ has a nontrivial intersection with every projective line on $P_{n+1}(C)$.

3.4. An $[s]$-plane q lies on $Q_n(C)$ if and only if $q \subset q^{\perp_f}$. Hence $s \leq n - s$, i.e., $s \leq [n/2]$. If $s < [n/2]$, it follows from 3.2 and 3.3 that q is contained in an $[s+1]$-plane lying on $Q_n(C)$. Thus the maximal planes on $Q_n(C)$ are $[n/2]$-dimensional, and any plane lying on $Q_n(C)$ can be imbedded in a maximal one.

3.5. $A_{s+1}^{(2s+2)} = [s]$-planes on $Q_{2s}(C)$ consists of two connected components or irreducible subvarieties V_0 and V_1, each of which is homeomorphic to I_{s+1}. The dimension of intersection of two $[s]$-planes on $Q_{2s}(C)$ is congruent to s (mod 2) if they belong to the same component, and to $s - 1$ (mod 2) if they belong to different components.

3.6. It is a direct consequence of 3.4 and 3.5 that given an $[s-1]$-plane q on $Q_{2s}(C)$, there exist unique $[s]$-planes $q_0 \in V_0$ and $q_1 \in V_1$ such that $q = q_0 \cap q_1$, $q^{\perp_f} = q_0 \vee q_1$, $Q_{2s}(C) \cap q^{\perp_f} = q_0 \cup q_1$.

3.7. Let $Q_{2s-1}(C) \subset Q_{2s}(C)$ be an inclusion of nonsingular quadrics. Then by 3.6 above, each $[s-1]$-plane q on $Q_{2s-1}(C)$ corresponds to a unique $q_0 \in V_0$, $q_0 \supset q$, and each $q_0 \in V_0$ necessarily intersects $Q_{2s-1}(C)$ in an $[s-1]$-plane q. This establishes a homeomorphism between V_0 and $A_s^{(2s+1)} = [s-1]$-planes on $Q_{2s-1}(C)$.

Let P_q be the unique point of q_0 which is m-orthogonal to q. Define a continuous map $f: V_0 \to Q_{2s}(C)$ by $f(q_0) = P_q$. Let E, F, ξ be the canonical C^{s+1}-, C^s-, C^1-bundles over $V_0, A_s^{(2s+1)}$ and $Q_{2s}(C)$ respectively. Then, since $q_0 = q \vee P_q$, we have $E = F \oplus f^*(\xi)$. $P_q \notin Q_{2s-1}(C)$ by definition, and hence the map f

factors through the open contractible space $Q_{2s}(C) - Q_{2s-1}(C)$, and is thus null homotopic. Hence the pull-back $f^*(\xi)$ of f to V_0 is trivial, i.e., $f^*(\xi) = 1$ and $E = F \oplus 1$.

3.8. Let $q_1 \subset q_2$ be an inclusion of projective $[s]$- and $[s+1]$-planes lying on $Q_n(C)$. Let $P \in (q_2 - q_1)$. Then $q_2^{\perp_f} = q_1^{\perp_f} \cap P^{\perp_f}$. Let h be a hyperplane in $q_1^{\perp_f}$ not passing through P and thus intersecting the hyperplane $q_2^{\perp_f}$ (containing P) in an $[n-s-2]$-plane h_0. Central projection through P establishes a homeomorphism between $(h - h_0)$ and $Q_n(C) \cap (q_1^{\perp_f} - q_2^{\perp_f})$. Thus the latter is an open cell of complex dimension $n - s - 1$.

3.9. Let q_0 be a fixed $[s-1]$-plane in $P_{n-1}(C)$, and $S_t(q_0) = (q \in G^c_{n,k} \mid \dim (q \cap q_0) = t - 1)$ for $t \leq \min(s, k)$. Then the map $S_t(q_0) \to G^c_{n,t}$ defined by $q \to q \cap q_0$ is continuous.

3.10. Let $O_0 \in Q_1(C)$ and $P_1(C)$ be the hyperplane in $P_2(C)$ which is f-orthogonal to O_0. Let $C^3 = (e_0, e_1, e_2)$, $Q^c(z) = z_0^2 + z_1^2 + z_2^2$, $O_0 = [e_0 + ie_1]$. Then the curves $a(t) = [(\cos t)e_0 + ie_1 + (\sin t)e_2]$ in $Q_1(C)$ and $b(t) = [(\cos t)e_0 + (i \cos t)e_1 + (\sin t)e_2]$ in $P_1(C)$ both starting at O_0 have a common tangent vector $e_2 \in S^5$ at this point. Hence $Q_1(C)$ and $P_1(C)$ have a "double" intersection at O_0.

3.11. For $k = a + b$, decompose a $[k-1]$-plane q_0 into a disjoint join $q_0 = q_a \vee q_b$ of an $[a-1]$-plane q_a and a $[b-1]$-plane q_b. Let S_a and S_b be the submanifolds of $G^c_{n,k}$ of $[k-1]$-planes containing q_a and q_b respectively. $q \in S_a$ intersects $q_a^{\perp_m} = [n-a-1]$ at $[b-1]$, and the intersection uniquely determines q. Hence $S_a = G^c_{n-a,b}$, and similarly $S_b = G^c_{n-b,a}$. $\dim_c S_a + \dim_c S_b = (n-a-b)b + (n-b-a)a = (n-k)k$, i.e., S_a and S_b are of complementary dimensions in $G^c_{n,k}$. They also intersect transversally at the single point q_0. This gives a direct sum decomposition for the tangent plane to $G^c_{n,k}$ at q_0: $T_{q_0}(G^c_{n,k}) = T_{q_0}(S_a) \oplus T_{q_0}(S_b)$.

4. Topology of $Q_n(C)$

Let $[p]$ be a maximal plane of dimension $p = [n/2]$ lying on $Q_n(C)$, $[p] \supset [p-1] \supset \cdots \supset [1] \supset [0]$ be a cellular decomposition for $[p]$ by its sub-projective-spaces, and

$$[n+1] \supset [0]^{\perp_f} \supset [1]^{\perp_f} \supset \cdots \supset [n-p-1]^{\perp_f}$$
$$\supset [p] \supset [p-1] \supset \cdots \supset [1] \supset [0]$$

be the corresponding cellular decomposition for $P_{n+1}(C)$.

Define $Q_k(C) = Q_n(C) \cap [n-k-1]^{\perp_f}$ for $k > p$. Then $Q_k(C) \supset [n-k-1]$, and is thus an $[n-k-1]$-degenerate subquadric of $Q_n(C)$. It follows from 3.8 that $\{Q_k(C) - Q_{k-1}(C)\}$ is an open cell of complex dimension k for $k > p+1$, and that $\{Q_{p+1}(C) - Q_n(C) \cap [n-p-1]^{\perp_f}\}$ is an open $[p+1]$-cell.

For $n = 2p + 1$, $Q_n(C) \cap [n - p - 1]^{\perp_f} = Q_{2p+1}(C) \cap [p]^{\perp_f} = [p]$, and thus

$$\begin{aligned} Q_{2p+1}(C) \supset Q_{2p}(C) \supset \cdots \supset Q_{p+1}(C) \\ \supset [p] \supset [p-1] \supset \cdots \supset [1] \supset [0] \end{aligned}$$

forms a cellular decomposition for $Q_{2p+1}(C)$.

For $n = 2p$, assume without loss of generality that $[p] = [p]_0 \in V_0$. Then by 3.6 there exists a unique $[p]_1 \in V_1$ such that $Q_n(C) \cap [n - p - 1]^{\perp_f} = Q_{2p}(C) \cap [p-1]^{\perp_f} = [p]_0 \cup [p]_1$. Thus

$$\begin{aligned} Q_{2p}(C) \supset Q_{2p-1}(C) \supset \cdots \supset Q_{p+1}(C) \supset [p]_0 , \\ [p]_1 \supset [p-1] \supset \cdots \supset [1] \supset [0] \end{aligned}$$

is a cell decomposition for $Q_{2p}(C)$.

5. *CW*-structure of $A_{s+1}^{(n+2)}$

Define, for $q \in A_{s+1}^{(n+2)}$ and $t \in Z^+$, $q_t = q \cap$ complex t-dimensional cell of $Q_n(C)$, i.e.,

$$q_t = \begin{cases} q \cap [t] & \text{for } t < n/2 , \\ q \cap Q_t(C) & \text{for } t > n/2 , \end{cases}$$

$$q_{p_0} = q \cap [p]_0 , \quad q_{p_1} = q \cap [p]_1 \qquad \text{for } p = n/2 .$$

Observation. (i) *q_t is a subspace of q.*

(ii) *The sequence (q_t) forms a filtration:*

For $n = 2p + 1$,

$$q = q_{2p+1} \supset q_{2p} \supset \cdots \supset q_{p+1} \supset q_p \supset \cdots \supset q_1 \supset q_0 .$$

For $n = 2p$, either

$$q = q_{2p} \supset q_{2p-1} \supset \cdots \supset q_{p+1} \supset q_{p_0} \supset q_{p-1} \supset \cdots \supset q_0, \; q_{p_1} = q_{p-1} ,$$

or

$$q = q_{2p} \supset q_{2p-1} \supset \cdots \supset q_{p+1} \supset q_{p_1} \supset q_{p-1} \supset \cdots \supset q_1 \supset q_0, \; q_{p_0} = q_{p-1} ,$$

by subspaces whose dimensions decrease at most 1 *at each step.*

Proof. (i) Obviously, $q_t = q \cap [t]$ for $t \leq n/2$ is a subspace, and

$$q_t = q \cap Q_t(C) = q \cap Q_n(C) \cap [n - t - 1]^{\perp_f} = q \cap [n - t - 1]^{\perp_f}$$

for $t > n/2$ is also a subspace.

(ii) For $t \leq n/2$,

$$\dim q_t = \dim (q \cap [t]) \leq \dim (q \cap [t-1]) + 1 = \dim q_{t-1} + 1 \ .$$

For $t > n/2$,

$$\begin{aligned} \dim q_{t+1} &= \dim (q \cap Q_{t+1}(C)) \\ &= \dim (q \cap [n-t-2]^{\perp_f}) \leq \dim (q \cap [n-t-1]^{\perp_f}) + 1 \\ &= \dim (q \cap Q_t(C)) + 1 = \dim q_t + 1 \ . \end{aligned}$$

If $n = 2p + 1$, then

$$\begin{aligned} \dim q_{p+1} &= (\dim q \cap Q_{p+1}(C)) \\ &= \dim (q \cap [p-1]^{\perp_f}) \leq \dim (q \cap [p]^{\perp_f}) + 1 \\ &= \dim (q \cap Q_{2p+1}(C) \cap [p]^{\perp_f}) + 1 \\ &= \dim (q \cap [p]) + 1 = \dim q_p + 1 \ . \end{aligned}$$

Thus

$$q = q_{2p+1} \supset q_{2p} \supset \cdots \supset q_{p+1} \supset q_p \supset \cdots \supset q_1 \supset q_0$$

is the required filtration.

If $n = 2p$, then

$$\begin{aligned} q = [p-1]^{\perp_f} &= q \cap Q_{2p}(C) \cap [p-1]^{\perp_f} \\ &= q \cap ([p]_0 \cup [p]_1) = q_{p_0} \cup q_{p_1} \end{aligned}$$

is a subspace, and thus either $q \cap [p-1]^{\perp_f} = q_{p_0} \supset q_{p_1}$ or $q \cap [p-1]^{\perp_f} = q_{p_1} \supset q_{p_0}$. If $q \cap [p-1[^{\perp_f} = q_{p_0} \supset q_{p_1}$, then

$$\begin{aligned} q_{p_1} &= q_{p_0} \cap q_{p_1} = q \cap ([p]_0 \cap [p]_1) = q \cap [p-1] = q_{p-1} \ , \\ \dim q_{p+1} &= \dim (q \cap [p-2]^{\perp_f}) \leq \dim (q \cap [p-1]^{\perp_f}) + 1 \\ &= \dim q_{p_0} + 1 \ . \end{aligned}$$

Thus

$$q = q_{2p} \supset q_{2p-1} \supset \cdots \supset q_{p+1} \supset q_{p_0} \supset q_{p-1} \supset \cdots \supset q_1 \supset q_0$$

is the required filtration.

Similarly, if $q \cap [q-1]^{\perp_f} = q_{p_1} \supset q_{p_0}$, then we have $q_{p_0} = q_{p-1}$ and

$$q = q_{2p} \supset q_{2p-1} \supset \cdots \supset q_{p+1} \supset q_{p_1} \supset q_{p-1} \supset \cdots \supset q_1 \supset q_0$$

is the required filtration. q.e.d.

For $0 \leq a_0 < a_1 < \cdots < a_s \leq n$, we introduce the closed Schubert cell

$$\Omega_{a_0 a_1 \cdots a_s} = (q \in A_{s+1}^{(n+2)} \,|\, \dim q_{a_t} \geq t) \ .$$

An immediate corollary of the preceding observation is the following.

Corollary. $A^{(n+2)}_{s+1} = \bigcup \Omega_{a_0a_1\cdots a_s}$.

However, some of the cells in this covering are "superfluous", and the next lemma shows that $A^{(n+2)}_{s+1}$ can be covered by a smaller class of Schubert cells $(\Omega_{a_0a_1\cdots a_s} \mid a_i + a_j \neq n \text{ for } i < j)$.

Notation. For $a = (a_0, a_1, \cdots, a_s)$ and $b = (b_0, b_1, \cdots, b_s) \in (Z^+)^{s+1}$, we write: $b \leq a$ if and only if $b_j \leq a_j$, $0 \leq j \leq s$; $b = a$ if and only if $b_j = a_j$, $0 \leq j \leq s$; $b < a$ if and only if $b \leq a$, $b \neq a$.

Lemma. $\Omega_{a_0a_1\cdots a_s} = \bigcup_{b \leq a} (\Omega_{b_0b_1\cdots b_s} \mid b_i + b_j \neq n \textit{ for } i < j)$.

Proof. Suppose $a_i + a_j = n$ for some $i < j$; otherwise, the lemma follows trivially. There are two cases to consider.

1. $\dim q_{a_{i-1}} = \dim q_{a_i} \geq i$. Define $b_k = \min(a_k; a_i - i + k - 1)$ for $0 \leq k \leq i - 1$. Then $\dim q_{b_k} \geq k$, i.e., $q \in \Omega_{b_0b_1\cdots b_{i-1}a_i-1a_{i+1}\cdots a_s}$.

2. $\dim q_{a_{i-1}} = \dim q_{a_i} - 1$. Then $[a_i] = q_{a_i} \vee [a_i - 1]$.

(i) $q_{a_j} \perp_f q_{a_i}$ since $q \subset Q_n(C)$.

(ii) $q_{a_j} \perp_f [n - a_j - 1] = [a_i - 1]$, and thus by the above

$$q_{a_j} \subset Q_n(C) \cap [a_i]^{\perp_f} = Q_n(C) \cap [n - a_j]^{\perp_f} = Q_{a_j-1}(C) ,$$

i.e., $\dim q_{a_{j-1}} = \dim q_{a_j} \geq j$. Define $c_k = \min(a_k; a_j - j + k - 1)$ for $0 \leq k \leq j - 1$. Then $\dim q_{c_k} \geq k$, i.e., $q \in \Omega_{c_0c_1\cdots c_{j-1}a_j-1a_{j+1}\cdots a_s}$. Thus

$$\Omega_{a_0a_1\cdots a_s} = \Omega_{b_0\cdots b_{i-1}a_i-1a_{i+1}\cdots a_s} \cup \Omega_{c_0\cdots c_{j-1}a_j-1a_{j+1}\cdots a_s} ,$$

where $b_k \leq a_k$ for $1 \leq k \leq i - 1$, and $c_k \leq a_k$ for $1 \leq k \leq j - 1$. Hence the lemma follows by induction on $\sum_{j=0}^{s} a_j = a_0 + a_1 + \cdots + a_s$. q.e.d.

We now define the open Schubert cell $\Omega^{\text{open}}_{a_0a_1\cdots a_s}$ for $a_i + a_j \neq n$, $i < j$:

$$\Omega^{\text{open}}_{a_0a_1\cdots a_s} = (q \in A^{(n+2)}_{s+1} \mid \dim q_t = j \text{ for } a_j \leq t < a_{j+1}) .$$

The basis of our CW-structure theorem is the following.

Proposition. *$\Omega^{\text{open}}_{a_0a_1\cdots a_s}$ is an open topological cell of complex dimension $d_c = \sum_{j=0}^{s} a_j - s(s+1) + e$, where e is the number of pairs (a_i, a_j), $i < j$, $a_i + a_j < n$. For $a_j \leq n/2$ and $0 \leq j \leq s$, $\Omega^{\text{open}}_{a_0a_1\cdots a_s}$ is the ordinary Schubert cell $(\Omega^c)^{\text{open}}_{a_0a_1\cdots a_s}$ of the complex Grassmann manifold $G^c_{[n/2]+1,s+1} (\subset A^{(n+1)}_{s+1})$, in which case, $e(\Omega_{a_0a_1\cdots a_s}) = \frac{1}{2}s(s+1)$ and $d_c(\Omega_{a_0a_1\cdots a_s}) = \sum_{j=0}^{s} a_j - \frac{1}{2}s(s+1)$.*

Proof. We use induction on s. For $s = 0$, $A^{(n+1)}_1 = Q_n(C)$, and the open Schubert cells of $A^{(n+2)}_1$ are precisely the open cells of $Q_n(C)$ as determined in § 4. Let $s \geq 1$, and assume the induction hypothesis for $s - 1$. We define an onto map $F: \Omega^{\text{open}}_{a_0a_1\cdots a_s} \to \Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$ by $F(q) = q_{a_{s-1}}$. It follows from 3.9 that F is continuous. Let F_q be the fiber of F at an arbitrary $[s-1]$-plane $q \in \Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$. We have two cases to consider.

1. $a_s \leq n/2$. Then $\Omega^{\text{open}}_{a_0a_1\cdots a_s}$ is precisely the ordinary Schubert cell $(\Omega^c)^{\text{open}}_{a_0a_1\cdots a_s}$ in the Grassmann manifold $G^c_{a_s+1,s+1}$. $w \in F_q$ cuts $q^{\perp_m} \cap ([a_s] - [a_s - 1])$ at a single point P_w which uniquely determines w. Hence F_q is

homeomorphic to $q^{\perp_m} \cap ([a_s] - [a_s - 1])$ which is an open cell of complex dimension $d_c = a_s - s$. Let O_j be the unique point in $[j]$ which is m-orthogonal to $[j-1]$, and $\tilde{q} = [O_{a_0}, O_{a_1}, \cdots, O_{a_{s-1}}]$ the distinguished element of $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$. By the induction hypothesis, $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$ is an open cell and thus contractible. Hence the principal bundle $U(a_{s-1}+1) \to G^c_{a_{s-1}+1,s}$ is "trivial" over $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$, i.e., admits a cross section $t\colon \Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}} \to U(a_{s-1}+1)$. t_q maps $\tilde{q}$ onto q, and hence $\tilde{q}^{\perp_m}$ onto $q^{\perp_m}$ isomorphically. Also, t_q transforms $[a_s]$ and $[a_s - 1]$ isomorphically onto themselves. It thus induces a homeomorphism $t_q\colon F_{\tilde{q}} = \tilde{q}^{\perp_m} \cap ([a_s] - [a_s - 1]) \to q^{\perp_m} \cap ([a_s] - [a_s - 1]) = F_q$. Hence $(q, P) \mapsto t_q(P)$ yields a "trivialization" for F. Thus $\Omega^{\text{open}}_{a_0a_1\cdots a_s}$ is a product bundle $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}} \times F_{\tilde{q}}$ over $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$ and, by the induction hypothesis, is an open topological cell of complex dimension $d_c = \sum_{j=0}^{s} a_j - \frac{1}{2}s(s+1)$.

2. $a_s > n/2$. $w \in F_q$ again cuts $q^{\perp_m} \cap ([n - a_s - 1]^{\perp_f} - [n - a_s]^{\perp_f})$ at a single point P_w which uniquely determines w. It follows from 3.2 that $w \in A^{(n+2)}_{s+1}$ if and only if $P_w \in Q_n(C) \cap q^{\perp_f}$. Thus the fiber F_q is homeomorphic to

$$F_q = Q_n(C) \cap q^{\perp_m} \cap q^{\perp_f} \cap ([n - a_s - 1]^{\perp_f} - [n - a_s]^{\perp_f}) .$$

We now observe the following.

(i) By 3.1 (iv), Q^c is nonsingular on the join $q \vee c(q)$. Thus the restriction of $Q_n(C)$ to its f-orthogonal complement, i.e., to the plane $q^{\perp_m} \cap q^{\perp_f}$ is a nonsingular quadric $Q_{n-2s}(C)$.

(ii) Let e_s be the number of indices a_t such that $t < s$, $a_t + a_s < n$, or equivalently, such that $a_t \leq n - a_s - 1$. Then by the definition of $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$ we have $\dim (q \cap [n - a_s - 1]) = e_s - 1$. Since $a_t \neq n - a_s \ \forall\, t$, $q \cap [n - a_s] = q \cap [n - a_s - 1]$, i.e., $\dim (q \cap [n - a_s]) = e_s - 1$.

(iii) $q \subset Q_{a_s}(C) = Q_n(C) \cap [n - a_s]^{\perp_f}$, i.e., q and $[n - a_s]$ both lie on $Q_n(C)$ and are mutually f-orthogonal. Thus the join $q \vee [n - a_s]$ lies on $Q_n(C)$. Since $\dim (q \vee [n - a_s]) = \dim q + \dim [n - a_s] - \dim (q \cap [n - a_s]) = n - a_s - s - e_s$, the subspace $q \vee [n - a_s - 1]$ of the join also lies on $Q_n(C)$ and is of (complex) dimension $n - a_s - s - e_s - 1$.

(iv) Let h_q and k_q be the m-orthogonal complements of q in $q \vee [n - a_s]$ and $q \vee [n - a_s - 1]$ respectively. Then $h_q \subset Q_n(C) \cap q^{\perp_f}$ since $q \vee [n - a_s]$ lies on $Q_n(C)$. Thus

$$h_q \subset Q_n(C) \cap q^{\perp_f} \cap q^{\perp_m} = Q_{n-2s}(C) ,$$

$$\dim h_q = n - a_s - e_s , \qquad \dim k_q = n - a_s - e_s - 1 ,$$

$$q^{\perp_f} \cap [n - a_s]^{\perp_f} = (q \vee [n - a_s])^{\perp_f} = (q \vee h_q)^{\perp_f} = q^{\perp_f} \cap h_q^{\perp_f} .$$

Similarly,

$$q^{\perp_f} \cap [n - a_s - 1]^{\perp_f} = q^{\perp_f} \cap k_q^{\perp_f} .$$

(v) $$F_q = Q_n(C) \cap q^{\perp_m} \cap q^{\perp_f} \cap (k_q^{\perp_f} - h_q^{\perp_f}),$$

i.e.,

$$F_q = Q_{n-2s}(C) \cap (k_q^{\perp_f} - h_q^{\perp_f}) ,$$

where $\perp_f$ now denotes f-orthogonal complements in the plane $q^{\perp_m} \cap q^{\perp_f}$. Hence it follows from 3.8 that F_q is an open topological cell of (complex) dimension

$$\begin{aligned} d_c &= n - 2s - (n - a_s - e_s - 1) - 1 = a_s - 2s + e_s \\ &= (n - 2s) - \dim h_q \geq \tfrac{1}{2}(n - 2s) . \end{aligned}$$

(a) If n is *even* and $a_s - 2s + e_s = \frac{1}{2}(n - 2s)$, then h_q is a maximal plane on $Q_{n-2s}(C)$, and k_q is of codimension 1 in h_q. It follows from 3.6 that there exists a unique maximal plane h'_q belonging to the opposite variety containing h_q such that $h_q \cap h'_q = k_q$ and $Q_{n-2s}(C) \cap k_q^{\perp_f} = h_q \cup h'_q$. Thus

$$F_q = Q_{n-2s}(C) \cap k_q^{\perp_f} - Q_{n-2s}(C) \cap h_q^{\perp_f} = h_q \cup h'_q - h_q = h'_q - k_q$$

is an *open projective space*.

(b) If $a_s - 2s + e_s > \frac{1}{2}(n - 2s)$, then $Q_{n-2s}(C) \cap k_q^{\perp_f}$ is an $[n - a_s - e_s]$-degenerate quadric $Q_{a_s-2s+e_s}(C)$, and hence $F_q = Q_{a_s-2s+e_s}(C) - Q_{n-2s}(C) \cap h_q^{\perp_f}$ is an *open quadric*. Let

$$\begin{aligned} \Delta_{s,n-a_s-e_s,1} = SO(n+2)/U(s) \times U(n - a_s - e_s) \\ \times U(1) \times SO(a_s - s + e_s - n) \end{aligned}$$

be the flag manifold of triplets of ordered mutually m-orthogonal $[s - 1]$, $[n - a_s - e_s - 1]$ and $[0]$-subspaces of $[n - a_s + s]$-spaces lying on $Q_n(C)$. Define $\theta\colon \Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}} \to \Delta_{s,n-a_s-e_s,1}$ by $\theta(q) = (q, k_q, r_q)$ where r_q is the unique point in h_q which is m-orthogonal to k_q. Continuity of θ follows from 3.9. By the induction hypothesis, $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$ is an open contractible cell, and thus θ admits a lifting t to $SO(n + 2)$, i.e.,

$$\begin{tikzcd} & SO(n+2) \arrow[d] \\ \Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}} \arrow[ur, dashed, "t"] \arrow[r, "\theta"] & \Delta_{s,n-a_s-e_s,1} \end{tikzcd} .$$

Let O_j be the unique point of $[j]$, m-orthogonal to $[j - 1]$, $O'_{n-j} = c(O_{n-j})$ the unique point of $Q_j(C)$, m-orthogonal to $Q_{j-1}(C)$, $0 \leq x \leq s - 1$ the largest integer such that $a_x \leq n/2$, $\tilde{q} = [O_{a_0}, \cdots, O_{a_x}, O'_{n-a_x+1}, \cdots, O'_{n-a_s}]$ the distinguished element of $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$, and $\theta(\tilde{q}) = (\tilde{q}, \tilde{k}_q, \tilde{r}_q)$ the distinguished element of $\Delta_{s,n-a_s-e_s,1}$. t_q maps $\tilde{q}$ isomorphically onto q, and therefore the plane $\tilde{q}^{\perp_f} \cap \tilde{q}^{\perp_m}$ isomorphically onto the plane $q^{\perp_f} \cap q^{\perp_m}$. Hence t_q maps $\tilde{Q}_{n-2s}(C)$ homeomorphically onto $Q_{n-2s}(C)$. Also, t_q is an isomorphism of $\tilde{h}_q$ and $\tilde{k}_q$ onto h_q and k_q, and thus of $\tilde{h}_q^{\perp_f}$ and $\tilde{k}_q^{\perp_f}$ onto $h_q^{\perp_f}$ and $k_q^{\perp_f}$ respectively, and therefore induces a homeomorphism

$$t_q\colon F_{\tilde q} = \tilde Q_{n-2s}(C) \cap (\tilde k_{\tilde q}^{\perp f} - \tilde h_{\tilde q}^{\perp f}) \to Q_{n-2s}(C) \cap (k_q^{\perp f} - h_q^{\perp f}) = F_q \,.$$

Thus $(q, P) \mapsto t_q(P)$ yields a "trivialization"

$$\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}} \times F_{\tilde q} \xrightarrow{=} \Omega^{\text{open}}_{a_0a_1\cdots a_s} \,.$$

Hence $\Omega^{\text{open}}_{a_0a_1\cdots a_s}$ is a product bundle over $\Omega^{\text{open}}_{a_0a_1\cdots a_{s-1}}$ and, by the induction hypothesis, is an open topological cell of (complex) dimension

$$\begin{aligned} d_c &= \sum_{j=0}^{s-1} a_j - (s-1)s + e(\Omega_{a_0a_1\cdots a_{s-1}}) + a_s - 2s + e_s \\ &= \sum_{j=0}^{s} a_j - s(s+1) + e(\Omega_{a_0a_1\cdots a_s}) \,. \quad \text{q.e.d.} \end{aligned}$$

Suppose $\dim q_{a_j} \geq j$ and $\dim q_{a_{j-1}} < j$. Since $\dim q_{a_j} \leq \dim q_{a_{j-1}} + 1$, it follows that $\dim q_{a_j} = j$ and $\dim q_{a_{j-1}} = j - 1$. Hence we have the standard identity

$$\Omega^{\text{open}}_{a_0a_1\cdots a_s} = \Omega_{a_0a_1\cdots a_s} - \bigcup_{a_{j-1}<a_j-1} \Omega_{a_0\cdots(a_j-1)\cdots a_s} \,,$$

or, equivalently,

$$\Omega^{\text{open}}_{a_0a_1\cdots a_s} = \Omega_{a_0a_1\cdots a_s} - \bigcup_{b<a} \Omega_{b_0b_1\cdots b_s} \,,$$

which, by applying the lemma of § 5, this can be strengthened to read:

$$\Omega^{\text{open}}_{a_0a_1\cdots a_s} = \Omega_{a_0a_1\cdots a_s} - \bigcup_{c<a} \Omega_{c_0\cdots c_s} \quad \text{with} \quad c_i + c_j \neq n,\ i < j \,.$$

It follows from the preceeding proposition (by induction on the dimension) that $\Omega_{a_0a_1\cdots a_s}$, $a_i + a_j \neq n$, $i < j$, is a topological cell attached to the Schubert cells $(\Omega_{c_0c_1\cdots c_s} \mid c < a,\ c_i + c_j \neq n,\ i < j)$ lying on its boundary. This immediately yields the following CW-structure theorem which is the main result of this paper.

***CW*-structure theorem.** *$A_{s+1}^{(n+2)}$ is a CW-complex consisting of Schubert cells $\Omega_{a_0a_1\cdots a_s}$ for $0 \leq a_0 < a_1 < \cdots < a_s \leq n$, $a_i + a_j \neq n$, $i < j$, $\Omega_{a_0a_1\cdots a_s}$ is the variety of $[s]$-planes on $Q_n(C)$ which intersect the complex a_j-dimensional cell of $Q_n(C)$ at a plane of complex dimension j, $0 \leq j \leq s$, and*

$$\dim(\Omega_{a_0a_1\cdots a_s}) = 2\left(\sum_{j=0}^{s} a_j - s(s+1) + e\right) ,$$

where e is the number of pairs (a_i, a_j), $i < j$, $a_i + a_j < n$.

Demonstration. As a demonstration of the CW-structure theorem, we now present the following examples.

1. $A_3^{(8)} = [2]$-planes on $Q_6(C)$

Ω_{012} 0	Ω_{013_0} 2	Ω_{013_1} 2	Ω_{014} 4	Ω_{023_0} 4	Ω_{023_1} 4	Ω_{025} 6	Ω_{03_04} 6
Ω_{456} 18	Ω_{3_156} 16	Ω_{3_056} 16	Ω_{256} 14	Ω_{3_146} 14	Ω_{3_046} 14	Ω_{146} 12	Ω_{23_16} 12

Ω_{08_14} 6	Ω_{03_05} 8	Ω_{03_15} 8	Ω_{045} 10	Ω_{123_0} 6	Ω_{123_1} 6	Ω_{13_04} 8	Ω_{13_14} 8
Ω_{23_06} 12	Ω_{13_16} 10	Ω_{13_06} 10	Ω_{126} 8	Ω_{3_145} 12	Ω_{3_045} 12	Ω_{23_15} 10	Ω_{23_05} 10

Dual cells appear in the same column, and the number in the corner indicates the dimension of the cell. (Refer to § 8 for duality.)

2. $A_2^{(7)} =$ [1]-planes on $Q_5(C)$

Ω_{01} 0	Ω_{02} 2	Ω_{03} 4	Ω_{04} 6	Ω_{12} 4	Ω_{13} 6
Ω_{45} 14	Ω_{35} 12	Ω_{25} 10	Ω_{15} 8	Ω_{34} 10	Ω_{24} 8

Corollary. *The inclusion map* $j\colon A_s^{(n)} \subset A_{s+1}^{(n+1)}$ *is "cellular", and* $A_s^{(n)}$ *is the subcomplex of* $A_{s+1}^{(n+2)}$ *consisting of Schubert cells* $\Omega_{a_0\cdots a_s}$ *for which* $a_0 = 0$. *In particular,* $Q_{n-2s}(C) = A_1^{(n-2s+2)}$ *is the subcomplex of* $A_{s+1}^{(n+2)}$ *consisting of Schubert cells for which* $a_j = 0$, $j < s$.

6. Homology and cohomology of $A_{s+1}^{(n+2)}$

Since $A_{s+1}^{(n+2)}$ admits a triangulation by *even* dimensional cells only, the boundary and coboundary operators are zero, and each Schubert cell represents a distinct homology (cohomology) class. Hence $A_{s+1}^{(n+2)}$ is simply connected, $H^*(A_{s+1}^{(n+2)}; Z)$ is torsion free and vanishes in odd dimensions. $H^{2i}(A_{s+1}^{(n+2)}; Z)$ is the free abelian group on Schubert cells $\Omega_{a_0a_1\cdots a_s}$ for which $\dim \Omega_{a_0a_1\cdots a_s} = 2i$.

The Euler-Poincaré characteristic

$$\chi(A_{s+1}^{(n+2)}) = \text{Total number of cells} = 2^{s+1}\cdot\binom{[n/2]+1}{s+1}.$$

It follows from Proposition 2.5.2 of [1] that $K^1(A_{s+1}^{(n+2)}) = 0$ and $K^0(A_{s+1}^{(n+2)})$ is the free abelian group on $\chi(A_{s+1}^{(n+2)})$ generators.

7. Maximal planes on $Q_n(C)$

The special case of the CW-structure theorem for $s = [n/2]$ reduces to Ehressmann's triangulation in [5] of the variety of maximal planes on $Q_n(C)$.

(i) For $n = 2s$ the indices $(a_0, a_1, \cdots, a_s)$ of a Schubert cell $\Omega_{a_0 \cdots a_s}$ are picked one from each column of

$$\begin{pmatrix} 0 & 1 & \cdots s-1 & s_0 \\ 2s & 2s-1 & \cdots s+1 & s_1 \end{pmatrix},$$

since $a_i + a_j \neq n$. Thus once $(a_0, \cdots, a_{x-1})$ are chosen (where $0 \leq x \leq s$ is the largest integer such that $a_x \leq n/2$), a_x is either s_0 or s_1, and the rest of the indices $(a_{x+1}, \cdots, a_s)$ are the elements in the 2nd-row of the complementary columns. Let $V_j = I_{s+1}$ be the irreducible subvariety of $A_{s+1}^{(2s+2)}$ containing $[s]_j$ for $j = 0, 1$. Then it follows from 3.5 that $\Omega_{a_0 a_1 \cdots a_s}$ lies in V_0 if and only if

$$a_x = \begin{cases} s_0 & \text{for } x \equiv s \pmod 2, \\ s_1 & \text{for } x \equiv s-1 \pmod 2, \end{cases}$$

and in V_1 if and only if

$$a_x = \begin{cases} s_1 & \text{for } x \equiv s \pmod 2, \\ s_0 & \text{for } x \equiv s-1 \pmod 2. \end{cases}$$

Thus the Schubert cells of $A_{s+1}^{(2s+2)}$ are evenly divided between V_0 and V_1, and each $\Omega_{a_0 a_1 \cdots a_s}$ is uniquely determined by the indices $(a_0, a_1, \cdots, a_{x-1})$, i.e., by the dimensions of intersection with the decomposition $[s-1] \supset [s-2] \supset \cdots \supset [1] \supset [0]$. We thus put $\Omega_{a_0 a_1 \cdots a_s} = [a_0, a_1, \cdots, a_{x-1}]$ and

$$\begin{aligned} e(\Omega) &= \tfrac{1}{2}x(x+1) + (2s - a_s) + (2s - a_{s-1} - 1) + \cdots \\ &\quad + (2s - a_{x+1} - (s - x - 1)), \\ \dim_c(\Omega) &= \sum_{j=0}^{s} a_j - s(s+1) + \tfrac{1}{2}x(x+1) + 2s(s-x) \\ &\quad - \sum_{j-x+1}^{s} a_j - \tfrac{1}{2}(s-x)(s-x-1), \end{aligned}$$

i.e.,

$$\dim_c [a_0, a_1, \cdots, a_{x-1}] = \sum_{j=0}^{x-1} a_j + \tfrac{1}{2}s(s - 2x + 1).$$

(ii) For $n = 2s + 1$ the indices of a Schubert cell $\Omega_{a_0 a_1 \cdots a_s}$ are picked one from each column of

$$\begin{pmatrix} 0 & 1 & \cdots s-1 & s \\ 2s+1 & 2s & \cdots s+2 & s+1 \end{pmatrix}.$$

Thus once the first set indices $(a_0, a_1, \cdots, a_x)$ are given, the rest $(a_{x+1}, \cdots, a_s)$ are simply elements of the 2nd-row of the complementary columns. Hence $\Omega_{a_0a_1\cdots a_s}$ is uniquely determined by the dimensions of intersection with the decomposition $[s] \supset [s-1] \supset \cdots \supset [1] \supset [0]$. We thus denote $\Omega_{a_0a_1\cdots a_s} = [a_0, a_1, \cdots, a_x]$,

$$\begin{aligned} e(\Omega) &= \tfrac{1}{2}x(x+1) + (2s+1-a_s) + (2s-a_{s-1}) + \cdots \\ &\quad + (2s+1-a_{x+1}-(s-x-1)) , \\ \dim_c(\Omega) &= \sum_{j=0}^{s} a_j - s(s+1) + \tfrac{1}{2}x(x+1) + (s-x)(2s+1) \\ &\quad - \sum_{j=x+1}^{s} a_j - \tfrac{1}{2}(s-x)(s-x-1) , \end{aligned}$$

i.e.,

$$\dim_c [a_0, a_1, \cdots, a_x] = \sum_{j=0}^{x} a_j + \tfrac{1}{2}(s+1)(s-2x) .$$

(iii) Let $h\colon A_s^{(2s+1)} \xrightarrow{=} V_0$ be the canonical homeomorphism of 3.7 between the variety $A_s^{(2s+1)}$ of maximal planes on $Q_{2s-1}(C)$ and the irreducible subvariety V_0 of maximal planes on $Q_{2s}(C)$. Let $[s-1] \supset [s-2] \supset \cdots \supset [1] \supset [0]$ be the cellular decomposition of the maximal plane $[s-1]$ on $Q_{2s-1}(C)$, and $[s]_0 \supset [s-1] \supset \cdots \supset [1] \supset [0]$ the cellular decomposition of $[s]_0 = h[s-1]$. Then using the notation introduced above, we can identify the Schubert cells $[a_0, a_1, \cdots, a_t]$ of V_0 and $[a_0, a_1, \cdots, a_t]$ of $A_s^{(2s+1)}$ for $0 \leq a_0 < a_1 < \cdots < a_t \leq s-1$ through the homeomorphism h.

8. Duality theory for $A_{s+1}^{(n+2)}$

We first briefly summarize the standard duality theory for $G^c_{n+2,s+1}$. (For details see [8, Chapter III].) Let

$$(1) \qquad [n+1] \supset [n] \supset \cdots \supset [1] \supset [0]$$

be a cellular decomposition for $P_{n+1}(C)$, and

$$(2) \qquad [n+1] \supset [0]^{\perp m} \supset [1]^{\perp m} \supset \cdots \supset [n]^{\perp m}$$

the dual cellular decomposition by m-complementary planes. Let P_j be the unique point of $[j]$ which is m-orthogonal to $[j-1]$. Let $(\Omega^c_{a_0a_1\cdots a_s})$ and $(\bar{\Omega}^c_{b_0\cdots b_s})$ be the two systems of Schubert cells of $G^c_{n+2,s+1}$ arising from (1) and (2) respectively. $\bar{\Omega}^c_{n-a_s\cdots n-a_0}$ is called the dual cell of $\Omega^c_{a_0a_1\cdots a_s}$. The duality theory for $G^c_{n+2,s+1}$ states that two Schubert cells $\Omega^c_{a_0a_1\cdots a_s}$ and $\bar{\Omega}^c_{b_0b_1\cdots b_s}$ of complementary dimensions intersect transversally at a single point $q = [P_{a_0}P_{a_1} \cdots P_{a_s}]$ if they are dual, and are disjoint if not.

We saw in § 4 that if $[p] \supset [p-1] \supset \cdots \supset [1] \supset [0]$ is the cellular decomposition of a maximal plane $[p]$ on $Q_n(C)$, then the corresponding cellular decomposition

$$(3) \qquad \begin{aligned} &[n+1] \supset [0]^{\perp_f} \supset [1]^{\perp_f} \supset \cdots \supset [n-p-1]^{\perp_f} \\ &\qquad \supset [p] \supset [p-1] \supset \cdots \supset [1] \supset [0] \end{aligned}$$

of $P_{n+1}(C)$ gives rise to a cellular decomposition for $Q_n(C)$:

$$\begin{aligned} &Q_{2p+1}(C) \supset Q_{2p}(C) \supset \cdots \supset Q_{p+1}(C) \\ &\qquad \supset [p] \supset [p-1] \supset \cdots \supset [1] \supset [0] \qquad \text{for } n = 2p+1 , \\ &Q_{2p}(C) \supset Q_{2p-1}(C) \supset \cdots \supset Q_{p+1}(C) \supset [p]_0 , \\ &\qquad [p]_1 \supset [p-1] \supset \cdots \supset [0] \qquad \text{for } n = 2p . \end{aligned}$$

Let

$$(4) \qquad \begin{aligned} &[n+1] \supset [0]^{\perp_m} \supset \cdots \supset [n-p-1]^{\perp_m} \\ &\qquad \supset \{[p]^{\perp_f}\}^{\perp_m} \supset \cdots \supset \{[0]^{\perp_f}\}^{\perp_m} \end{aligned}$$

be the dual decomposition of $P_{n+1}(C)$ by m-complementary planes. Since, $[k]^{\perp_m} = c([k])^{\perp_f}$ and $([k]^{\perp_f})^{\perp_m} = c([k])$, $0 \leq k \leq p$, (4) is precisely the cellular decomposition

$$(5) \qquad \begin{aligned} &[n+1] \supset c([0])^{\perp_f} \supset \cdots \supset c([n-p-1])^{\perp_f} \\ &\qquad \supset c([p]) \supset \cdots \supset c([0]) \end{aligned}$$

corresponding to the maximal plane $c([p])$ on $Q_n(C)$, and thus induces a cellular decomposition for $Q_n(C)$. We put

$$\begin{aligned} [\bar{k}] &= c([k]) \qquad \text{for } 0 < k < p , \\ \overline{Q_k(C)} &= c([n-k-1])^{\perp_f} \cap Q_n(C) , \\ [\bar{k}] &= [\overline{n-k}]^{\perp_f} = c([n-k])^{\perp_f} \qquad \text{for } k > p . \end{aligned}$$

For $n = 2p$, $[p]_j$ is disjoint from $c([p]_j)$ for $j = 0, 1$. It follows from 3.5 that

$$\begin{aligned} c([p]_0) \in V_1 \quad &\text{and} \quad c([p]_1) \in V_0 \qquad \text{for } p \text{ even} , \\ c([p]_0) \in V_0 \quad &\text{and} \quad c([p]_1) \in V_1 \qquad \text{for } p \text{ odd} . \end{aligned}$$

Thus we put

$$[\bar{p}]_0 = \begin{cases} c([p]_1) & \text{for } p \text{ even} , \\ c([p]_0) & \text{for } p \text{ odd} , \end{cases} \qquad [\bar{p}]_1 = \begin{cases} c([p]_0) & \text{for } p \text{ even} , \\ c([p]_1) & \text{for } p \text{ odd} . \end{cases}$$

Also for $n = 2p+1$, put $[\bar{p}] = c([p])$.

With this notation, the induced cellular decomposition of $Q_n(C)$ reads as:

$$\begin{aligned} Q_{2p+1}(C) &\supset \overline{Q_{2p}(C)} \supset \cdots \supset \overline{Q_{p+1}(C)} \\ &\quad \supset [\bar{p}] \supset [\overline{p-1}] \supset \cdots \supset [\bar{0}] \qquad \text{for } n = 2p+1 \ , \\ Q_{2p}(C) &\supset \overline{Q_{2p-1}(C)} \supset \cdots \supset \overline{Q_{p+1}(C)} \supset [\bar{p}]_0 \ , \\ &\qquad [\bar{p}]_1 \supset [\overline{p-1}] \supset \cdots \supset [\bar{0}] \qquad \text{for } n = 2p \ . \end{aligned}$$

The Schubert cells, arising from this decomposition, will be denoted by $\bar{\Omega}_{a_0 \cdots a_s}$. It is clear that the two cellular decompositions of $Q_n(C)$ (obtained from (1) and (2) are congruent under the action of $SO(n+2)$, and thus the corresponding Schubert cells $\Omega_{a_0 a_1 \cdots a_s}$ and $\bar{\Omega}_{a_0 a_1 \cdots a_s}$ represent the same homology class. Let also $(\Omega^c_{b_0 \cdots b_s})$ and $(\bar{\Omega}^c_{b_0 \cdots b_s})$ be the two systems of ordinary Schubert cells of the Grassmann variety $G^c_{n+2,s+1}$ corresponding to (3) and (4) respectively.

Definition. $\Omega^t_{a_0 a_1 \cdots a_s} = \bar{\Omega}_{n-a_s n-a_{s-1} \cdots n-a_0}$ is called the dual cell of $\Omega_{a_0 a_1 \cdots a_s}$ with the following convention:

If $n = 2p$, then put, for $a_j = p_0$,

$$n - a_j = \begin{cases} p_0 & \text{for } p \text{ even} \ , \\ p_1 & \text{for } p \text{ odd} \ , \end{cases}$$

and, for $a_j = p_1$,

$$n - a_j = \begin{cases} p_1 & \text{for } p \text{ even} \ , \\ p_0 & \text{for } p \text{ odd} \ . \end{cases}$$

$$\begin{aligned} e(\Omega_{a_0 a_1 \cdots a_s}) &= \text{number of pairs } (a_i, a_j),\ i < j,\ a_i + a_j < n \ . \\ e(\Omega^t_{a_0 a_1 \cdots a_s}) &= \text{number of pairs } (a_i, a_j),\ i < j,\ a_i + a_j > n \ . \end{aligned}$$

Thus $e(\Omega) + e(\Omega^t) = \frac{1}{2}s(s+1)$, and by the CW-structure theorem,

$$\dim_c(\Omega) + \dim_c(\Omega^t) = \tfrac{1}{2}(s+1)(2n-3s) = \dim_c A^{(n+2)}_{s+1} \ .$$

Also $\Omega_{a_0 a_1 \cdots a_s} \longmapsto \Omega^t_{a_0 a_1 \cdots a_s}$ is a bijection between Schubert cells of a fixed dimension and those of complementary dimension.

Lemma. *There exists a minimal imbedding J of the system $(\Omega_{a_0 a_1 \cdots a_s})$ of $A^{(n+2)}_{s+1}$ into the system $(\Omega^c_{b_0 b_1 \cdots b_s})$ of $G^c_{n+2,s+1}$, and a minimal embedding $\bar{J}$ of $(\bar{\Omega}_{a_0 a_1 \cdots a_s})$ into $(\bar{\Omega}^c_{b_0 b_1 \cdots b_s})$ such that*

(i) $\Omega_{a_0 a_1 \cdots a_s} \subset J(\Omega_{a_0 a_1 \cdots a_s})$ *and* $\bar{\Omega}_{a_0 a_1 \cdots a_s} \subset \bar{J}(\bar{\Omega}_{a_0 a_1 \cdots a_s})$, *and* $\Omega_{a_0 a_1 \cdots a_s} \subset \Omega_{b_0 b_1 \cdots b_s}$ *in* $A^{(n+1)}_{s+1}$ *if and only if* $J(\Omega_{a_0 \cdots a_s}) \subset J(\Omega_{b_0 \cdots b_s})$ *in* $G^c_{n+2,s+1}$ *(and a similar condition for $\bar{J}$).*

(ii) $\Omega_{a_0 a_1 \cdots a_s}$ *and* $\bar{\Omega}_{b_0 b_1 \cdots b_s}$ *are "dual in* $A^{(n+2)}_{s+1}$ *if and only if* $J(\Omega_{a_0 a_1 \cdots a_s})$ *and* $\bar{J}(\bar{\Omega}_{b_0 b_1 \cdots b_s})$ *are "dual" in* $G^c_{n+2,s+1}$.

(iii) $J(\Omega_{a_0 a_1 \cdots a_s}) \cap A^{(n+2)}_{s+1} = \Omega_{a_0 a_1 \cdots a_s}$ *except for* $n = 2p$ *and* $a_j = p_1$ *for* *some j, in which case $J(\Omega_{a_0\cdots p_1\cdots a_s}) \cap A_{s+1}^{(n+2)} = \Omega_{a_0\cdots p_1\cdots a_s} \cup \Omega_{a_0\cdots p_0\cdots a_s}$ (and a similar condition for $\bar{J}$).*

Proof. We first construct imbeddings j and $\bar{j}$ of the cells of $Q_n(C)$ into those of $P_{n+1}(C)$ as defined by (3) and (4) respectively by putting:

$$\begin{aligned}
j([k]) &= [k] \qquad \text{for } 0 < k < n/2 \ , \\
j([p]_0) &= [p] \quad \text{and} \quad j([p]_1) = [p+1] = [p-1]^{\perp_f} \qquad \text{for } n = 2p \ , \\
j(Q_k(C)) &= [k+1] = [n-k-1]^{\perp_f} \qquad \text{for } k > n/2 \text{; similarly ,} \\
\bar{j}([\bar{k}]) &= [\bar{k}] \qquad \text{for } 0 \le k < n/2, \text{ and for } n = 2p \ , \\
\bar{j}([p]_0) &= \begin{cases} \overline{[p-1]}^{\perp_f} = \overline{[p+1]} & \text{for } p \text{ even ,} \\ [\bar{p}] & \text{for } p \text{ odd ,} \end{cases} \\
\bar{j}([p]_1) &= \begin{cases} [\bar{p}] & \text{for } p \text{ even ,} \\ \overline{[p-1]}^{\perp_f} = \overline{[p+1]} & \text{for } p \text{ odd ,} \end{cases} \\
\bar{j}(\overline{Q_k(C)}) &= \overline{[n-k-1]}^{\perp_f} = \overline{[k+1]} \qquad \text{for } k > n/2 \ .
\end{aligned}$$

Define J and $\bar{J}$ by

$$\begin{aligned}
J(\Omega_{a_0a_1\cdots a_s}) &= \Omega^c_{j(a_0)j(a_1)\cdots j(a_s)} \ , \\
\bar{J}(\bar{\Omega}_{a_0a_1\cdots a_s}) &= \bar{\Omega}^c_{\bar{j}(a_0)\bar{j}(a_1)\cdots \bar{j}(a_s)} \ .
\end{aligned}$$

Properties (i), (ii) and (iii) are easily verified from the definition. q.e.d.

This lemma enables us to develop a duality theory for $A_{s+1}^{(n+2)}$ from the standard duality theory for $G^c_{n+2,s+1}$.

Proposition. (i) *$\Omega_{a_0a_1\cdots a_s} \cap \Omega^t_{b_0b_1\cdots b_s} = \emptyset$ unless*

$$\Omega_{a_0a_1\cdots a_s} \supset \Omega_{b_0b_1\cdots b_s} \qquad (\text{i.e., } a \ge b) \ .$$

(ii) *Let O_j be the unique point of $[j]$ which is m-orthogonal to $[j-1]$, and let $O'_j = c(O_j)$, $0 \le j \le s$. Let $0 \le x \le s$ be the largest integer such that $a_x \le n/2$. Then $\Omega_{a_0a_1\cdots a_s}$ and $\bar{\Omega}_{b_0b_1\cdots b_s}$ of complementary dimension intersect transversally at a single $[s]$-plane $\tilde{q} = [O_{a_0}, \cdots, O_{a_x}, O'_{n-a_{x+1}}, \cdots, O'_{n-a_s}]$ if they are "dual", and are disjoint if not.*

Proof. Suppose $\Omega_{a_0\cdots a_s} \not\supset \Omega_{b_0b_1\cdots b_s}$. Then $J(\Omega_{a_0\cdots a_s}) \not\supset J(\Omega_{b_0\cdots b_s})$ by part (i) of the lemma, and it follows from the duality theory for $G^c_{n+2,s+1}$ that $J(\Omega_{a_0\cdots a_s}) \cap J(\Omega_{b_0\cdots b_s})^t = \emptyset$. Also $J(\Omega_{b_0\cdots b_s})^t = \bar{J}(\Omega^t_{b_0\cdots b_s})$ by Part (ii) of the lemma. Thus $J(\Omega_{a_0\cdots a_s})$, $\bar{J}(\Omega^t_{b_0\cdots b_s})$ and their subsets $\Omega_{a_0\cdots a_s}$, $\Omega^t_{b_0\cdots b_s}$ are disjoint, respectively, by the lemma.

(ii) It follows from Part (ii) of the lemma that if $\Omega_{a_0\cdots a_s}$ and $\bar{\Omega}_{b_0\cdots b_s}$ are dual in $A_{s+1}^{(n+2)}$, so are $J(\Omega_{a_0\cdots a_s})$ and $\bar{J}(\bar{\Omega}_{b_0\cdots b_s})$ in $G^c_{n+2,s+1}$, and $J(\Omega_{a_0\cdots a_s})$ and $\bar{J}(\bar{\Omega}_{b_0\cdots b_s})$ intersect transversally at a single $[s]$-plane $\tilde{q} = [O_{a_0}, \cdots, O_{a_x}, O'_{n-a_{x+1}}, \cdots, O'_{n-a_s}]$ by the duality theory for $G^c_{n+2,s+1}$.

Obviously, $\tilde{q} \in \Omega_{a_0\cdots a_s} \cap \bar{\Omega}_{b_0\cdots b_s}$, and the subset $\Omega_{a_0\cdots a_s}$ of $J(\Omega_{a_0\cdots a_s})$ and subset the $\bar{\Omega}_{b_0\cdots b_s}$ of $\bar{J}(\bar{\Omega}_{b_0\cdots b_s})$ also intersect transversally at $\tilde{q}$. If $\Omega_{a_0\cdots a_s}$ and $\bar{\Omega}_{b_0\cdots b_s}$ are not dual, then it follows from Part (i) of the proposition that they are disjoint. q.e.d.

This can be best expressed in a single theorem:

Intersection theorem. *Homology classes* $\{\Omega_{a_0\cdots a_s}\}$ *and* $\{\Omega_{b_0\cdots b_s}\}$ *of complementary dimension intersect in* 1 *if they are "dual" and in* 0 *if not.*

9. Chern classes

An immediate application of the duality theory for $A_s^{(n)}$ is the computation of the Chern classes of the principal $U(s)$-bundle $V_{n,2s}(A_s^{(n)}; U(s))$.

Theorem. *"Stability" for Chern classes is attained at* $n = 2s + 3$, *and the* ith *Chern class* $c_i = \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s}$ *for* $n \geq 2s + 3$. *As for the unstable cases:*

(i) *For* $n = 2s + 2$, $c_i = \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s_0} + \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s_1}$.

(ii) *For* $n = 2s + 1$, $c_i = 2[01 \cdots s - i - 1, s - i + 1 \cdots s - 1]^*$.

(iii) *For* $n = 2s$, $c_s = 0$ *and* $c_i = 2[01 \cdots s - i - 2, s - i \cdots s - 2]^*$, $1 \leq i \leq s - 1$.

Proof. For $n \geq 2s + 3$, let $j: A_s^{(n)} \to G^c_{n,s}$ be the "inclusion", $\dim_c (\Omega_{a_0a_1\cdots a_s}) = i$, and

$$j_*(\Omega_{a_0\cdots a_s}) = k_{a_0\cdots a_s}\Omega^c_{01\cdots s-i-1\ s-i+1\cdots s}$$
$$+ \text{ linear combinations of other } [i]\text{-cells of } G^c_{n,s} .$$

Taking "intersections" of both sides with $(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}$ yields

$$k_{a_0\cdots a_s} = j_*(\Omega_{a_0\cdots a_s})\cdot(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} .$$

$n \geq 2s + 3$ implies that $n - 1 - s > \frac{1}{2}(n - 2)$, and thus $[\bar{a}_j] \cap Q_{n-2}(C) = \bar{Q}_{a_j-1}(C)$ for $a_j \geq n - 1 - s$. Hence

$$\begin{aligned} A_s^{(n)} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} &= A_s^{(n)} \cap \bar{\Omega}^c_{n-1-s\cdots n+i-s-2\ n+i-s\cdots n-1} \\ &= \bar{\Omega}_{n-2-s\cdots n+i-s-3\ n+i-s-1\cdots n-2} \\ &= \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s} , \end{aligned}$$

which implies that

$$\begin{aligned} \Omega_{a_0\cdots a_s} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} &= \Omega_{a_0\cdots a_s} \cap A_s^{(n)} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} \\ &= \Omega_{a_0\cdots a_s} \cap \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s} . \end{aligned}$$

It follows from the duality theory for $A_s^{(n)}$ that $k_{a_0\cdots a_s}$ except $k_{01\cdots s-i-1\ s-i+1\cdots s}$ all vanish. By the proposition of § 5

$$\Omega_{01\cdots s-i-1\ s-i+1\cdots s} = \Omega^c_{01\cdots s-i-1\ s-i+1\cdots s} ,$$

and thus

$$k_{01\cdots s-i-1\ s-i+1\cdots s} = \Omega^c_{01\cdots s-i-1\ s-i+1\cdots s}\cdot(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} = 1$$

by the duality theory for $G^c_{n,s}$. Hence the dual map j^* on the cohomology level satisfies

$$j^*(\Omega^c)^*_{01\cdots s-i-1\ s-i+1\cdots s} = \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s}$$

and $c_i = \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s}$ by "naturality" for Chern classes.

(i) For $n = 2s+2$, again let $j\colon A_s^{(2s+2)} \to G^c_{2s+2,s}$ be the inclusion. Then

$$j^*(\Omega^c)^*_{01\cdots s-i-1\ s-i+1\cdots s} = \sum_{\dim_c(\Omega_a)=i} k_{a_0\cdots a_s}\Omega^*_{a_0\cdots a_s},$$

$[\overline{s+1}] \cap Q_{2s}(C) = [\bar{s}_0] \cup [\bar{s}_1]$, $[\bar{a}_j] \cap Q_{2s}(C) = \bar{Q}_{a_j-1}(C)$, for $a_j \geq s+2$,

$$\begin{aligned} &A_s^{(2s+2)} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} \\ &\quad = A_s^{(2s+2)} \cap \bar{\Omega}^c_{s+1\cdots s+i\ s+i+2\cdots 2s+1} \\ &\quad = \bar{\Omega}_{s_0\ s+1\cdots s+i-1\ s+i+1\cdots 2s} \cup \bar{\Omega}_{s_1\ s+1\cdots s+i-1\ s+i+1\cdots 2s} \\ &\quad = \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s_0} \cup \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s_1}, \end{aligned}$$

and thus

$$\begin{aligned} &\Omega_{a_0\cdots a_s} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} \\ &\quad = \Omega_{a_0\cdots a_s} \cap A_s^{(2s+2)} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} \\ &\quad = \Omega_{a_0a_1\cdots a_s} \cap (\Omega^t_{01\cdots s-i-1\ s-i+1\cdots s_0} \cup \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s_1}). \end{aligned}$$

Hence $k_{a_0\cdots a_s}$ except $k_{01\cdots s-i-1\ s-i+1\cdots s_0}$ and $k_{01\cdots s-i-1\ s-i+1\cdots s_1}$ all vanish.

$\Omega^c_{01\cdots s-i-1\ s-i+1\cdots s}$ and $(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}$ intersect transversally at a single $[s-1]$-plane $\tilde{q} = [O_0, \cdots, O_{s-i-1}, O_{s-i+1}, \cdots, O_s]$ and $\tilde{q} \in \Omega_{01\cdots s-i-1\ s-i+1\cdots s_0} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}$, and thus their subsets

$$\Omega_{01\cdots s-i-1\ s-i+1\cdots s_0}, \qquad (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}$$

also intersect transversally at $\tilde{q}$. Hence $k_{01\cdots s-i-1\ s-i+1\cdots s_0} = 1$, and similarly $k_{01\cdots s-i-1\ s-i+1\cdots s_1} = 1$.

$$j^*(\Omega^c)^*_{01\cdots s-i-1\ s-i+1\cdots s} = \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s_0} + \Omega^*_{01\cdots s-i-1\ s-i+1\cdots s_1},$$

and, by naturality, the result follows.

(ii) For $n = 2s+1$,

$$[\bar{s}] \cap Q_{2s-1}(C) = [\overline{s-1}], \quad [\bar{a}_j] \cap Q_{2s-1}(C) = \bar{Q}_{a_j-1}(C) \quad a_j > s,$$

$$\begin{aligned} A_s^{(2s+1)} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} &= A_s^{(2s+1)} \cap \bar{\Omega}^c_{s\cdots s+i-1\ s+i+1\cdots 2s} \\ &= \bar{\Omega}_{s-1\ s\cdots s+i-2\ s+i\cdots 2s-1}, \end{aligned}$$

$a_0 + a_1 = (s-1) + s = 2s-1$, and repeatedly using the method of the proof of the lemma in § 5 we obtain

$$\begin{aligned}\bar{\Omega}_{s-1\ s\ s+1\cdots s+i-2\ s+i\cdots 2s-1} &= \bar{\Omega}_{s-2\ s\ s+1\cdots s+i-2\ s+i\cdots 2s-1}\\ &\ \ \vdots\\ &= \bar{\Omega}_{s-i\ s\ s+1\cdots s+i-2\ s+i\cdots 2s-1}\\ &= \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1}\ ,\end{aligned}$$

and therefore

$$\begin{aligned}&\Omega_{a_0\cdots a_s} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}\\ &\quad= \Omega_{a_0\cdots a_s} \cap A_s^{(2s+1)} \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}\\ &\quad= \Omega_{a_0\cdots a_s} \cap \Omega^t_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1}\ .\end{aligned}$$

Thus $k_{a_0\cdots a_s}$ except $k_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1}$ all vanish.

$\Omega_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1}$ and $(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}$ intersect at a single $[s-1]$-plane $\tilde{q} = [O_0, \cdots, O_{s-i-1}, O_{s-i+1}, \cdots, O_{s-1}, O'_{s-i}]$, and $k_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1}$ is the degree of intersection at this point. Let $a = [O_0, \cdots, O_{s-i-1}, O_{s-i+1}, \cdots, O_{s-1}]$, and let S_a and S_0 be the submanifolds of $G^c_{2s+1,s}$ of planes passing through a and O'_{s-i} respectively. Then by 3.11 we have a direct sum decomposition of tangent planes

$$T_q(G^c_{2s+1,s}) = T_q(S_a) \oplus T_q(S_0)\ . \tag{6}$$

Also

$$\begin{aligned}S_a \cap \Omega_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1} &= \Omega_{01\cdots s-i-1\ s-i+1\cdots s-1}\ ,\\ S_0 \cap \Omega_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1} &= Q_1(C)\ ,\end{aligned}$$

where $Q_1(C)$ is the nonsingular quadric on the 2-plane (O_{s-i}, Y, O'_{s-i}), Y being the unique point of $[s-1]^{\perp_f}$ which is m-orthogonal to $[s-1]$. Since

$$\dim \Omega_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1} = \dim \Omega_{01\cdots s-i-1\ s-i+1\cdots s-1} + \dim Q_1(C)\ ,$$

we obtain a subdecomposition of (6):

$$\begin{aligned}&T_q(\Omega_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1})\\ &\quad= T_q(\Omega_{01\cdots s-i-1\ s-i+1\cdots s-1}) \oplus T_q Q_1(C)\ .\end{aligned} \tag{7}$$

Also

$$S_a \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} = (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s-1}\ ,$$

where t on the right hand side denotes "dual" in the Grassmann manifold $G^c_{2s,s-1} = [s-2]$-planes on $(O'_{s-i})^{\perp_m}$, and

$$S_0 \cap (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} = [\overline{s-1}]^{\perp_f} ,$$
$$\dim (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s}$$
$$= \dim (\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s-1} + \dim [\overline{s-1}]^{\perp_f} .$$

Thus we obtain

$$T_q(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} = T_q(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s-1} \oplus T_q[\overline{s-1}]^{\perp_f} . \tag{8}$$

Since (7) and (8) are subdecompositions of the same direct sum decomposition (6),

$$\begin{aligned} &T_q(\Omega_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1}) \cap T_q(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s} \\ &\quad = T_q(\Omega_{01\cdots s-i-1\ s-i+1\cdots s-1}) \cap T_q(\Omega^c)^t_{01\cdots s-i-1\ s-i+1\cdots s-1} \\ &\quad \oplus T_qQ_1(C) \cap T_q[\overline{s-1}]^{\perp_f} . \end{aligned} \tag{9}$$

The first summand is zero by the duality theory for $G^c_{2s,s-1}$. Let $P_1(C) = (O'_{s-i})^{\perp_f}$ in the 2-plane (O_{s-i}, Y, O'_{s-i}). Then $P_1(C) \subset [\overline{s-1}]^{\perp_f}$, and it follows from 3.10 that $\dim T_qQ_1(C) \cap T_qP_1(C) = 1$. Since $T_qQ_1(C) \not\subset T_q[\overline{s-1}]^{\perp_f}$, we have $\dim T_qQ_1(C) \cap T_q[\overline{s-1}]^{\perp_f} = 1$, and it follows from (9) that

$$k_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1} = 2 , \qquad \text{i.e.,}$$
$$\begin{aligned} c_i = j^*(\Omega^c)^*_{01\cdots s-i-1\ s-i+1\cdots s} &= 2\Omega^*_{01\cdots s-i-1\ s-i+1\cdots s-1\ s+i-1} \\ &= 2[01 \cdots s-i-1, s-i+1 \cdots s-1]^* \end{aligned}$$

by the notation of § 7.

(iii) For $n = 2s$, let $V_0 = I_s$ be an irreducible subvariety of $A_s^{(2s)}$. The principal $U(s)$-bundle $f_s^{(2s)} : V_{2s,2s} \to A_s^{(2s)}$ is two disjoint copies of the canonical $U(s)$-bundle E over V_0. By 3.7, E splits into a direct sum $E = 1 \oplus F$ of a trivial line bundle 1 and the canonical $U(s-1)$-bundle F over $A_{s-1}^{(2s-1)}$, or equivalently $f_{s-1}^{(2s-1)} : V_{2s-1,2s-2} \to A_{s-1}^{(2s-1)}$. Thus $c_s(E) = 0$ and

$$c_i(E) = c_i(F) = 2[01 \cdots s-i-2, s-i \cdots s-2]^* \qquad \text{for } 1 \leq i \leq s-1$$

by (ii) above and (iii) of § 7.

10. Applications

A 2-form w of constant rank $2s$ on a trivial R^n-bundle E (over B) can be represented (after suitable normalization) as a map $w_1 : B \to A_s^{(n)}$, and decomposing w into a sum $w = y_1 \wedge y_{s+1} + \cdots + y_s \wedge y_{2s}$ of products of 1-forms (y_i) on E is equivalent to lifting w_1 to $V_{n,2s}$. (Refer to [4] for details.) We thus obtain

Proposition. *A necessary condition for the decomposability of a 2-form w of constant rank $2s$ on a trivial R^n-bundle E (over B) is that $w_1^*(c_i) = 0$ in $H^{2i}(B; Z)$ where $c_i \in H^{2i}(A_s^{(n)}; Z)$ are as given by the theorem of the preceding section.*

If the total bundle E is not trivial, then a necessary condition for a 2-form w on E of constant rank $2s$ to decompose is that the $2s$-dimensional subbundle S_w of E, on which w is a 2-form of maximal rank, is trivial. Using the triviality of S_w, w is represented as a map $w_1: B \to I_s$. Then w decomposes if and only if w_1 lifts to $SO(2s)$. By (iii) of the theorem of the preceding section, a necessary condition for the existence of such a lift is

$$2w_1^*([01 \cdots s-i-2, s-i \cdots s-2]^*) = 0 \qquad \text{for } 1 \leq i \leq s-1 .$$

It can be verified (although we shall not go into the ring structure of $H^*(A_s^{(n)}; Z)$ here) that $([01 \cdots s-i-2, s-i \cdots s-2]^*, 1 \leq i \leq s-1)$ form a homogenous system of generators for $H^*(I_s; Z)$, and this immediately yields

Proposition. *A necessary condition for the decomposability of a 2-form w of constant rank $2s$ on an R^n-bundle E (over B) is:*

1. *S_w is a trivial bundle,*
2. *Image $w_1^* \subset$ 2-torsion in $H^*(B; Z)$.*

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