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THE TWO-PIECE-PROPERTY AND CONVEXITY FOR SURFACES WITH BOUNDARY

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0. Introduction

For smooth manifolds immersed in an n-dimensional Euclidean space R^n, one has the notions of the Two-Piece-Property (T.P.P.), introduced by Banchoff, of the minimal total absolute curvature, defined by Chern and Lashof, and of tightness, defined by Kuiper. The three notions are equivalent for closed surfaces in R^3. The definition of the T.P.P. applies equally well to manifolds with boundary. A closed surface M in R^3 has minimal total absolute curvature if $\frac{1}{2\pi}\int_M |K|\, dM = 4 - \chi(M)$, where K is the Gaussian curvature of M, dM is the area element of M, and $\chi(M)$ is the Euler characteristic of M. This definition does not adapt itself to surfaces with boundary, although we will show that the following formula holds for a surface M with boundaries $C_1, \cdots, C_n$ in R^3 having the T.P.P.:

$$\int_{M,K>0} K\, dM + \sum_{i=1}^{n} \int_{C_i} |k|\, ds + \int_{C_i} k_g\, ds = 4\pi ,$$

where k is the curvature of the boundary curve, k_g is its geodesic curvature, and s its arc-length.

Definition. An immersion $f\colon M \to R^n$ of an m-dimensional manifold (with or without boundary) has the T.P.P. if for any hyperplane H in R^n the set $f^{-1}(R^n - H)$ has at most two connected components.

For the standard definition of tightness using height functions and the relationships between these concepts see Kuiper [5]. We will use a definition which is equivalent for surfaces and makes sense for manifolds with boundary.

Definition. An immersion $f\colon M \to R^n$ is said to be k-tight if for almost all z in S^{n-1} and real numbers c, the inclusion map $i\colon M(z, c) = \{x\colon z\cdot f(x) \leq c\} \to M$ induces monomorphisms in homology $0 \to H_k(M(z, c)) \xrightarrow{i_*} H_k(M)$ for some field of coefficients. It is tight if it is k-tight for all k; see Kuiper [4, p. 53].

For an immersion of a topological sphere into R^3, the T.P.P. or tightness or minimal total absolute curvature implies that the immersion is an embedd-

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ing onto the boundary of a convex body in R^3; see Chern-Lashof [2, p. 307]. In § 1, using the identity mentioned above we will prove the following theorem.

Theorem 1. *If M is a manifold with boundary ∂M topologically equivalent to a sphere with p disjoint discs removed, then for any immersion $f: (M, \partial M) \to R^3$ with the T.P.P. we have*

(a) *$f \mid \partial M$ consists of planar convex curves,*

(b) *f is an embedding of M into the boundary of its convex hull, or $f(M)$ lies in a plane.*

In §2 we show that the results of Theorem 1 are valid for immersions into arbitrary Euclidean spaces; we say that an immersion $f: M \to R^n$ is substantial if $f(M)$ is not contained in any hyperplane.

Theorem 2. *If $f: (M, \partial M) \to R^n$ is an immersion with the T.P.P., where $(M, \partial M)$ is a two sphere with p disjoint open discs removed, then f is not substantial for $n \geq 4$.*

We apply the analysis used in this section to tight immersions of surfaces with boundary.

Theorem 3. *If $f: (M, \partial M) \to R^n$ is a smooth tight immersion of a compact connected surface with nonvoid boundary, then f is not substantial for $n \geq 4$.*

One example of a nonplanar surface with the T.P.P. is a half-sphere (which is topologically a disc); however, it can be shown that a tight disc in R^3 must actually be contained in some plane as the convex hull of a convex curve. Therefore the T.P.P. and tightness are not equivalent for manifolds with boundary, although tightness implies the T.P.P.

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1. Surfaces with boundary in R^3 having the T.P.P.

In this section we will consider compact connected surfaces with boundary differentiably immersed in R^3. We will study some consequences of the T.P.P., and at the end of the section using some equivalent formulations of the T.P.P. we will show that any immersion of a topological sphere minus some discs having the T.P.P. is convex. If we let $C: [a, b] \to R^3$ be an arc-length parametrization of a component of ∂M, then we have a natural tangent vector $X_x = C'(t_0)$, where $x = C(t_0)$ is a boundary point. Let Y_x be the tangent vector perpendicular to X_x which points away from M. Then $Z = X \times Y$ is the unit normal, and we can extend the frame (X, Y, Z) globally along a neighborhood of the curve C.

Given any unit vector z, we consider the height function $z \cdot f$, where $(\cdot)$ is the usual Euclidean inner product in R^3. For a surface without boundary N, open or closed, $z \cdot f$ has a critical point at a point y if and only if z is normal to N at y. Suppose that $g: U \to S^2$ is the Gauss map. Then we know that $z \cdot f$

has a critical point at x if $g(x) = z$. The critical point at x is nondegenerate if and only if the Jacobian of g at x is not zero, so, by Sard's theorem, $z \cdot f$ has nondegenerate critical points in U for z in S^2 minus a set of measure zero; see Milnor [9].

At a boundary point, we say that $z \cdot f$ has a critical point if $z \cdot (f \mid \partial M) = z \cdot C$ has one, and that it is nondegenerate if (i) it is nondegenerate as a critical point of $z \cdot C$, and (ii) $z \cdot Y_x \neq 0$. A function ϕ on M can be locally expressed at an interior nondegenerate critical point y as $\phi(x) = \phi(y) \pm (x_1)^2 \pm (x_2)^2$, and its index is equal to the number of minus signs in that expression. Thus ϕ has index 2 if and only if y is a local maximum. At a nondegenerate critical boundary point x, we say $z \cdot f$ has index 2 if $z \cdot C''(t_0) < 0$ and $z \cdot Y_x > 0$, index 0 if $z \cdot C''(t_0) > 0$ and $z \cdot Y_x < 0$, index one otherwise. We say that $z \cdot f$ is a nondegenerate function if it has only nondegenerate critical points at either boundary or interior points.

We have that $z \cdot f$ is nondegenerate for almost all z. For a nondegenerate function, let $\beta(z)$ equal the number of critical points in ∂M with index 2, and $\gamma(z)$ equal the number of critical points in the interior of M with index 2. The following lemma justifies our definition of a critical point of index 2 at a boundary point.

Lemma 1. *Let x in ∂M be a nondegenerate critical point of the function $z \cdot f$. Then the point x is a strict local maximum of $z \cdot f$ if and only if x is a critical point of index* 2.

Proof. If x is a strict local maximum of $z \cdot f$, then it is also a strict local maximum of $z \cdot C$. Hence $z \cdot C''(t_0) < 0$, and $z \cdot (-Y)$ cannot be greater than zero, because, in that case, if $D\colon [0, \varepsilon] \to M$ is a curve in M starting at x with $D'(0) = -Y$, then $(z \cdot D)' = z \cdot D' = z \cdot (-Y) > 0$ and therefore $z \cdot f(D(\varepsilon)) > z \cdot f(D(0)) = z \cdot f(x)$, contradicting the fact that x is a local maximum of $z \cdot f$. Conversely, identifying X with $f_*(X)$ we consider the map $p\colon R^3 \to \{X\} \oplus \{z\}$ given by orthogonal projection. The derivative p_* of p is given by $p_* = p$ because p is linear. Now consider the map $p \circ f$; note that $z \cdot f = z \cdot (p \circ f)$, because z is in $p(R^3)$. Therefore it is sufficient to show that $z \cdot (p \circ f)$ has a strict local maximum. The derivative of $p \circ f$ is $(p \circ f)_* = p_* f_*$. It is an immersion because $p_* f_*(X) = p_*(X) \neq 0$ and $p_* f_*(Y) = p_*(Y) = p(Y) = (Y \cdot X)X + (Y \cdot z)z = (Y \cdot z)z \neq 0$, by our assumption. Since $p \circ f$ is an immersion, $p(f(M))$ must be locally to one side of $p \circ C$; it lies to the side in the direction of $-z$, since $z \cdot Y > 0$ implies $z \cdot (p \circ f_* Y) > 0$. Moreover, $z \cdot (p \circ C)'' = z \cdot (p \circ C'') = z \cdot C'' < 0$, because z is in $p(R^3)$. Therefore $z \cdot (p \circ C)$ has a strict local maximum at x, and $z \cdot (p \circ f)$ has a strict local maximum at x also. q.e.d.

Any function must have a maximum on a compact set, so that if $z \cdot f$ is nondegenerate, it must have a strict local maximum at an interior or at a boundary point. In any case, we have that $\beta(z) + \gamma(z) \geq 1$, and

$$\int_{S^2} (\beta(z) + \gamma(z))\, dS^2 \geq 4\pi \ .$$

The next lemma shows that we get equality if and only if $f: M, \partial M \to R^3$ has the T.P.P.

Lemma 2. *An immersion $f: M, \partial M \to R^n$ of a surface with boundary has the T.P.P. if and only if any nondegenerate function of the form $z \cdot f$ has only one maximum*, i.e., $\beta(z) + \gamma(z) = 1$.

Proof. Suppose that $z \cdot f$ had two nondegenerate maxima at x_1 and at x_2, with $c = z \cdot f(x_1) \leq z \cdot f(x_2)$. Then, for any small $\varepsilon > 0$, we consider the half-space $H = \{y \text{ in } R^n : z \cdot y > c - \varepsilon\}$; but since x_1 is a strict local maximum of $z \cdot f, f^{-1}(H)$ would contain two different components of M.

Conversely, suppose that f does not have the T.P.P.; then, for some vector z and real number c, the set $\{x \text{ in } M : z \cdot f(x) > c\}$ consists of at least two connected components A and B. Let x_1 in A and x_2 in B be the maxima of $z \cdot f$, with $D = z \cdot f(x_1) \leq z \cdot f(x_2)$. Then given any $\varepsilon > 0$ there exists δ such that $D(z') = \{x \text{ in } M : z' \cdot f(x) > c + \varepsilon\} \subset A \cup B$ if $\|z - z'\| < \delta$. Now $z' \cdot f(x_1) = z \cdot f(x_1) - (z \cdot f(x_1) - z' \cdot f(x_1)) \geq D - |z \cdot f(x_1) - z' \cdot f(x_1)| > D - \delta(\sup\{\|f(x)\|\}) > c + \varepsilon$, for small ε and δ. Similarly, $z' \cdot f(x_2) > c + \varepsilon$, which implies $D(z') \cap A \neq \phi$ and $D(z') \cap B \neq \phi$. Consequently $D(z')$ has at least two connected components if $\|z - z'\| < \delta$. Since arbitrarily close to z there is a z' such that $z' \cdot f$ has only nondegenerate maxima, we have a z' with a critical point of index 2 in $D(z') \cap A$ and another in $D(z') \cap B$. q.e.d.

Now we want to calculate the contribution of the $\gamma(z)$'s for any immersed surface in R^3.

Lemma 3. *At an interior point x, the fact that a nondegenerate function $z \cdot f$ has a local strict maximum implies that the Gaussian curvature $K(x)$ is greater than zero. Conversely, if $K(x) > 0$, then $z \cdot f$ has a maximum at x, where z is equal to one of the normals $\pm Z_x$.*

Proof. See Kuiper [4].

We see therefore that if g is the Gauss map, then $\int_{M,K>0} g^*(dS^2) = \int_{S^2} \gamma(z)\, dS^2$. On the other hand, we know that the Jacobian of the Gauss map is equal to the Gaussian curvature K. Therefore the contribution of the $\gamma(z)$'s is $\int_{S^2} \gamma(z) dS^2 = \int_{M,K>0} K\, dM$. The following is a formula similar to that of minimal total curvature characterizing the T.P.P. for surfaces in R^3:

$$(2) \qquad \int_{M,K>0} K\, dM + \int_{S^2} \beta(z)\, dS^2 = 4\pi \ .$$

If $f: M, \partial M \to R^3$ has the T.P.P., and $z \cdot f$ is a nondegenerate height function with x as a local maximum, then x must be a global maximum by Lemma

2, i.e., $z \cdot f(y) \leq z \cdot f(x)$ for every y in M. If we denote the plane $\{y \text{ in } R^3: w \cdot y = c\}$ by $P(w, c)$, then we see that $f(M)$ is on one side of $P(z, z \cdot f(x))$, while still touching it. For any immersion we define the following.

Definition. $P(z, c)$ is called a top-plane of f if $z \cdot f(y) \leq c$ for all y in M, and $z \cdot f(x) = c$ for some x in M.

Definition. If $P(z, c)$ is a top-plane of f, then $f^{-1}(c)$ is called a topset.

Note that if $z \cdot f$ is nondegenerate with maximum x and $z \cdot f(x) = c$, then the corresponding topset is just the single point $\{x\}$. Thus almost all topsets consist of single points. The following proposition will be useful later.

Lemma 4. *The limit of top-planes is also a top-plane.*

Proof. Suppose that $P(z_n, c_n)$ is a sequence of top-plane with $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty} c_n = c$; then we want to show that $P(z, c)$ is a top-plane. Let x_i be a global maximum of $z_i \cdot f$, $i = 1, 2, 3, \cdots$; take a converging subsequence of x_i's. Rename the sequence such that $\lim_{n\to\infty} x_n = x$. Then $c = \lim_{n\to\infty} c_n = \lim_{n\to\infty} z_n \cdot f(x_n) = z \cdot f(x)$. The point x must be a global maximum; if not, there exists a point y in M such that $z \cdot f(y) > c$. But then there exists an integer k such that $c_n \geq z_n \cdot f(y) > z \cdot f(y) - \varepsilon > c$, if $n \geq k$; and $\lim_{n\to\infty} c_n > c$, a contradiction. q.e.d.

If y is an interior point, then the only possible top-plane is the tangent plane TM_y; however, in general, this will not be the situation for a boundary point. In the case of a flat disc contained in some plane in R^3, every plane containing the tangent vector $C'(t_0) = X_x$ at some boundary point x is a top-plane. On the other hand, if M is a hemisphere, then only half the possible z's give rise to top-planes $P(z, z \cdot f(x))$ at a boundary point x. If $C(t)$ is an arc-length parametrization of a component of ∂M immersed in R^3, then we have already denoted $C'(t)$ by X. If $C''(t) = X_t' \neq 0$, let $N_t = C''(t)/\|C''(t)\|$, the curvature normal to the curve. Let $B_t = X_t \times N_t$, the binormal to the curve. The frame (X, N, B) along $C(t)$ is the Frenet frame of the boundary curve. Now we are ready to define the function which will tell us how many normal vectors give rise to functions $z \cdot f$ with local maxima (half of the vectors which give rise to supporting planes). Let $\theta: C \to R$ be defined by letting $\theta(C(t))$ equal the angle between B_t and Z_t, that is, the smallest angle. If $C''(t) = 0$, let $\theta(C(t)) = 0$.

Note that $0 \leq \theta(x) \leq \pi$, and $\cos \theta(C(t)) = B_t \cdot Z_t$; even though B and Z depend on the direction of C, $B_t \cdot Z_t$ does not. The angle $\theta(C(t))$ is the same as the angle between N_t and Y_t, i.e., $\cos \theta(C(t)) = N_t \cdot Y_t$.

At a boundary point x, if $\theta(x)$ is not equal to zero or π, then $N_x \neq Y_x$, and we can choose the normal Z_x with $Z_x \cdot N_x < 0$. Consider the vectors $W_\alpha = (\cos \alpha) Z_x + (\sin \alpha) Y_x$. The next lemma justifies the above definition of θ.

Lemma 5. *The function $W_\alpha \cdot f$ has a nondegenerate critical point of index 2 at x if and only if $0 < \alpha < \theta(x) \leq \pi$.*

Proof. $N_x = (\cos \theta) Y_x - (\sin \theta) Z_x$, by the definition of θ and the choice of Z. Then $W_\alpha \cdot Y = \sin \alpha > 0$ if and only if $0 < \alpha < \pi$, and $W_\alpha \cdot N = \cos \theta \sin \alpha$

$-\sin\theta\cos\alpha = \sin(\alpha-\theta) < 0$ if and only if $-\pi < \alpha - \theta < 0$, or $\alpha < \theta$. Since $W_\alpha \cdot C''(t) = \|C''(t)\| W_\alpha \cdot N$, our proof is complete.

An important consequence is the following.

Corollary 6. *If x in ∂M is not in a top-set of an immersion $f\colon M, \partial M \to R^3$ having the T.P.P., then $\theta(x) = 0$, i.e., $C''(t_0)\cdot Z_x = 0$.*

Proof. If $\theta(x) \neq 0$, then there exists an α, $0 < \alpha < \theta(C(t_0))$; and $W_\alpha \cdot f$ has a local maximum at x and hence a global maximum. Therefore $P(W_\alpha, W_\alpha \cdot f(x))$ is a top-plane, and x is in a topset. q.e.d.

A curve $C(t)$ on a surface with $C''(t_0)\cdot Z_x = 0$ is said to be asymptotic at x. In terms of the second fundamental form h, this means that $h(X, X) = 0$. The simplest example of an asymptotic curve is a plane curve. Note that the image under f of any topset must be contained in the boundary of the convex hull of $f(M)$, if f has the T.P.P.; therefore Corollary 6 says that if a portion of the boundary curve is in the interior of the convex hull $H(M)$ of $f(M)$, then it must be an asymptotic curve.

The planes $P(W_0, W_0\cdot f(x))$ and $P(W_{\theta(x)}, W_{\theta(x)}\cdot f(x))$ are the tangent plane to the surface and the osculating plane of the curve, respectively. If $\theta(x) \neq 0$, we showed that every $W_\alpha \cdot f$ had a global maximum at x, if f has the T.P.P. Therefore, if f has the T.P.P. and $\theta(x) \neq 0$, then both planes are top-planes by Lemma 4. This fact gives the following corollary which we will use later.

Corollary 7. *If $C(t)$, $t_1 \leq t \leq t_2$, is a portion of a boundary curve which is contained in the boundary of the convex hull $\partial H(M)$ of an immersion with the T.P.P., and $C''(t) \neq 0$, then $C(t)$ is a plane curve.*

Proof. If $C(t)$ is in a topset and $\theta(C(t)) \neq 0$, then, as we remarked above, the osculating plane is a top-plane. In case $\theta(x) = 0$, then either the curvature of the curve is equal to zero or the osculating plane is the tangent plane and the only possible top-plane. Since $C''(t) \neq 0$, the osculating planes are all top-planes, implying that the curve is locally on one side of the osculating plane. But this in turn implies that the torsion $\tau(C(t)) = 0$. Hence the curve is a plane curve for $t_1 \leq t \leq t_2$. q.e.d.

At this time, we want to go back to calculate the contribution of the $\beta(z)$'s to the "total curvature" of the immersion $f\colon M, \partial M \to R^3$.

Proposition 8. *If f has the T.P.P., then*

$$\int_{S^2} \beta(z)\, dS^2 = \sum_{i=1}^{n} \int_{C_i} (1 - \cos\theta(s))|k(s)|\, ds \,, \tag{3}$$

where $k(s)$ is the curvature of the curve, i.e., $\|C''(t_0)\|$. Indeed, for any immersion, if each boundary curve consists of portions which are either planar or asymptotic, then (3) *still holds.*

Proof. By Lemma 5, the total contribution of $\int_{S^2} \beta(z)\, dS^2$ is equal to the sum of the contributions of the $(W_\alpha)_x$'s from each curve. Hence it is sufficient to prove that the area in S^2 of $\{(W_\alpha)_x : x \text{ in } C, 0 < \alpha < \theta(x)\}$ is equal to $\int_C (1 - \cos\theta(s))|k(s)|ds$ for each boundary curve C. More precisely, since integrals and areas are additive, we can consider a small segment $C(t)$, $t_1 \leq t \leq t_2$, on which the map $(x, a) \to (W_a)_x$ in S^2 is one-to-one; it is enough to show that the area of $\{(W_\alpha)_{C(t)} : t_1 \leq t \leq t_2;\ 0 < \alpha < \theta(C(t))\}$ is equal to $\int_{C[t_1,t_2]} (1 - \cos\theta(s))|k(s)|ds$.

If x in C is also in the interior of the convex hull of $f(M)$, then $\theta(x) = 0$, and there is no W_α contribution, by Corollary 6; and, on the other side, $1 - \cos\theta = 0$. Since $k(x) = 0$ implies that $\theta(x) = 0$, we can also assume that $k(x) \neq 0$. Therefore, by Corollary 7, we can assume that $C(t)$, $t_1 \leq t \leq t_2$, is a plane curve with $k(x) \neq 0$.

Consider now spherical coordinates for S^2, $F: [0, 2\pi] \times [0, \pi] \to S^2$, $F(\phi, r) = (\sin r \cos\phi, \sin r \sin\phi, \cos r)$. Let $q(\phi) = F(\phi, \pi/2)$ in S^1 in $(R^2 \times \{0\} \cap S^2)$ in R^3. Since C is planar, we can parametrize C as follows: $[a, b] \xrightarrow{q} S^1 \xrightarrow{h^{-1}} C$, where $h(s) = -N_s$ is the "Gauss map" of the curve in the plane. The map h is one-to-one since $k(s) \neq 0$. Naming this parametrization by the same C, we have that $s = C(\phi) = h^{-1}q(\phi)$. Since we can assume that C lies in $R^2 \times \{0\}$ such that $(0,0,1)\cdot f(x) \leq 0$, with $q(\phi) = -N_{C(\phi)}$, it is clear that $F(\phi, r) = (W_{(\theta-r)})_{C(\phi)}$. Therefore the area contributed by the $(W_\alpha)_s$'s of C is equal to the area of $F(D)$, where $D = \{(\phi, \alpha) : a < \phi < b, 0 < \alpha < \theta(C(\phi))\}$. That is,

$$\int_{F(D)} dS^2 = \int_D F^*(dS^2) = \int_D \sin r\, dr\, d\phi = \int_a^b \left[\int_0^{\theta(C(\phi))} \sin r\, dr\right] d\phi$$
$$= \int_a^b \left[-\cos r|_0^{\theta(C(\phi))}\right] d\phi = \int_a^b (1 - \cos\theta(C(\phi)))\, d\phi\ .$$

On the other hand,

$$\int_{C([a,b])} (1 - \cos\theta(s))|k(s)|ds$$
$$= \int_{C([a,b])} (1 - \cos\theta(C(C^{-1}(s)))) \left|\frac{\partial C^{-1}}{\partial s}\right| ds\ ,$$

because

$$\left|\frac{\partial C^{-1}}{\partial s}\right| = \left|\frac{\partial h}{\partial s}\right| \cdot \left|\frac{\partial q^{-1}}{\partial r}\right|_{r=q(\phi)} = |k(s)| \cdot \left|\frac{\partial q^{-1}}{\partial r}\right| = |k(s)|\ ,$$

since q is length-preserving. Finally, the last integral is equal to

$$\int_{a=C^{-1}(C(a))}^{b=C^{-1}(C(b))} (1 - \cos\theta(C(\phi)))\, d\phi\ ,$$

by a change of variables. q.e.d.

We observe that the geodesic curvature k_g is equal to the length of the projection of the vector $C''(s)$ on the tangent plane, positive in the inward direction, that is, $k_g(s) = C''(s)\cdot(-Y) = -|k(s)|(N_s\cdot Y_s)$. Since $-\cos\theta(s) = -N_s\cdot Y_s$, the following proposition follows directly from formula (2) and Proposition 8.

Proposition 9. *If $f\colon M, \partial M \to R^3$ has the T.P.P., then*

$$\int_{M,K>0} K\,dM + \sum_{i=1}^{n}\int_{C_i}(|k| + k_g)\,ds = 4\pi\,. \tag{4}$$

Conversely, if each boundary curve consists of portions which are either planar or asymptotic, and if (4) *holds, then f has the T.P.P.*

With the aid of formula (4), we will be able to characterize the immersions of M with the T.P.P., when M is topologically a sphere minus a finite number of disjoint discs. In the remainder of this section, we let $(M, \partial M)$ be topologically equivalent to $\left(S^2 - \bigcup_{i=1}^{n} D_i, \bigcup_{i=1}^{n} C_i\right)$, where the D_i's are open discs with boundaries C_i's, and $C_i \cap C_j = \varnothing$ if $i \neq j$.

Proof of part (*a*) *of Theorem* 1. First, we recall the Gauss-Bonnet formula

$$\int_M K\,dM + \int_{\partial M} k_g\,ds = 2\pi\chi(M)\,, \tag{5}$$

where $\chi(M)$ is the Euler characteristic, K is the Gaussian curvature, and k_g is the geodesic curvature of the boundary curves. Recall now that the Euler characteristic of M is $\chi(M) = 2 - n$. Since f has the T.P.P., combining formulas (4) and (5) we have

$$\begin{aligned}
4\pi &= \int_{M,K>0} K\,dM + \sum_{i=1}^{n}\int_{C_i}(|k| + k_g)\,ds \geq \int_M K\,dM + \sum_{i=1}^{n}\int_{C_i}(|k| + k_g)\,ds \\
&= 2\pi\chi(M) - \int_{\partial M} k_g\,ds + \sum_{i=1}^{n}\int_{C_i}(|k| + k_g)\,ds \\
&= 2\pi(2 - n) + \sum_{i=1}^{n}\int_{C_i}|k|\,ds \geq 4\pi\,,
\end{aligned}$$

since by Fenchel's theorem, $\int_{C_i}|k(s)|\,ds \geq 2\pi$, with equality only if C_i is a plane convex curve. Since we have equalities, the boundary curves are convex, and $\int_{M,K>0} K\,dM = \int_M K\,dM$, which implies that $K \geq 0$ on M.

Proof of part (*b*) *of Theorem* 1. We know that the boundary curves are convex, i.e., they bound a convex disc in some plane. We extend f to a map $\tilde{f}$ on the closed sphere by mapping each D_i homeomorphically onto the convex disc with boundary C_i. The new map $\tilde{f}$ will not be smooth in general. but it will be continuous. Moreover, if $z \cdot f$ is a nondegenerate function on M, then $z \cdot \tilde{f}$ will have exactly the same isolated critical points as $z \cdot f$, since there are no new ones in $\bigcup_{i=1}^{n} D_i$. In particular, $z \cdot \tilde{f}$ has only one maximum for almost all z in S^2. If we can show that $\tilde{f}$ is a topological immersion, then we can apply Theorem 4 in [5, p. 227], which concludes that $\tilde{f}$ is an embedding onto the boundary of a convex body in R^3. We will show that either $\tilde{f}$ is an immersion or that $f(M)$ is contained in some plane. If $\tilde{f}$ is not an immersion at a point x in a boundary curve C, then C is contained in TM_x, and $-Y_x$ points towards the interior of D. Let Z_x be the normal which has x as a global maximum of $Z_x \cdot f$.

For a smooth convex curve, such as C, we have a well defined interior normal N_x at every point x of C even if $k(x) = 0$. If $k(x) \neq 0$ then $\cos \theta(x) = N_x \cdot Y_x = -N_x \cdot N_x = -1$, and $\theta(x) = \pi$. Thus $-Z_x \cdot f$ also has a global maximum at x, and $f(M)$ is contained in $f_*(TM_x)$. Even if $k(x) = 0$ we have $-Y_x = N_x$. If $C(t_0) = x$, let $[a, b]$ be the largest interval containing t_0 on which $C''(t) = 0$; then $C(b)$ is the limit of points x_n in C such that $k(x_n) \neq 0$. Now we observe that $C\colon [a, b] \to R^3$ is an asymptotic curve in M, but, as it was shown in the proof of part (b), $K \geq 0$ on M. This implies that $\tau = \sqrt{-K} = 0$ along $C([a, b])$, and that Z_y and Y_y are parallel along $C([a, b])$. Thus $Y_y = -N_y$ along $C([a, b])$ and, since $\lim_{n\to\infty} -Y_{x_n} = -Y_{C(b)} = N_{C(b)}$ and $k(x_n) \neq 0$, we have that $\lim \theta(x_n) = \pi$. We observe that the half-spaces $H(W_{\theta(x_n)}, W_{\theta(x_n)} f(x_n)) = \{y \in R^3\colon W_{\theta(x_n)} \cdot y \leq W_{\theta(x_n)} \cdot f(x_n)\}$ all contain $f(M)$, since f has the T.P.P. Looking at the proof of Lemma 4, we could also have proven that the limit of the half-spaces, $H\left(\lim_{n\to\infty} W_{\theta(x_n)}, \lim_{n\to\infty} W_{\theta(x_n)} \cdot f(x_n)\right) = H(-Z_{C(b)}, -Z_{C(b)} \cdot C(b)) = H(-Z_x, -Z_x \cdot f(x))$, also contain $f(M)$, since they are closed. Therefore x is a global maximum of both $Z_x \cdot f$ and $-Z_x \cdot f$, and hence $f(M)$ is contained in the plane perpendicular to Z_x.

2. Surfaces with the T.P.P. and tightness in higher dimensions

In § 1 we studied immersions in R^3 of spheres with discs removed with the T.P.P. In this section we will show that any immersion in R^n of a 2-sphere with discs removed having the T.P.P. must actually be contained in some three-dimensional affine subspace of R^n, so that we do not obtain any more examples in higher codimension. The analysis involved will give similar results for the case of any tight immersions of surfaces with boundary in R^n, that is, they must be contained in three-dimensional affine subspaces of R^n.

Let $f\colon M, \partial M \to R^n$ be a smooth immersion of a compact surface with boundary into R^n. Let TM_x and $TM_x^\perp$ denote the tangent plane and the normal space at x, with S_x and $S_x^\perp$ the unit tangent circle and unit normal sphere, respectively. As before, let X be the unit tangent vector to the boundary curves and let Y be the tangent vector to M, perpendicular to X, which points out from M. At each point x of M we have a vector-valued symmetric bilinear map $\alpha_x: TM_x \times TM_x \to TM_x^{\perp}$, the second fundamental form.

In what follows, we assume that $f: M, \partial M \to R^n$ is an isometric immersion with the T.P.P.

The locus of points $\alpha(X, X)$ with X in S_x can be shown to be an ellipse in the normal plane called the curvature ellipse. This concept was introduced by Wilson and Moore [12] and further studied by Little [7]. A radial line in the normal space is a half-line emanating from the origin.

Lemma 10. *If $f: M \to R^n$ is an immersion of a surface (with or without boundary) into R^n such that at a point x the curvature ellipse B_x is contained in a radial line without touching the origin, then f is not substantial for $n \geq 4$.*

Proof. Fix $\beta = \alpha(X, X)$, X in TM_x. If z is a vector of the open half-sphere $\{z \text{ in } S_x^{\perp}: z \cdot \beta < 0\}$, then $z \cdot f$ has a strict local maximum at x and therefore a global maximum at x since f has the T.P.P.. Therefore, in the limit, we see that the equatorial $(n-4)$-sphere S_x' is composed of vectors which satisfy $w \cdot f(x) \geq w \cdot f(y)$ for all y in M. If both $w \cdot f$ and $(-w) \cdot f$ have global maxima at x, then $f(M)$ is contained in the hyperplane perpendicular to w under the assumption for simplicity that $f(x) = 0$. Hence $f(M) \subset S_x'^{\perp}$, a three-plane in R^n.

Lemma 11. *If p is in ∂M and $\alpha_p(X, X) \neq 0$, where X is tangent to the boundary, then $f(M) \subset H \subset R^n$ for some three-plane H.*

Proof. Let $z \in S_x^{\perp}$ with $z \cdot \alpha(X, X) < 0$. Let $p: R^n \to R^3$ be the orthogonal projection onto the three-plane $TM_x \oplus \{z\}$, and $C(t)$ the boundary curve with $C(t_1) = x$. If $D = p \circ C$, then the surface $g = p \circ f$ has $D''(t_1) \cdot z = [(p \circ C)''(t_1)] \cdot z = [p(C''(t_1)] \cdot z = p(\alpha(X, X)) \cdot z = \alpha(X, X) \cdot z < 0$ since $z \in p(R^n)$. Therefore, by Lemma 1, $w \cdot g$ has a local maximum at x, where $w = z + \varepsilon Y_x$ for small positive ε. Since w is in $p(R^n)$, we have that $w \cdot g = w \cdot f$ so that $w \cdot f$ has a strict global maximum at x because f has the T.P.P. Thus, in the limit, when ε tends to zero, $z \cdot f$ has a global maximum at x, and the same is true for all vectors in the half-sphere $\{\beta \text{ in } S_x: \beta \cdot \alpha(X, X) < 0\}$. Therefore, as in the above lemma, all vectors in the equatorial sphere $S_x' = \{\beta \text{ in } S_x: \beta \cdot \alpha(X, X) = 0\}$ give rise to global support hyperplanes. Hence $f(M) \subset S_x'^{\perp}$, a three-plane in R^n. q.e.d.

We recall now that the Gauss-Bonnet formula for a region with boundary M is an intrinsic formula with $k_g = \nabla_X X \cdot (-Y)$, where ∇ is the covariant derivative in M, X is the unit tangent vector to the boundary curve, and Y is the tangent vector perpendicular to X and pointing away from M. We are considering surfaces immersed in R^n and, for those with the T.P.P., we can obtain an extrinsic formulation as follows.

Proposition 12. *If $f: M, \partial M \to R^n$ is substantial, then $k_g = -|k|$ for a boundary point x.*

Proof. By definition, $|k| = \|\tilde{\nabla}_X X\|$ where $\tilde{\nabla}$ is the covariant derivative in R^n. Using Lemma 11 and Gauss's formula for submanifolds, we have that

$\tilde{\nabla}_X X = \nabla_X X + \alpha(X, X) = \nabla_X X$. On the other hand, $\nabla_X X = \pm\|\nabla_X X\| Y$; if the sign were negative, then $(z + \varepsilon Y)\cdot f$ would have a local maximum at x for any z in $S_x^\perp$ since $\tilde{\nabla}_X X = \nabla_X X$. Thus, if we assume $f(x) = 0$, then any $z^\perp$ would be a global support plane, and $f(M) \subset TM_x$, so that f is not substantial. Hence $\nabla_X X = -\|\nabla_X X\|(-Y) = -|k|(-Y)$, and $k_g = \nabla_X X\cdot(-Y) = -|k|$. q.e.d.

Thus formula (5) becomes

$$\int_M K\, dM = 2\pi\chi(M) + \int_{\partial M} |k|\, ds\ . \tag{6}$$

Now we consider further the local geometry of the immersion $f\colon M \to R^n$.

Definition. Given z in $S_p^\perp$, the Lipschitz-Killing curvature $G(p, z)$ in the direction of z is the determinant of the symmetric bilinear form $\alpha\cdot z$, i.e., $G(p, z) = (\alpha(X, X)\cdot z)(\alpha(Y, Y)\cdot z) - (\alpha(X, Y)\cdot z)^2$, where X and Y are orthonormal tangent vectors.

If we let $g\colon N(f) \to S^{n-1}$ be the Gauss map of M, where $N(f)$ is the normal bundle on the interior of M, with $g(p, z) = z$, then $G(p, z)$ is equal to the determinant of the Jacobian of g at (p, z). Therefore, by Sard's theorem, all z, except for a set of measure zero on S^{n-1}, are regular values of g, i.e., for almost all z, $G(p, z) \neq 0$ if $g(p, z) = z$.

Now, for any z in S^{n-1}, $z\cdot f$ has a maximum at some point x in M, since M is compact. Let $p(z) = 1$ if the maximum is attained at a boundary point, $=0$ if at an interior point; let $\gamma(z) = 1 - \rho(z)$. Then

$$c_{n-1} = \int_{S^{n-1}} dS^{n-1} = \int_{S^{n-1}} \gamma(z)\, dS^{n-1} + \int_{S^{n-1}} \rho(z)\, dS^{n-1}\ ,$$

where c_{n-1} denotes the volume of the $(n - 1)$-sphere.

Since M is a manifold, $z\cdot(f\,|\,\partial M)$ has a strict maximum for almost all z, implying that $\tilde{\nabla}_X X\cdot z < 0$. By Proposition 12, $\tilde{\nabla}_X X = \nabla_X X = |k|Y$, but then $z\cdot(-Y) > 0$, which contradicts the fact that $(-Y)$ is the tangent vector pointing towards the inside of M. We have shown

$$\int_{S^{n-1}} \rho(z)\, dS^{n-1} = 0\ . \tag{7}$$

Let $\tilde{\gamma}(z) = 1$ if and only if $G(p, z) > 0$ and p in the interior of M is a maximum with (p, z) in $g^{-1}(z)$. Since, by Sard's theorem, $G(p, z) \neq 0$ for all (p, z) in $g^{-1}(z)$, for all z except for a set of measure zero we have that

$$\int_{S^{n-1}} \gamma(z)\, dS^{n-1} = \int_{S^{n-1}} \tilde{\gamma}(z)\, dS^{n-1}\ .$$

Therefore by the change of variables suggested above we get

$$
\begin{aligned}
c_{n-1} &= \int_{S^{n-1}} \tilde{\gamma}(z)\, dS^{n-1} = \int_{\substack{g(N(f))\\ G(p,z)>0}} \frac{1}{2}\, dS^{n-1} \\
&= \frac{1}{2}\int_{\substack{N(f)\\ G(p,z)>0}} g^*\, dS^{n-1} = \frac{1}{2}\int_{\substack{N(f)\\ G(p,z)>0}} G(p,z)\, dN(f) \ ,
\end{aligned} \tag{8}
$$

where we use the fact that the number of maxima is equal to one-half the number of maxima and minima of $z \cdot f$, which in turn is equal to one-half the number of points p with $G(p, z) > 0$.

We now show that we can obtain the Gaussian curvature as an average of the Lipschitz-Killing curvatures. Let c_n denote the volume of the n-sphere S^n.

Lemma 13. *If $f: M \to R^n$ is an isometric immersion, then*

$$
\int_{S_p^\perp} G(p, z)\, dS_p^\perp = \frac{c_{n-3}}{n-2} K(p) \ . \tag{9}
$$

Proof. Let $A(v) = \alpha \cdot v$, a real-valued symmetric bilinear form defined for every normal vector v. Note that $G(p, v) = \det A(v)$. If $v_1, \cdots, v_{n-2}$ is an orthonormal frame of normal vectors in TM_p, then $K(p) = \det A(v_1) + \cdots + \det A(v_{n-2})$. On the other hand, $A(v) = a_1 A(v_1) + \cdots + a_{n-2} A(v_{n-2})$ if $v = a_1 v_1 + \cdots + a_{n-2} v_{n-2}$. Since $A(v)$ is a 2×2 matrix, $\det A(v) = a_1^2 \det A(v_1) + \cdots + a_{n-2}^2 \det A(v_{n-2}) + \sum_{i \neq j} a_i a_j B_{ij}$, where B_{ij} consists of a sum of products of the coefficients of the $A(v_i)$'s. We claim that $\int_{S_p^\perp} a_i a_j B_{ij}\, dS_p^\perp = 0$. Let S_s^0 be the small sphere obtained by intersecting $S_p^\perp$ with the hyperplane in the normal space, $a_j = s$. Then

$$
\int_{S_p^\perp} a_i a_j B_{ij}\, dS_p^\perp = B_{ij} \int_{-1}^{1} \frac{s}{(1-s^2)^{1/2}} \left(\int_{S_s^0} a_i \right) ds = 0 \ ,
$$

since a_i is an odd function on S_s^0. Therefore

$$
\int_{S_p^\perp} G(p, v)\, dS_p^\perp = \sum_{i=1}^{n-2} \det A(v_i) \int_{S_p^\perp} a_i^2\, dS_p^\perp \ ,
$$

the theorem is proven if we show that $\int_{S_p^\perp} a_i^2\, dS_p^\perp = \dfrac{c_{n-3}}{n-2}$. Since $\int_{S_p^\perp} a_i^2\, dS_p^\perp = \int_{S_p^\perp} a_j^2\, dS_p^\perp$, we have

$$
(n-2)\int_{S_p^\perp} a_i^2\, dS_p^\perp = \sum_{j=1}^{n-2} \int_{S_p^\perp} a_j^2\, dS_p^\perp = \int_{S_p^\perp} \sum_{j=1}^{n-2} a_j^2\, dS_p^\perp = \int_{S_p^\perp} dS_p^\perp = c_{n-3} \ .
$$

Corollary 14 (*Gauss-Bonnet theorem*). *If $f: M \to R^n$ is an isometric immersion of a closed surface M, then $\int_M K\, dM = 2\pi\chi(M)$.*

Proof. If $g: N(f) \to S^{n-1}$ is the Gauss map, then, as we saw above,

$$\int_{N(f)} g^* \, dS^{n-1} = \int_M \int_{S_p} G(p, z) \, dz \, dM = \int_M \frac{c_{n-3}}{n-2} K(p) \, dM .$$

At the same time, the left-hand integral is equal to dc_{n-1}, where d is the degree of the Gauss map; this degree is equal to the Euler characteristic of M (see Kobayashi-Nomizu [3, p. 359]). Therefore $c_{n-1}\chi(M) = \dfrac{c_{n-2}}{n-2} \displaystyle\int_M K(p) \, dM$, and the theorem follows from the fact that the ratio c_{n-1}/c_{n-3} of the volumes of spheres is equal to $2\pi/(n-2)$ (see Otsuki [10, p. 64]). q.e.d.

Let $\mu(p) = \min \{G(p, z) : z \text{ in } S_p^\perp\}$ and $\lambda(p) = \max \{G(p, z) : z \text{ in } S_p^\perp\}$. Note that $\mu(p) \leq 0$, since $G(p, z) \leq 0$ if z is perpendicular to $\alpha(Z, Z)$ for some tangent vector Z. It can also be pointed out that $\mu(p) = 0$ if and only if the curvature ellipse B_p is contained in some line in the normal space starting at the origin, which is called a radial line. The following proposition gives a formula similar to that of Proposition 9 of § 1.

Proposition 15. *If $f: M, \partial M \to R^n$ has the T.P.P., then*

$$4\pi = \int_{p \in \mathrm{Int}\, M} \left(\frac{n-2}{c_{n-3}} \int_{\substack{z \in S_p^\perp \\ G(p,z) \geq 0}} G(p, z) \, dS_p^\perp \right) dM \geq \int_{\substack{M \\ \lambda(p) > 0}} K(p) \, dM \tag{10}$$

with equality only if $\mu(p) = 0$ for every p in M.

Proof. We get the first equality by multiplying both sides of (3) by $2(n-2)/c_{n-3}$, noting, as in the proof of Corollary 14, that $(n-2)c_{n-1}/c_{n-3} = 2\pi$. The last inequality follows from Lemma 13.

Proof of Theorem 2. Combining formulas (6) and (10) we obtain

$$\begin{aligned} 4\pi &\geq \int_{\substack{M \\ \lambda(p) \geq 0}} K(p) \, dM \geq \int_M K \, dM = \sum_{i=1}^r \int_{C_i} |k| \, ds + 2\pi(2 - r) \\ &= 4\pi + \sum_{i=1}^r \left(\int_{C_i} |k| \, ds - 2\pi \right) \geq 4\pi . \end{aligned}$$

The second inequality holds since $K < 0$ on $\{p \text{ in } M : \lambda(p) < 0\}$ by Lemma 13, and the last inequality is true by Fenchel's theorem. Since we have equalities, $\mu(p) = 0$. But at some point p we must have $K(p) > 0$, and this implies that $\alpha_p(Z, Z) \neq 0$ for all Z in S_p; otherwise, we would have $K(p) = \alpha(Z, Z) \cdot \alpha(W, W) - \alpha(Z, W) \cdot \alpha(Z, W) = -\alpha(Z, W)^2 \leq 0$. Therefore the curvature ellipse is contained in a radial line without touching the origin, and the theorem follows from Lemma 10. q.e.d.

Banchoff has given an example of a tight polyhedral Moebius band in R^4: it is the union of the triangles $e_i e_{i+1} e_{i+2}$ (i mod 5) of a simplex $e_1 e_2 e_3 e_4 e_5$ in

R^4. Kuiper has shown that any tight topologically and substantially embedded Moebius band in R^4 is equal to Banchoff's example (see Kuiper [6]). From this it follows that there is no smooth embedded tight Moebius band in R^4. Using formula (8) we can quickly show that there is no smooth tight substantial immersion in R^4 of a surface with nonvoid boundary.

Proof of Theorem 3. We recall that for a connected surface M with or without boundary, $f: M \to R^n$ is tight if and only if for any vector z in S^{n-1} and real number c, the inclusion map $H_*(M(z,c)) \to H_*(M)$ is injective, where $M(z,c) = \{x \text{ in } M: z \cdot f(x) \leq c\}$. If $G(p,z) > 0$ for some p in the interior of M, then $z \cdot f$ or $(-z) \cdot f$ has a strict local maximum at p; say it is $z \cdot f$. Then, for small ε, $M(-z, -z \cdot f(p) + \varepsilon)$ contains a small isolated disc D with boundary curve C, which obviously bounds in M. However, $C \subset M(z, z \cdot f(p) - \varepsilon)$, but it cannot bound in $M(z, z \cdot f(p) - \varepsilon)$, since in that case M would contain a component without boundary, contradicting the fact that M is connected. Hence $G(p,z) \leq 0$ for every p and z, contradicting formula (3), since f has the T.P.P.

Another simple consequence of our formulas is the following.

Proposition 16. *Let $f: M, \partial M \to R^4$ have the T.P.P., where M is equal to a flat torus minus r discs. If f is substantial, the boundary curves C_i, $i = 1, \cdots, r$, are plane convex curves.*

Proof. In this case, formula (2) becomes $0 = 2\pi(-r) + \sum_{i=1}^{r} \int_{C_i} |k|\, ds$. Since $\int_{C_i} |k|\, ds \geq 2\pi$, we have equalities, and hence by Fenchel's theorem each C_i is a plane convex curve. q.e.d.

By studying further the functions $\mu(p)$ and $\lambda(p)$, introduced above, we can get some insight into the extrinsic geometry of surfaces in higher dimensions. In what follows, we will restrict ourselves to four dimensions. The following lemma is proven by Otsuki who studied this problem in [10].

Lemma 17. *If $f: M \to R^4$ is an isometric immersion. then*

$$K(p) = \lambda(p) + \mu(p) .$$

Proof. $G(p,z) = \det A(z)$, where $A(z) = \alpha \cdot z$; and, given an orthonormal base v_1, v_2 in $TM_p^\perp$, then $G(p, a_1 v_1 + a_2 v_2) = \det(a_1 A(v_1) + a_2 A(v_2))$ is a quadratic form in a_1 and a_2. Hence we can "diagonalize" G, i.e., there exist orthonormal vectors w_1 and w_2 in $TM_p^\perp$ which are eigenvectors of $G(p,w)$ with eigenvalues $G(p,w_1)$ and $G(p,w_2)$. Therefore, one of them, say $G(p,w_1)$, is equal to $\min\{G(p,z): z \text{ in } S_p^\perp\} = \mu(p)$, and $G(p,w_2) = \lambda(p)$. On the other hand, $G(p,w_i) = \det A(w_i)$; however, as we pointed out above, $K(p) = \det A(w_1) + \det A(w_2)$ if w_1 and w_2 are orthonormal. q.e.d.

Note that if $\mu(p) \neq \lambda(p)$ then we have two distinct normal lines. Using this observation, we obtain the following global proposition.

Proposition 18. *There exists no immersion having the T.P.P. of the projective plane P^2 in R^4, with Gaussian curvature $K(p) > 0$ at every point.*

Proof. Since Kuiper in [4] has shown that there is none in R^3, it is sufficient to show that if $K > 0$ throughout P^2, then the immersion is not substantial. We have remarked previously that $\mu(p) \leq 0$. Therefore, in this case, $\lambda(p) > 0$ by Lemma 17.

Case 1: $\mu(p) < 0$ for all p in P^2. Then $\det A(w_1) = G(p, w_1) = \mu(p) < 0$ for all p, that is, p has negative Lipschitz-Killing curvature in the direction of w_1 (or $-w_1$); as in the case of hyperbolic point in R^3, this means that there exist two distinct "asymptotic" lines Z and W in TM_p with $\alpha(Z, Z)\cdot z = \alpha(W, W)\cdot z = 0$. Hence there exists a continuous tangent line field throughout P^2, implying that $\chi(P^2) = 0$, a contradiction.

Case 2: $\mu(p) = 0$ at some point p. As we have remarked before, this implies that the curvature ellipse B_p is contained in a radial line. On the other hand, $\lambda(p) > 0$ implies that B_p does not meet the origin. The theorem follows now from Lemma 10.

3. Remarks

It is well known that two isometric closed convex surfaces in R^3 are congruent. However in general, this is not true for surfaces with boundary, see Leibin [8]. In our case of isometric immersions with the T.P.P., we can use some partial results of Pogorelov [11] to obtain rigidity theorems.

Proposition. *Given two isometric immersions $f_1, f_2\colon M, \partial M \to R^3$ with the T.P.P., then the surfaces differ by a rigid motion of R^3 if either one of the following holds:*

(a) *there exists an isometry τ of R^3 with $f_1 | \partial M = \tau \circ f_2 | \partial M$ and $k_g \neq -|k|$ for one curve $C \subset \partial M$,*

(b) *∂M consists of one curve and $f_i(M)$ can be projected in a one-to-one fashion to some plane.*

Proof. Condition (a) implies that, after a rigid motion, the immersed surfaces can be put in a position such that they are visible from the origin and that the distances from the origin to corresponding points on the boundary are equal. But with this last property, Pogorelov [11, Th. 4, p. 181] concludes that the surfaces agree globally.

Similarly, if condition (b) holds, since our boundary curves are planar, after a motion of R^3 we can put them in a position such that they are in the half plane $\{z \in R^3 : z \cdot v \geq 0\}$ and $v \cdot f_1(x) = v \cdot f_2(x)$ for every $x \in \partial M$, and we can apply Theorem 2 of Pogorelov [11, p. 178].

In § 1 we showed that for the sphere with discs removed the T.P.P. implies convex boundary curves. But for surfaces of higher genus there exist examples with nonplanar boundary curves, which must be asymptotic. Also, not all examples of surfaces with boundary having the T.P.P. are obtained from closed tight surfaces. Although there are polyhedral examples of Moebius bands in R^3 with the T.P.P., there seems to be no smooth example. Banchoff [1] has studied the T.P.P. and tightness for n-manifolds with boundary in R^n.

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