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CONVEXITY CONDITIONS RELATED WITH 1/2 ESTIMATE IN BOUNDARY PROBLEMS WITH SIMPLE CHARACTERISTICS. II

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Choose a submanifold (not necessarily closed) $\mathcal{N}^\lambda$ of S^*U, which is transversal to C^λ and intersects C^λ only at $(x^0, \zeta^\lambda(x^0))$. Pick a nonzero u in $W^\lambda(x^0, \zeta^\lambda(x^0)) =$ the image of $\rho_1^\lambda(x^0, \zeta^\lambda(x^0))$. Then the function

$$f_u(x, \xi) = |a(x, \xi)\rho_1^\lambda(x, \xi)u|^2$$

viewed as a function on $\mathcal{N}^\lambda$ is of class C^∞ and nonnegative, and $(x^0, \zeta^\lambda(x^0))$ is its isolated zero, i.e., an isolated critical point of f_u on $\mathcal{N}^\lambda$.

Definition 2.3. Assume that the characteristics of A is smooth. We say that a characteristic $(x^0, \zeta^\lambda(x^0))$ of A is nondegenerate if and only if $(x^0, \zeta^\lambda(x^0))$ is a nondegenerate critical point of f_u on $\mathcal{N}^\lambda$ for all nonzero u in $W^\lambda(x^0, \zeta^\lambda(x^0))$. We say that the characteristics of A are nondegenerate when each characteristic is so.

Since f_u on $\mathcal{N}^\lambda$ takes the minimum value at $(x^0, \zeta^\lambda(x^0))$, the above condition means that the Hessian of f_u on $\mathcal{N}^\lambda$ at $(x^0, \zeta^\lambda(x^0))$ is positive definite. In terms of a chart $(\theta_1, \cdots, \theta_k)$ of $\mathcal{N}^\lambda$ with center $(x^0, \zeta^\lambda(x^0))$, this means that the $k \times k$-matrix $(\partial^2 f_u / \partial\theta_\gamma \partial\theta_{\gamma'})\,(0)$ is positive definite. If $(x^0, \zeta^\lambda(x^0))$ is nondegenerate for a choice of a pair of $\mathcal{N}^\lambda$ and a local trivialization of E, it is also so for any other such choice. We can check this by writing down how f_u and its Hessian change when we make a different choice. Note on this connection that $a(x^0, \zeta^\lambda(x^0)) \cdot \rho_1^\lambda(x^0, \zeta^\lambda(x^0)) = 0$.

Because of (9) and (10), $\{(w, (\zeta^\lambda(x^0) + \chi)/(1 + |\chi|^2)^{\frac{1}{2}});\ w \in N^\lambda$ and $\chi \perp \zeta^\lambda(x^0)\}$ forms a submanifold ζ^λ as above. Hence by (11) and (15), the nondegeneracy condition means that $F^\lambda(x^0; w, \chi) \,|\, W^\lambda(x^0, \zeta^\lambda(x^0))$ is injective for all $w \in T_{\tau^0}N^\lambda$ and $\chi \perp \zeta^\lambda(x^0)$. Thus we have

Proposition 2.1. *Assume that the characteristics of A are smooth and the projection $C^\lambda \to {}'C^\lambda$ is bijective, and further that $(x^0, \zeta^\lambda(x^0))$ is a nondegenerate characteristic. Then $F^\lambda(x^0; w, \chi)$, restricted to $W(x^0, \zeta^\lambda(x^0))$, is injective for sufficiently small $w \in N^\lambda$ and any $\chi \perp \zeta^\lambda(x^0)$ provided $(w, \chi) \neq 0$.*

Lemma 2.8. *Under the assumptions in Proposition 2.1, for any $\varepsilon > 0$ we*

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can find $\delta_\varepsilon > 0$ satisfying the following condition: For any $\delta_\varepsilon > \delta > 0$ there is $C_{\varepsilon,\tau}$ such that

$$\mathscr{R}\langle K^\lambda_{\delta 2}(x, D)u, u\rangle + C_{\varepsilon,\delta}\|Q(u)\|^2 \geq -\varepsilon\|u\|^2_{\frac{1}{2}}$$

for all $u \in C^\infty_0(U, U \times L_0)$, provided we choose U sufficiently small and φ^λ with its support sufficiently close to C^λ.

Proof. By Proposition 2.1, $F^\lambda(y; w, \chi)$ on $W^\lambda(y, \zeta^\lambda(y))$ is injective when $(w, \chi) \neq 0$ and $\chi \perp \zeta^\lambda(y)$ provided $|y|$ and $|w|$ are sufficiently small. On the other hand, $H^\lambda_2(x, \xi)$ is of order at least 3 in $(w, \chi(x, \xi))$. Hence for any $\delta > 0$ there is $\delta(\varepsilon) > 0$ such that

$$\langle \delta F^\lambda(y; w\langle\zeta^\lambda(y), \xi\rangle, \chi)^* F^\lambda(y; w\langle\zeta^\lambda(y), \xi\rangle, \chi) + H^\lambda_2(x, \xi)u, u\rangle > 0 ,$$

where $\chi = \chi(y, \xi)$ for all nonzero $u \in W^\lambda(y, \zeta^\lambda(y))$ provided $|y|, |w|$ and $|\xi|^{-1}\chi(y, \xi)$ are less than $\delta(\varepsilon)$. Since $W^\lambda(y, \zeta^\lambda(y))$ is the image of $\rho^\lambda_1(y, \xi)$, we see by (22) that

$$\langle K^\lambda_{\delta 2}(x, \xi)u, u\rangle > 0$$

for all $u \in L_0$, provided we choose U sufficiently small and φ^λ with its support sufficiently close to C^λ. Hence by Theorem 1.4,

$$\langle K^\lambda_{\delta 2}(x, D)u, u\rangle \geq -\langle L(x, D)u, u\rangle ,$$

where

$$\begin{aligned} L(x, \xi) = \sum \tfrac{1}{2}(1 + |\xi|^2)^{\frac{1}{2}} a_{jk}\partial^2 K^\lambda_{\delta 2}(x, \xi)/\partial\xi_j\partial\xi_k \\ + (1 + |\xi|^2)^{-\frac{1}{2}} b_{jk}\partial^2 K^\lambda_{\delta 2}(x, \xi)/\partial x_j\partial x_k + \text{terms of lower orders,} \end{aligned}$$

and a_{jk}, b_{jk} are given in (12) of § 1. By (22) and (23) we see easily that each component of the matrices $\partial^2 K^\lambda_{\delta 2}(x, \xi)/\partial\xi_j\partial\xi_k$ and $\partial^2 K^\lambda_{\delta 2}(x, \xi)/\partial x_j\partial x_k$ can be written as $\varphi^\lambda(x, \xi)^2(\delta t(x, \xi) + h(x, \xi)) + s(x, \xi)$, where t and h are independent of the choice of φ^λ and $\operatorname{Supp} s(x, \xi)$ does not touch the characteristics. Moreover, $h(x, \xi)$ is of order at least 1 in $w_1, \cdots, w_{n-1}, \chi_1(x, \xi), \cdots, \chi_{n-1}(x, \xi)$. Thus we can suppose that $|\varphi^\lambda(x, \xi)^2(\delta t(x, \xi) + h(x, \xi))|/|\xi|$ is as small as we wish, when δ and U are sufficiently small and $\operatorname{Supp}\varphi^\lambda(x, \xi)$ is sufficiently close to C^λ. From this together with Lemma 2.3, we therefore see that

$$|\mathscr{R}\langle L(x, D)u, u\rangle| \leq \varepsilon(\|u\|_{\frac{1}{2}})^2 + CQ(u) .$$

Hence $\mathscr{R}\langle K^\lambda_{\delta 2}(x, D)u, u\rangle + CQ(u) \geq -\varepsilon(\|u\|_{\frac{1}{2}})^2$. q.e.d.

By Lemmas 2.7 and 2.8, we have

Lemma 2.9. *Under the assumptions in Lemma 2.8, for any $\varepsilon > 0$ we can find δ_ε such that for any $\delta_\varepsilon > \delta > 0$ we have*

$$C_{\varepsilon,\delta}Q(u) + \varepsilon\|u\|^2_{\frac{1}{2}} \geq (1 - \delta)\|F^\lambda\{\varphi^\lambda(x, D)\rho^\lambda_1(x, D)\}u\|^2 \qquad (u \in C^\infty_0(U, U \times L_0)) ,$$

provided $C_{\varepsilon,\delta}$ *is sufficiently large,* U *is sufficiently small, and* Supp φ^λ *is sufficiently close to* C^λ.

We further reduce the problem to a case of a differential operator with constant coefficient in $w\langle\zeta^\lambda(y), D\rangle$ and $\chi(x, D)$.

Lemma 2.10. *Under the assumptions in Proposition* 2.1, *for any* $\varepsilon > 0$ *we can find* δ_ε *such that for any* $0 < \delta < \delta_\varepsilon$ *we have the following: If* U *is sufficiently small and* Supp φ^λ *is sufficiently close to* C^λ, *then*

$$\begin{aligned}&\|F^\lambda(y;\, w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|^2\\&\quad - (1-\delta)\|F^\lambda(0;\, w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|^2\\&\qquad \geq \varepsilon\|u\|_{\frac{1}{2}}^2 - C\|u\|^2 - \|R(x, D)u\|^2\end{aligned}$$

for all $u \in C_0^\infty(U, U \times L_0)$, *where* $R(x, \xi)$ *depends on* φ^λ, *is of order* 1, *and* Supp R *is outside of the characteristics.*

Proof. By Proposition 2.1 there is a constant $c > 0$ such that

$$|F^\lambda(y;\, w, \chi)u|^2 \geq c(|w|^2 + |\chi|^2)|u|^2\,,$$

provided y, w are sufficiently small, $\chi \perp \zeta^\lambda(y)$, and $u/|u|$ is in a sufficiently small neighborhood of $W(x^0, \zeta^\lambda(x^0))$. Since $F^\lambda(y;\, w, \chi)$ is linear in w and χ, it follows

$$\begin{aligned}&|\varphi^\lambda(x, \xi)F^\lambda(y;\, w\langle\zeta^\lambda(y), \xi\rangle, \chi(y, \xi))\rho_1^\lambda(x, \xi)u|^2\\&\quad \geq c(|w|^2\langle\zeta^\lambda(y), \xi\rangle^2 + |\chi(y, \xi)|^2)\varphi^\lambda(x, \xi)^2|\rho_1^\lambda(x, \xi)u|^2\end{aligned}$$

for all $u \in L_0$ and $x \in U$, provided U is sufficiently small and Supp φ^λ is sufficiently close to C^λ. For any $\varepsilon_1 > 0$ we may also assume that U is so small that

$$||F^\lambda(y;\, w, \chi)u|^2 - |F^\lambda(0;\, w, \chi)u|^2| \leq \varepsilon_1(|w|^2 + |\chi|^2)|u|^2\,.$$

Set

$$\begin{aligned}G(x, \xi) &= F^\lambda(y;\, w\langle\zeta^\lambda(y), \xi\rangle, \chi(y, \xi))\,,\\G_0(x, \xi) &= F^\lambda(0;\, w\langle\zeta^\lambda(y), \xi\rangle, \chi(y, \xi))\,.\end{aligned}$$

Then

$$\begin{aligned}&|G(x, \xi)\varphi^\lambda(x, \xi)\rho_1^\lambda(x, \xi)u|^2 - (1-\delta)|G_0(x, \xi)\varphi^\lambda(x, \xi)\rho_1^\lambda(x, \xi)u|^2\\&\quad = \delta|G(x, \xi)\varphi^\lambda(x, \xi)\rho_1^\lambda(x, \xi)u|^2\\&\qquad + (1-\delta)(|G(x, \xi)\varphi^\lambda(x, \xi)\rho_1^\lambda(x, \xi)u|^2 - |G_0(x, \xi)\varphi^\lambda(x, \xi)\rho_1^\lambda(x, \xi)u|^2)\\&\quad \geq (\delta c - (1-\delta)\varepsilon_1)\varphi^\lambda(x, \xi)^2(|w\langle\zeta^\lambda(y), \xi\rangle|^2 + |\chi(y, \xi)|^2)|\rho_1^\lambda(x, \xi)u|^2\,.\end{aligned}$$

For a given $\delta > 0$ we choose ε_1 so small that $\delta c - (1-\delta)\varepsilon_1 > 0$. Then $\langle\varphi^\lambda(x, \xi)^2 J(x, \xi)u, u\rangle \geq 0$, where

$$J(x, \xi) = \rho_1^\lambda(x, \xi)(G(x, \xi)^* G(x, \xi) - (1 - \delta)G_0(x, \xi)^* G_0(x, \xi))\rho_1^\lambda(x, \xi) .$$

Hence by Theorem 1.4,

$$\langle\{\varphi^\lambda(x, D)^2 J(x, D)\}u, u\rangle \geq -\mathscr{R}\langle L(x, D)u, u\rangle ,$$

where

$$\begin{aligned} L(x, \xi) = \varphi^\lambda(x, \xi)^2 \sum_{j,k} (\tfrac{1}{2}(1 + |\xi|)^{\frac{1}{2}} a_{jk} \partial^2 J(x, \xi)/\partial\xi_j \partial\xi_k \\ + \tfrac{1}{2}(1 + |\xi|)^{-\frac{1}{2}} b_{jk} \partial^2 J(x, \xi)/\partial x_j \partial x_k) + R_1(x, \xi) \\ + \text{terms of lower orders,} \end{aligned}$$

where $R_1(x, \xi)$ is a sum of terms containing derivatives of $\varphi^\lambda(x, \xi)$ and hence its support does not touch the characteristics. By (12) of § 1, we may choose $g(x)$ in Theorem 1.4 in such a way that the absolute value of each component of the matrix $\sum \frac{1}{2}(1 + |\xi|^2)^{-\frac{1}{2}} b_{jk} \partial^2 J(x, \xi)/\partial x_j \partial x_k$ is less than $\varepsilon'|\xi|$. $\partial^2 J(x, \xi)/\partial\xi_j\partial\xi_k$ is a sum of terms which contain as a factor $G(x, \xi)^* G(x, \xi) - (1 - \delta)G_0(x, \xi)^* \cdot G_0(x, \xi)$ or its partial derivatives in ξ. Since these partial derivatives can enter only through partial derivatives of $\langle\zeta^\lambda(y), \xi\rangle$ or of $\chi(y, \xi)$, each term contains a factor of the form $(a(y) - (1 - \delta)a(0))b(x, \xi)$, so that if we choose δ and U sufficiently small, its absolute value can be made to be less than $\varepsilon'|\xi|$. Thus for any $\varepsilon > 0$ we find for a sufficiently small choice of ε', δ, and U that

$$|\langle\{\varphi^\lambda(x, D)^2 L(x, D)\}u, u\rangle| \leq \varepsilon\|u\|_{\frac{1}{2}}^2 + |\langle R_1(x, D)u, u\rangle|$$

for all $u \in C_0^\infty(U, U \times L_0)$. Therefore

$$\begin{aligned} &\|F^\lambda(y; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|^2 \\ &\quad - (1 - \delta)\|F^\lambda(0; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|^2 \\ &= \langle(\{\varphi^\lambda(x, D)^2 J(x, D)\} + B(x, D))u, u\rangle \\ &\geq \langle B(x, D)u, u\rangle - \varepsilon\|u\|_{\frac{1}{2}}^2 - |\langle R_1(x, D)u, u\rangle| , \end{aligned}$$

where $B(x, \xi)$ is of order 1 and can be calculated by means of the formula for the symbols of compositions and adjoints of pseudo-differential operators. Hence it remains to show that we may assume

$$|\mathscr{R}\langle B(x, D)u, u\rangle| \leq \varepsilon\|u\|_{\frac{1}{2}}^2 + |\langle R_2(x, D)u, u\rangle| + |\langle T_0 u, u\rangle| , \tag{26}$$

where $R_2(x, \xi)$ is of order ≤ 1 and does not touch the characteristics. We see easily that each term of the 1st order part of $B(x, \xi)$ contains either $\varphi^\lambda(x, \xi)^2 \cdot w\langle\zeta^\lambda(y), \xi\rangle$, $\varphi^\lambda(x, \xi)^2\chi(y, \xi)$, a factor of the form $(a(y) - (1 - \delta)a(0))$, or a derivative of $\varphi^\lambda(x, \xi)$. The sum of the terms of the last type is $R_2(x, \xi)$. We may assume that the absolute values of other terms are less than $\varepsilon'|\xi|$. Hence for a sufficiently small choice of ε' we have the formula (26). q.e.d.

By Lemmas 2.9 and 2.3 we have the following:

Proposition 2.2. *Assume that the characteristics of A are smooth, $C^\lambda \to {}'C^\lambda$ is bijective, and the characteristics are nondegenerate at $(x^0, \zeta^\lambda(x^0))$. Then, for any $\varepsilon > 0$ by a choice of a sufficiently small neighborhood U of x^0 and $\varphi^\lambda(x, \xi)$ with its support sufficiently close to C^λ,*

$$C_\varepsilon \|Q(u)\|^2 + \varepsilon \|u\|_{\frac{1}{2}}^2 \geq \|F^\lambda(x^0; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|^2$$

for all $u \in C_0^\infty(U, U \times L_0)$ provided C_ε is sufficiently large.

Expanding $a(x, \xi)\rho_1^\lambda(x, \xi)\rho_2^\lambda(x, \xi) = 0$ in Taylor series in (w, χ) at $(y, \langle\xi, \zeta^\lambda(y)\rangle\zeta^\lambda(y))$ and noting that $a(y, \zeta^\lambda(y))\rho_1^\lambda(y, \zeta^\lambda(y)) = 0$ we find that

$$F^\lambda(y; w, \chi)\rho_2^\lambda(y, \zeta^\lambda(y)) = 0 .$$

In particular,

$$F^\lambda(x^0; w, \chi)\rho_2^\lambda(x^0, \zeta^\lambda(x^0)) = 0 , \tag{27}$$

and we may consider that

$$F^\lambda(x^0; w, \chi) \in \mathrm{Hom}\,(W^\lambda(x^0, \zeta^\lambda(x^0)), E_0) . \tag{28}$$

Definition 2.4. Assume that the characteristics of A are smooth. We say that the characteristics are of fiber dimension 0 if and only if, for each point x^0 of M and for each component C^λ of the characteristics passing over x^0, $\pi\colon C^\lambda \to {}'C^\lambda$ is bijective.

Proposition 2.3. *Assume that the characteristics of A are smooth, of fiber dimension 0, and nondegenerate, and further that there is $r \geq 0$ such that for each λ we have, for all sufficiently small $\theta > 0$,*

$$\begin{aligned}&\|F^\lambda(x^0; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))v\|^2 + C_\theta\|v\|^2 \\ &\quad \geq -C\theta^{r+\frac{1}{2}}\|v\|_{\frac{1}{2}}^2 + c_0\theta^r\langle\langle\zeta^\lambda(x^0), D\rangle v, v\rangle + \langle L_\theta\langle\zeta^\lambda(x^0), D\rangle v, v\rangle \\ &\qquad + \mathscr{R}\langle(T_1^{\lambda\theta}(x, D) + R^{\lambda\theta}(x; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D)))v, v\rangle\end{aligned}$$

for all $v \in C_0^\infty(U, W^\lambda(x^0, \zeta^\lambda(x^0)))$, where L_θ is an endomorphism of $W^\lambda(x^0, \zeta^\lambda(x^0))$ such that $\langle L_\theta v, v\rangle \geq 0$ for all v, $T_1^{\lambda\theta}$ is of order 1, $T_1^{\lambda\theta}(x^0, \xi) = 0$ for all ξ, and $R^\theta(x; w, \chi)$ is linear in (w, χ). Then, for a sufficiently small neighborhood U of x^0, $Q(u) \geq c\|u\|_{\frac{1}{2}}^2$ for all $u \in C_0^\infty(U, U \times L_0)$.

Proof. For simplicity we set $\rho^\lambda = \rho_1^\lambda(x^0, \zeta^\lambda(x^0))$. Clearly

$$\|\rho^\lambda\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|_{\frac{1}{2}} \leq C\|u\|_{\frac{1}{2}} + C_1\|u\| , \tag{29}$$

where C_1 may depend on φ^λ. Put

$$
\begin{aligned}
K(x,\xi) = |\xi|\varphi(x,\xi)^2 + \sum_\lambda \varphi^\lambda(x,\xi)^2(|\xi|\rho_2^\lambda(x,\xi) \\
+ \rho_1^\lambda(x,\xi)\langle\zeta^\lambda(x^0),\xi\rangle)\rho^\lambda\rho_1^\lambda(x,\xi) \\
+ \frac{1}{|\xi|}\rho_1^\lambda(x,\xi)F^\lambda(x^0;\, w\langle\zeta^\lambda(y),\xi\rangle, \chi(y,\xi))^*F^\lambda(x^0; \\
w\langle\zeta^\lambda(y),\xi\rangle, \chi(y,\xi))\rho_1^\lambda(x,\xi) \, .
\end{aligned}
$$

Since the characteristics are nondegenerate,

$$
\langle (K(x,\xi) - c_1|\xi|)u, u\rangle \geq 0 \tag{30}
$$

for a constant c_1. Therefore, since $K(x,\xi)$ is of order 1, Theorem 1.4 implies that

$$
\langle K(x,D)u, u\rangle \geq c_1\|u\|_{\frac12}^2 - C\|u\|^2 \, . \tag{31}
$$

On the other hand, by Lemmas 2.3 and 2.6 together with Proposition 2.2,

$$
\begin{aligned}
&\langle K(x,D)u,u\rangle \leq \varepsilon\|u\|_{\frac12} + C_\varepsilon Q(u) \\
&\quad + \sum_\lambda \langle\langle\zeta^\lambda(x^0),D\rangle\rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u, \rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u\rangle \, .
\end{aligned} \tag{32}
$$

Thus, by our assumption,

$$
\begin{aligned}
&\sum_\lambda \|F^\lambda(x^0;\, w\langle\zeta^\lambda(y),D\rangle, \chi(y,D))\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u\|^2 \\
&\quad = \sum_\lambda \|F^\lambda(x^0;\, w\langle\zeta^\lambda(y),D\rangle, \chi(y,D))\rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u\|^2 \, , \quad \text{(by (27))} \\
&\quad \geq \sum_\lambda (-C\theta^{r+\frac12}\|\rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u\|_{\frac12}^2 \\
&\qquad + c_0\theta^r\langle\langle\zeta^\lambda(x^0),D\rangle\rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u, \rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u\rangle \\
&\qquad + \mathscr{R}\langle(T_1^{\lambda\theta}(x,D) + R^{\lambda\theta}(x;\, w\langle\zeta^\lambda(y),D\rangle, \\
&\qquad\qquad \chi(y,D)))\rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u, \rho^\lambda\varphi^\lambda(x,D)\rho_1^\lambda(x,D)u\rangle) + \gamma \\
&\quad \geq -\theta^{r+\frac12}c'\|u\|_{\frac12}^2 + c_0\theta^r(\langle K(x,D)u,u\rangle - \varepsilon\|u\|_{\frac12} - C'_\varepsilon Q(u)) \\
&\qquad + \langle T_1^\theta(x,D)u,u\rangle + \gamma \, , \quad \text{(by (29) and (32))} \, ,
\end{aligned}
$$

where $T_1^\theta(x,\xi) = \sum_\lambda \rho_1^\lambda(x,\xi)(T_1^{\lambda\theta}(x,\xi) + R^{\lambda\theta}(x;\chi(x,\xi)))\varphi^\lambda(x,\xi)^2\rho^\lambda\rho_1^\lambda(x,\xi)$, and $\gamma = \langle L_\theta\langle\zeta^\lambda(x^0), D\rangle\rho^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u, \rho^\lambda\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\rangle$. By choosing $\operatorname{Supp}\varphi^\lambda$ sufficiently close to C^λ and U small, we may assume that $\langle L_\theta\langle\zeta^\lambda(x^0),\xi\rangle\varphi^\lambda(x,\xi)^2\rho^\lambda\rho_1^\lambda(x,\xi)u, \rho^\lambda\rho_1^\lambda(x,\xi)u\rangle \geq 0$, so that $\gamma \geq -C_\theta\|u\|^2$. Hence by (31),

$$
\begin{aligned}
&\sum_\lambda \|F^\lambda(x^0;\, w\langle\zeta^\lambda(y),D\rangle, \chi(y,D))\{\varphi^\lambda(x,D)\rho_1^\lambda(x,D)\}u\|^2 + C''_{\theta,\varepsilon}Q(u) \\
&\qquad \geq c_0\theta^r(c_1 - (\varepsilon + \theta^{\frac12}c'))\|u\|_{\frac12}^2 + \langle T_1^\theta(x,D)u,u\rangle \, .
\end{aligned}
$$

Choose θ_1 and ε so small that $c_1 - (\varepsilon + \theta_1^{\frac{1}{2}}c') > 0$. Then, for constants $c > 0$ and $C > 0$ (setting $T_1(x, \xi) = T_1^{\theta_1}(x, \xi)$),

$$\sum_\lambda \|F^\lambda(x^0; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))\{\varphi^\lambda(x, D)\rho_1^\lambda(x, D)\}u\|^2 + CQ(u) \tag{33}$$
$$\geq c\|u\|_{\frac{1}{2}}^2 + \langle T_1(x, D)u, u\rangle$$

for all $u \in C_0^\infty(U, U \times L_0)$, provided U is sufficiently small and Supp φ^λ is sufficiently close to C^λ. Now by applying Proposition 2.2 to the left hand side of (33) for a sufficiently small choice of ε, for constants $c > 0$ and $C > 0$ (where c is independent of but C is dependent on the choice of φ^λ) we find that

$$CQ(u) \geq c\|u\|_{\frac{1}{2}}^2 + \langle T_1(x, D)u, u\rangle \ .$$

Since $T_1(x, \xi) = \sum_\lambda (T_1^{\lambda\theta_1}(x, \xi) + R^{\lambda\theta_1}(x; \chi(x, \xi)))\varphi^\lambda(x, \xi)^2\rho^\lambda\rho_1^\lambda(x, \xi)$ and $T_1^{\lambda\theta}(x^0, \xi) = 0$, $T_1(x, \xi)/|\xi|$ can be made arbitrarily small by choosing U sufficiently small and Supp φ^λ sufficiently close to C^λ. Hence for such choice of U and φ^λ, $\frac{c}{2}\|u\|_{\frac{1}{2}}^2 \leq CQ(u)$.

Definition 2.5. $F^\lambda(x^0; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D))$ will be called the localized operator of A at x^0 for the characteristics C^λ. For $\eta = (\eta_1, \cdots, \eta_{2n-2}) \in \boldsymbol{R}^{2(n-1)}$, $g^\lambda(\eta) = F^\lambda(x^0; \eta_1, \cdots, \eta_{2n-2})$ will be called the indirect symbol of the localized operator. (We recall w and y are considered as functions of x.)

In order to make the writing easy, we fix λ once for all and set for $j = 1, \cdots, n-1$

$$X_j(x, \xi) = w_j(x)\langle\zeta^\lambda(y(x)), \xi\rangle \ , \qquad X_j = X_j(x, D) \ , \tag{34}$$

$$X_{n-1+j}(x, \xi) = \chi_j(x, \xi) \ , \qquad X_{n-1+j} = X_{n-1+j}(x, D) \ , \tag{35}$$

$$f^{\lambda j}(x^0) = f^j \ , \qquad g^{\lambda j}(x^0) = f^{n-1+j} \ . \tag{36}$$

Thus we can write

$$F^\lambda = F^\lambda(x^0; w\langle\zeta^\lambda(y), D\rangle, \chi(y, D)) = \sum_{s=1}^{2n-2} f^s X_s \ , \tag{37}$$

By direct calculation we find that for $s, t = 1, \cdots, 2n-2$

$$X_s^* X_t - X_t^* X_s = \frac{1}{i} c_{st}(x)\langle\zeta^\lambda(y), D\rangle + \sum_{i=1}^{2n-1} b_{st}^r(x) X_r \ , \tag{38}$$

where $c_{st}(x)$ is a real valued function, skew-symmetric in s, t, given by, for $j, k = 1, \cdots, n-1$,

$$
\begin{aligned}
&c_{jk}(x) = 0\ , \\
(39)\qquad &c_{n-1+j\ k}(x) \equiv iX_{n-1+j}w_k(x) \qquad (\operatorname{mod} w)\ , \\
&c_{n-1+j\ n-1+k}(x) \equiv i(X_{n-1+j}\zeta_k^\lambda(x) - X_{n+1+k}\zeta_j^\lambda(x))\ , \qquad (\operatorname{mod} \zeta^\lambda)\ .
\end{aligned}
$$

Another way of writing down these functions are as follows:

$$
\begin{aligned}
dw_j &\equiv \sum_k c_{b+j\ k}\chi_k(y, dx) \qquad (\operatorname{mod} \langle \zeta^\lambda(y), dx\rangle)\ , \\
d\zeta^\lambda &\equiv \sum_{j,k} \tfrac{1}{2} c_{n-1+k\ n-1+j}\chi_j(y, dx) \wedge \chi_k(y, dx)\ , \qquad (\operatorname{mod} \langle \zeta^\lambda(y), dx\rangle)\ ,
\end{aligned}
$$

where ζ^λ denotes the differential form $\langle \zeta^\lambda(y(x)), dx\rangle = \sum_j \zeta_j^\lambda(y(x))dx_j$.

3. Study of the characteristic parts

In this section we fix vector spaces W, V, and a linear mapping

$$(1) \qquad g\colon \boldsymbol{R}^{2n-2} \ni \eta \to g(\eta) \in \operatorname{Hom}(W, V)\ .$$

We write

$$(2) \qquad g(\eta) = \sum_{s=1}^{2n-2} g^s \eta_s\ ,$$

where $g^s \in \operatorname{Hom}(W, V)$. U will be as in § 2, T_j will denote as in § 2 the pseudo-differential operators of order j which may change from formulas to formulas, and $X_1, \cdots, X_{2n-2}$ will have the same meaning as in § 2. For $u \in C_0^\infty(U, W)$ we set

$$(3) \qquad g(X)u = \sum g^s X_s u\ ,$$

which is in $C_0^\infty(U, V)$. We are interested in an estimate of the type described in Proposition 2.3. To this end we apply our results to the case where $g(\eta) = F^\lambda(x^0; \eta)$, $W = W^\lambda(x^0, \zeta^\lambda(x^0))$ and $V = E_0$. Assume that we are given hermitian metrics on W, V, and set

$$(4) \qquad \Delta_g(x, \xi) = g(X(x, \xi))^* g(X(x, \xi))\ .$$

Lemma 3.1. *For $u \in C_0^\infty(U, W)$,*

$$
\begin{aligned}
\|g(X)u\|^2 = {} & \mathscr{R}\langle \Delta_g(x, D)u, u\rangle \\
& + \sum \tfrac{1}{2}\mathscr{R}\Big\langle g^{s*}g^t\Big(\frac{1}{i}c_{st}(x)\langle \zeta^\lambda(y), D\rangle + b_{st}^r(x)X_r\Big)u, u\Big\rangle \\
& + \langle g(X)u, g(h(x))u\rangle + \langle T_0(x, D)u, u\rangle\ ,
\end{aligned}
$$

where $h_s(x)$ is a C^∞ function, $c_{st}(x)$ and $b_{st}^r(x)$ are defined in § 2 of (38).

Proof. Clearly, $X_s^* = X_s + h_s(x)$, where $h_s(x)$ is a C^∞ function. Thus

$$g(X)^* g(X)u = \Delta_g(x, D)u + \sum_{s,t,j} g^{s*}g^t(\partial X_s(x, \xi)/\partial\xi_j)Y_{tj}(x, \xi)u + g(h(x))^* g(X)u ,$$

where $Y_{tj} = (1/i)\partial X_t(x, \xi)/\partial x_j$, and therefore

$$\|g(X)u\|^2 = \mathscr{R}\langle\Delta_g(x, D)u, u\rangle + \mathscr{R}\langle g(X)u, g(h(x))u\rangle + \mathscr{R} \sum \langle g^{s*}g^t(\partial X_s(x, \xi)/\partial\xi_j)Y_{tj}(x, D)u, u\rangle . \tag{5}$$

(Note that $\partial X_s/\partial\xi_j$ is a function of x.) On the other hand, since $Y_{tj}(x, \xi)$ has purely imaginary coefficients, we have

$$\begin{aligned}
&\mathscr{R} \sum \langle g^{s*}g^t(\partial X_s(x, \xi)/\partial\xi_j)Y_{tj}(x, D)u, u\rangle \\
&\quad = -\mathscr{R} \sum \langle g^{s*}g^t(\partial X_t(x, \xi)/\partial\xi_j)Y_{sj}(x, D)u, u\rangle + \mathscr{R}\langle T_0(x, D)u, u\rangle \\
&\quad = \tfrac{1}{2}\mathscr{R} \sum \langle g^{s*}g^t\{(\partial X_s(x, \xi)/\partial\xi_j)Y_{tj}(x, D) \\
&\qquad\qquad - (\partial X_t(x, \xi)/\partial\xi_j)Y_{sj}(x, D)\}u, u\rangle + \mathscr{R}\langle T_0(x, D)u, u\rangle \\
&\quad = \tfrac{1}{2}\mathscr{R} \sum \langle g^{s*}g^t(X_sX_t - X_tX_s)u, u\rangle + \mathscr{R}\langle T_0(x, D)u, u\rangle \\
&\quad = \tfrac{1}{2}\mathscr{R} \sum \langle g^{s*}g^t(X_s^*X_t - X_t^*X_s)u, u\rangle + \mathscr{R}\langle T_0(x, D)u, u\rangle \\
&\quad = \tfrac{1}{2}\mathscr{R} \sum \left\langle g^{s*}g^t\left(\frac{1}{i}c_{st}(x)\langle\zeta^\lambda(y), D\rangle + b_{st}^r(x)X_r\right)u, u\right\rangle + \mathscr{R}\langle T_0(x, D)u, u\rangle ,
\end{aligned}$$

which together with (5) thus implies our formula. q.e.d.

Let V' be a vector space with a hermitian metric, and assume that we have, for all θ with sufficiently small absolute value, a linear map

$$g_\theta : \boldsymbol{R}^{2n-2} \ni \eta \to g_\theta(\eta) \in \mathrm{Hom}\,(W, V') ,$$

which depends differentiably on θ.

Lemma 3.2. *Assume that for an integer $d \geq 1$ we have the following*:

(i) $\quad g^{s*}g^t = g_0^{s*}g_0^t \qquad (s, t = 1, \cdots, 2n-2)$,

$(\text{ii})_d \quad g(\eta)^*g(\eta) - g_\theta(\eta)^*g_\theta(\eta) = \theta^{d+1}h_\theta(\eta)$,

where $h_\theta(\eta)$ depends differentiably on θ, and

$$(\text{iii})_d \quad \sum \left\langle \frac{1}{i}c_{st}(x^0)(g^{s*}g^t - g_\theta^{s*}g_\theta^t)u, u\right\rangle \geq c_0\theta^d|u|^2$$

for all $u \in W$ and all sufficiently small $\theta > 0$. Assume further that

(iv) $\quad \langle\Delta_g(x, \xi)u, u\rangle \geq c_1(\sum_s |X_s(x, \xi)u|^2)$.

Then for sufficiently small $\theta > 0$

$$\|g(X)u\|^2 + C_\theta\|u\|^2 \geq -C\theta^{d+\frac{1}{2}}\|u\|_{\frac{1}{2}} + c_0\theta^d\langle\langle\zeta^\lambda(x^0), D\rangle u, u\rangle \\ + \langle L_\theta\langle\zeta^\lambda(x^0), D\rangle u, u\rangle + \langle(T_1^\theta(x, D) + R^\theta(x, X))u, u\rangle$$

for all $u \in C_0^\infty(U, W)$, *where* $T_1^\theta(x^0, \xi) = 0$ *for all* ξ, $R^\theta(x, \eta)$ *is linear in* η, *and* $L_\theta \in \operatorname{Hom}(W, W)$ *such that* $\langle L_\theta u, u\rangle \geq 0$ *for all* u.

Proof. By Lemma 3.1,

$$\|g_\theta(X)u\|^2 = \mathscr{R}\langle \varDelta_{g_\theta}(x, D)u, u\rangle + \mathscr{R}\sum \frac{1}{i}\Big\langle g_\theta^{s*}g_\theta^t\Big(\frac{1}{i}c_{st}(x^0)\langle\zeta^\lambda(y), D\rangle \\ + b_{st}^r X_r\Big)u, u\Big\rangle + \mathscr{R}\langle g_\theta(X)u, \boldsymbol{h}_\theta(h(x))u\rangle + \langle T_0^\theta(x, D)u, u\rangle\ .$$

Thus for $0 < \delta < 1$

$$\begin{aligned}\|g(X)u\|^2 &= \delta\|g(X)u\|^2 + (1-\delta)\|g_\theta(X)u\|^2 \\ &\quad + (1-\delta)(\|g(X)u\|^2 - \|g_\theta(X)u\|^2) \\ &= (1-\delta)\|g_\theta(X)u\|^2 + \mathscr{R}\langle(\delta\varDelta_g(x, D) + (1-\delta)(\varDelta_g(x, D) \\ &\qquad\qquad - \varDelta_{g_\theta}(x, D)))u, u\rangle \\ &\quad + \mathscr{R}\sum\Big\langle\frac{c_{st}}{2i}(\delta g^{s*}g^t + (1-\delta)(g^{s*}g^t - g_\theta^{s*}g_\theta^t))\langle\zeta^\lambda(y), D\rangle u, u\Big\rangle \\ &\quad + \mathscr{R}\langle(R^{\theta,\delta}(x, x) + T_0^{\theta,\delta}(x, D))u, u\rangle\ .\end{aligned}$$

Hence

$$\begin{aligned}&\|g(X)u\|^2 + C_\theta\|u\|^2 \\ &\quad\geq \mathscr{R}\langle(\delta\varDelta_g(x, D) + (1-\delta)(\varDelta_g(x, D) - \varDelta_{g_\theta}(x, D)))u, u\rangle \\ &\qquad + \mathscr{R}\sum\frac{1}{2i}\langle c_{st}(\delta g^{s*}g^t + (1-\delta)(g^{s*}g^t - g_\theta^{s*}g_\theta^t))\langle\zeta^\lambda(x), D\rangle u, u\rangle \\ &\qquad + \mathscr{R}\langle R^{\theta,\delta}(x, X)u, u\rangle\ .\end{aligned}$$

By $(\mathrm{ii})_d$ and (iv),

$$\langle(\delta\varDelta_g(x, \xi) + (1-\delta)(\varDelta_g(x, \xi) - \varDelta_{g_\theta}(x, \xi)))u, u\rangle \\ \geq \delta c_1\textstyle\sum_s|X_s(x, \xi)u|^2 - (1-\delta)\theta^{d+1}\langle h_\theta(X(x, \xi))u, u\rangle\ .$$

For $\theta > 0$ and $\delta = \theta^{d+\frac{1}{2}}$, the right hand side of the above inequality becomes

$$\theta^{d+\frac{1}{2}}(c_1\textstyle\sum_s|X_s(x, \xi)u|^2 - (1 - |\theta|^{d+\frac{1}{2}})\theta^{\frac{1}{2}} < h_\theta(X(s, \xi))u, u\rangle)\ .$$

Therefore for sufficiently small θ, by Theorem 1.4 we have

$$(7)\quad \langle(\delta\varDelta_g(x, D) + (1-\delta)(\varDelta_g(x, D) - \varDelta_{g_\theta}(x, D)))u, u\rangle \geq -C_1\theta^{d+\frac{1}{2}}\|u\|_{\frac{1}{2}}\ ,$$

where $\delta = \theta^{d+\frac{1}{2}}$. By (iii)$_d$ we have for $\delta = \theta^{d+\frac{1}{2}}$

$$\sum \left\langle \frac{1}{2i} c_{st}(x^0)(\delta g^{s*}g^t + (1-\delta)(g^{s*}g^t - g^{s*}_\theta g^t_\theta))\langle\zeta^\lambda(x^0), D\rangle u, u\right\rangle$$
$$\geq C_0\theta^d\langle\langle\zeta^\lambda(x^0), D\rangle u, u\rangle - \theta^{d+\frac{1}{2}}c'\|u\|^2_{\frac{1}{2}} + \langle L_\theta\langle\zeta^\lambda(x^0), D\rangle u, u\rangle ,$$

together with (6) and (7)

$$\|g(X)u\|^2 + C_\theta\|u\|^2 \geq c_0\theta^d\langle\langle\zeta^\lambda(x^0), D\rangle u, u\rangle - \theta^{d+\frac{1}{2}}c\|u\|^2_{\frac{1}{2}}$$
$$+ \mathscr{R}\langle R^{\theta,\delta}(x, X)u, u\rangle + \mathscr{R}\langle T^\theta_1(x, D)u, u\rangle$$
$$+ \langle L_\theta\langle\zeta^\lambda(x^0), D\rangle u, u\rangle ,$$

where $T^\theta_1(x^0, \xi) = 0$.

Theorem 3.1. *Let A be a pseudo-differential operator of order* 1 *mapping $C^\infty_0(M, L)$ into $C^\infty_0(M, E)$. Assume that the characteristics of A are smooth, of fiber dimension* 0, *and nondegenerate, and further that, for each $x^0 \in M$ and each component of the characteristics C^λ passing over x^0, the indirect symbol $g(\eta)$ of the localized operator of A at x^0 relative to C^λ satisfies conditions* (i), (ii)$_d$, *and* (iii)$_d$ *(for an integer $d \geq 1$) in Lemma* 3.2. *Then there is a constant $c > 0$ such that*

$$\|Au\|^2 + \|u\|^2 \geq c\|u\|^2_{\frac{1}{2}}$$

for all $u \in C^\infty(M, E)$.

Proof. If $g(\eta)$ satisfies the conditions in Lemma 3.2 for $d \geq 1$, and m is an integer $m \geq 1$, then $g(\eta)$ satisfies the conditions for dm, so that we may use the common d for the indirect symbols relative to the components $C^1, \cdots, C^\lambda, \cdots$. Hence our theorem is an immediate corollary of Proposition 2.3 and Lemma 3.2. q.e.d.

We further study the conditions in Lemma 3.2. For $g(\eta) = \sum g^s\eta_s$ they are conditions on $g^{s*}g^t \in \operatorname{Hom}(W, W)$. Thus if we have another $h(\eta) \in \operatorname{Hom}(W, E_2)$ such that $h^{s*}h^t = g^{s*}g^t$ for all $s, t = 1, \cdots, 2n-2$, and if $g(\eta)$ satisfies the conditions, then so does $h(\eta)$. g induces a linear mapping $g\colon W \otimes \boldsymbol{R}^{2n-2} \to E_1$, and vice versa, and g and g are related by

$$g(u \otimes e^s) = g^s u \qquad (u \in W) ,$$

where $\{e^s\}$ is the standard base of $\boldsymbol{R}^{2n-2}$. We impose the hermitian metric on $W \otimes \boldsymbol{R}^{2n-2}$ induced by that of W. Let h be the positive semidefinite hermitian square root of g^*g, and $h(\eta) \in \operatorname{Hom}(W, W \otimes \boldsymbol{R}^{2n-2})$ be defined by h as above. Then $\langle h^{s*}h^t u, u'\rangle = \langle h^t u, h^s u'\rangle = \langle h(u \otimes e^t), h(u' \otimes e^s)\rangle = \langle g(u \otimes e^t), g(u' \otimes e^s)\rangle = \langle g^{s*}g^t u, u'\rangle$, i.e., $h^{s*}h^t = g^{s*}g^t$. Thus we may replace g by h. Moreover

$$(8) \qquad \ker g = \ker h .$$

Hence we may assume without loss of generality that $V = W \otimes \boldsymbol{R}^{2n-2}$, and that g is a positive semidefinite hermitian metric. If g_θ as in Lemma 3.2 exists, we may assume also that $g_\theta(\eta) \in \operatorname{Hom}(W, W \otimes \boldsymbol{R}^{2n-2})$, or equivalently $g_\theta \in \operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$. Then the first condition says that $g^*g = g_0^*g_0$, i.e., $g = vg_0$ where v is a unitary transformation of $W \otimes \boldsymbol{R}^{2n-2}$. Replacing g_θ by vg_θ we may assume that $g_0 = g$.

We first study the conditions in Lemma 3.2 for the case $d = 1$. Write

$$g_\theta \equiv g + \theta r \qquad (\operatorname{mod} \theta^2) .$$

Then $(\mathrm{ii})_1$ and $(\mathrm{iii})_1$ are equivalent to

$$(\mathrm{ii})_1' \quad g^{s*}r^t + r^{s*}g^t + g^{t*}r^s + r^{t*}g^s = 0 ,$$

$$(\mathrm{iii})_1' \quad \left\langle \frac{1}{i} c_{st}^0 (g^{s*}r^t + r^{s*}g^t)u, u \right\rangle < 0$$

for all nonzero u in W, where $c_{st}^0 = c_{st}(x^0)$. In order to write these conditions more concisely, we introduce an automorphism τ of $\operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$ defined by

$$(9) \qquad \langle r^\tau(u \otimes e^s), u' \otimes e^t \rangle = \langle r(u \otimes e^t), u' \otimes e^s \rangle$$

for all $u, u' \in W$ and $s, t = 1, \cdots, 2n-2$. Then $(\mathrm{ii})_1'$ is equivalent to

$$(\mathrm{ii})_1'' \quad g^*r + r^*g + (g^*r)^\tau + (r^*g)^\tau = 0 ,$$

Let $J\colon \boldsymbol{R}^{2n-2} \to \boldsymbol{R}^{2n-2}$ be defined by

$$J(e^s) = \sum_{t=1}^{2n-2} c_{st}^0 e^t .$$

Then for $r, g \in \operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$,

$$\begin{aligned}\sum_{s,t} \langle c_{st}^0 r^{s*} g^t u, u' \rangle &= \sum \langle c_{st}^0 g(u \otimes e^t), r(u' \otimes e^s) \rangle \\ &= \sum \langle g(u \otimes c_{st}^0 e^t), r(u' \otimes e^s) \rangle \\ &= \sum \langle r^* g(I \otimes J)(u \otimes e^s), u' \otimes e^s \rangle ,\end{aligned}$$

where I is the identity map of W. This suggests us to introduce a linear mapping $tr_W\colon \operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2}) \to \operatorname{Hom}(W, W)$ defined by

$$(10) \qquad \langle (tr_W h)u, u' \rangle = \sum_s \langle h(u \otimes e^s), u' \otimes e^s \rangle .$$

Then $(\mathrm{iii})_1'$ can be written as

$$(\mathrm{iii})_1'' \quad tr_W(i(g^*r + r^*g)(I \otimes J)) > 0 .$$

Thus we have the following:

Lemma 3.3. *Conditions* (i), $(\mathrm{ii})_1$, $(\mathrm{iii})_1$ *in Lemma* 3.2 *are satisfied if and only if we can find* $r \in \mathrm{Hom}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$ *such that*

$$(\mathrm{ii})_1'' \quad g^*r + r^*g + (g^*r)^\tau + (r^*g)^\tau = 0\ ,$$
$$(\mathrm{iii})_1'' \quad tr_W(i(g^*r + r^*g)(I \otimes J)) > 0\ .$$

Let V, V_1 be vector spaces with hermitian metrics. Then we always consider the vector space $\mathrm{Hom}\,(V, V_1)$ with a hermitian metric defined by

$$\langle r, g\rangle = Tr\, g^*r \qquad (g, r \in \mathrm{Hom}\,(V, V_1))\ .$$

The subspace over $\boldsymbol{R}$ of $\mathrm{Hom}\,(V, V)$ consisting of all self-adjoint transformations of V will be denoted by $\mathrm{Her}\,(V, V)$, so that

$$\dim_{\boldsymbol{R}}(\mathrm{Her}\,(V, V)) = (\dim_{\boldsymbol{C}} V)^2\ .$$

τ defined by the formula (9) is a hermitian unitary transformation of order 2 of $\mathrm{Hom}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$ and preserves $\mathrm{Her}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$. We set

$$(11) \qquad S = \{\alpha \in \mathrm{Her}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2});\ \alpha^\tau = \alpha\}\ ,$$

$$(12) \qquad S^A = \{\beta \in \mathrm{Her}\,(W \oplus \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2});\ \beta^\tau = -\beta\}\ .$$

For a subspace F of V, ρ_F generally denotes the orthogonal projection of V to F. We can now rewrite Lemma 3.3 as follows:

Proposition 3.1. *Conditions* (i), $(\mathrm{ii})_1$, $(\mathrm{iii})_1$ *in Lemma* 3.2 *are satisfied if and only if there is* $\beta \in S^A$ *such that*

1) $\rho_K\beta\rho_K = 0$ *where* $K = \ker g$,
2) $tr_W(i\beta(I \otimes J)) > 0$.

Proof. Assume that there is r as in Lemma 3.3. Then $\beta = g^*r + r^*g$ is in S^A and satisfies 1) and 2). Conversely, assume that β satisfies 1) and 2). Since β is hermitian, the condition 1) implies that there is $r \in \mathrm{Hom}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$ such that $\beta = g^*r + r^*g$. Then this r clearly satisfies $(\mathrm{ii})_1''$ and $(\mathrm{iii})_1''$. q.e.d.

In order to study these conditions further, we define a linear map (over $\boldsymbol{R}$) $\theta\colon \mathrm{Hom}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2}) \to \mathrm{Her}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$ by

$$(13) \qquad \theta(r) = g^*r + r^*g\ .$$

r is in $\mathrm{Ker}\,\theta$ if and only if ir^*g is hermitian. Since ir^*g is zero on $\mathrm{Ker}\, g$, we thus have a linear map

$$\operatorname{Ker}\theta \longrightarrow \operatorname{Her}(\operatorname{Im} g, \operatorname{Im} g) ,$$

where $\operatorname{Im} g$ denotes the image of g (being hermitian, it is the orthogonal complement of $\operatorname{Ker} g$). It is easy to check that this map is surjective and the kernel is isomorphic to $\operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, \operatorname{Ker} g)$. Thus

$$\begin{aligned} \dim_{\boldsymbol{R}} \operatorname{Ker}\theta &= (\dim_{\boldsymbol{C}} \operatorname{Im} g)^2 + 2m(2n-2) \dim_{\boldsymbol{C}} \operatorname{Ker} g \\ &= ((2n-2)m)^2 + (\dim_{\boldsymbol{C}} \operatorname{Ker} g)^2 \qquad (m = \dim_{\boldsymbol{C}} w) . \end{aligned} \tag{14}$$

When V is a vector subspace of $W \otimes \boldsymbol{R}^{2n-2}$, $r \in \operatorname{Hom}(V, V)$ can be identified with an element in $\operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$, which coincides with r on V and is zero on the orthogonal complement of V. Thus we always consider $\operatorname{Hom}(V, V)$ as a subspace of $\operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$. We denote by π_S the projection to S of $S \oplus S^A = \operatorname{Her}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$. Clearly

$$\pi_S(h) = \tfrac{1}{2}(h + h^\tau) .$$

Lemma 3.4. $(\operatorname{Im}(\pi_S \circ \theta))^\perp \cap S = \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g) \cap S$, *where* $\perp$ *is taken in* $\operatorname{Her}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$.

Proof. α in S is in $(\operatorname{Im}(\pi_S \circ \theta))^\perp \cap S$ if and only if

$$\langle r^*g + g^*r + (r^*g)^\tau + (g^*r)^\tau, \alpha \rangle = 0 \tag{15}$$

for all $r \in \operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$. Since $\langle r, q \rangle = \langle r^\tau, q^\tau \rangle$ and $h(rq) = h(qr)$, we see easily that the right hand side is $4\mathscr{R}tr\, r^*g\alpha$. Therefore (15) is satisfied for all r if and only if $g\alpha = 0$. Since α is hermitian, it follows that the condition is equivalent to $\alpha \in \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g)$.

Lemma 3.5. $\operatorname{Im}\theta \cap S^A = \{\gamma - \gamma^\tau;\ \gamma \in \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g)\}^\perp \cap S^A$.

Proof. If $\beta \in \operatorname{Image}\theta \cap S^A$, then for any $\gamma \in \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g)$,

$$\begin{aligned} \mathscr{R}\langle \beta, \gamma - \gamma^\tau \rangle = 2\mathscr{R}\langle \beta, \gamma \rangle = 2\mathscr{R}tr\, \gamma\beta = 2\mathscr{R}tr\, \gamma(r^*g + g^*r) \\ = 4\mathscr{R}tr\,(r^*g\gamma) = 0 . \end{aligned}$$

Thus the left hand side is contained in the right hand side. We prove the equality by counting the dimension of both sides. Set $\Phi = \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g) \cap S$. Then the real dimension of the right hand side is equal to

$$\dim_{\boldsymbol{R}} S^A - (\dim_{\boldsymbol{C}} \operatorname{Ker} g)^2 + \dim_{\boldsymbol{R}} \Phi .$$

Since the left hand side is equal to the image by θ of $\operatorname{Ker} \pi_S \circ \theta$, its dimension is equal to

$$\begin{aligned} &\dim_{\boldsymbol{R}} (\operatorname{Ker} \pi_S \circ \theta) - \dim_{\boldsymbol{R}} (\operatorname{Ker}\theta) \\ &\quad = \dim_{\boldsymbol{R}} (\operatorname{Hom}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})) - \dim_{\boldsymbol{R}} (\operatorname{Im} \pi_S \circ \theta) \\ &\quad\quad - \dim_{\boldsymbol{R}} (\operatorname{Ker}\theta) \end{aligned}$$

$$= 2(m(2n-2))^2 - (\dim_{\boldsymbol{R}} S - \dim_{\boldsymbol{R}} \Phi) - ((m(2m-2))^2$$
$$+ (\dim_{\boldsymbol{C}} (\operatorname{Ker} g)^2) \qquad \text{(by Lemma 3.4 and (14))}$$
$$= (m(2n-2))^2 - \dim_{\boldsymbol{R}} S) + \dim_{\boldsymbol{R}} \Phi - (\dim_{\boldsymbol{C}} \operatorname{Ker} g)^2$$
$$= \dim_{\boldsymbol{R}} S^A - (\dim_{\boldsymbol{C}} \operatorname{Ker} g)^2 + \dim_{\boldsymbol{R}} \Phi . \quad \text{q.e.d.}$$

For $\gamma \in \operatorname{Her}(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$, set

$$C(\gamma) = tr_W(i\gamma(I \otimes J)) = \sum ic^0_{st}\gamma^{st} \in \operatorname{Hom}(W, W) . \tag{16}$$

Since $J^* = -J$, $C(\gamma)$ is hermitian. If $H \in \operatorname{Her}(W, W)$, then

$$\begin{aligned}\langle \gamma, H \otimes J\rangle &= tr(-(H \otimes J)\gamma) = i\, tr((H \otimes I)(I \otimes J)i\gamma) \\ &= i\, tr(H\, tr_W(i\gamma(I \otimes J))) = i\langle C(\gamma), H\rangle .\end{aligned}$$

Thus

$$\langle \gamma, H \otimes J\rangle = i\langle C(\gamma), H\rangle \qquad (H \in \operatorname{Her}(W, W)) . \tag{17}$$

Set

$$Z = \{\beta \in S^A;\ C(\beta) = 0\} . \tag{18}$$

Then by (17), $\beta \in S^A$ is in Z if and only if $\langle \beta, H \otimes J\rangle = 0$ for all $H \in \operatorname{Her}(W, W)$. Hence

$$S^A = Z \oplus (\operatorname{Her}(W, W) \otimes J) , \tag{19}$$

where $\oplus$ indicates an orthogonal decomposition.

Lemma 3.6. *Set* $G = \{\gamma - \gamma^\tau;\ \gamma \in \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g)\}$. *Then*

$$G = \rho_G Z \oplus (G \cap (\operatorname{Her}(W, W) \otimes J)) .$$

Proof. By (19), Z is orthogonal to $\operatorname{Her}(W, W) \otimes J$, so that $\rho_G Z$ is orthogonal to $G \cap (\operatorname{Her}(W, W) \otimes J)$. Let $v \in G$ be orthogonal to $\rho_G Z$. Then v is orthogonal to Z. Since v is in S^A, it is in $\operatorname{Her}(W \otimes W) \otimes J$ by (19) and hence in $G \cap (\operatorname{Her}(W, W) \otimes J)$. q.e.d.

Proposition 3.2. *Assume that not all* c^0_{st} *are zero, and define* $\mathscr{L} \subseteq \operatorname{Her}(W, W)$ *by*

$$\mathscr{L} \otimes J = \{\gamma - \gamma^\tau;\ \gamma \in \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g)\} \cap (\operatorname{Her}(W, W) \otimes J) .$$

Then conditions (i), $(\mathrm{ii})_1$, $(\mathrm{iii})_1$ *in Lemma* 3.2 *are satisfied if and only if there is a positive definite hermitian form on W orthogonal to* $\mathscr{L}$.

Proof. Assume that $\beta \in S^A$ satisfies conditions 1) and 2) in Proposition 3.1. By 1), $\langle \beta, \gamma^\tau\rangle = \langle \beta^\tau, \gamma\rangle = -\langle \beta, \gamma\rangle = -tr\, \gamma\beta = 0$ for all $\gamma \in \operatorname{Her}(\operatorname{Ker} g, \operatorname{Ker} g)$, so that β is orthogonal to G. Thus for $H \in \mathscr{L}, \langle C(\beta), H\rangle = -i\langle\beta, H \otimes J\rangle = 0$ (cf. (17)). Hence $C(\beta) = tr_W(i\beta \circ (I \otimes J))$ is orthogonal to $\mathscr{L}$ and is positive definite by 2). Conversely, assume that h is a positive definite hermitian form on W and is orthogonal to $\mathscr{L}$. Take $\beta_1 \in S^A$ such that $C(\beta_1) = h$. Then $\langle\beta_1, \mathscr{L} \otimes J\rangle = i\langle C(\beta_1), \mathscr{L}\rangle = 0$, so that β_1 is orthogonal to $G \cap (\mathrm{Her}\,(W, W) \otimes J)$. Thus by Lemma 3.5, $\beta_1 \in G^\perp + \rho_G Z$, where $G^\perp$ is the orthogonal complement of G in S^A. Since $\rho_G Z \subset Z + G^\perp$, it follows that $\beta_1 \in G^\perp + Z$. Write $\beta_1 = \beta + \zeta$, where $\beta \in G^\perp$ and $\zeta \in Z$. Then $C(\beta) = C(\beta_1) - C(\zeta) = C(\beta_1) = h$. Thus $C(\beta) > 0$. Since $\beta \in G^\perp$, for any $\gamma \in \mathrm{Her}\,(\mathrm{Ker}\, g, \mathrm{Ker}\, g)$ we have $\langle\beta, \gamma\rangle = \frac{1}{2}\langle\beta, \gamma - \gamma^\tau\rangle = 0$. Thus $tr\,\beta\gamma = 0$ for all $\gamma \in \mathrm{Her}\,(\mathrm{Ker}\, g, \mathrm{Ker}\, g)$, and hence β satisfies condition 2) in Proposition 3.1. q.e.d.

By considering the conditions in Lemma 3.2 for $d = 2$, we obtain a more general condition for half-estimate. We can write down these conditions parallel to Proposition 3.1 as follows:

Proposition 3.3. *Conditions* (i,) (ii)$_2$, (iii)$_2$ *in Lemma* 3.2 *are satisfied if and only if there is* $\beta \in S^A$ *such that*

1) $\rho_K\beta\rho_K \geq 0$ *where* $K = \ker g$,
2) $tr_W(i\beta(I \otimes J)) > 0$.

Proof. Assume that g_θ satisfies (i), (ii)$_2$, and (iii)$_3$, Write

$$g_\theta = g + \theta r + \theta^2 q \qquad (\mathrm{mod.}\ \theta^3)\ .$$

Then (ii)$_2$ and (iii)$_2$ are equivalent to

$$g^*r + r^*g + (g^*r + r^*g)^\tau = 0\ , \tag{20}$$

$$g^*q + q^*g + r^*r + (g^*q + q^*g + r^*r)^\tau = 0\ , \tag{21}$$

$$\langle i\, tr_W(\theta(g^*r + r^*g) + \theta^2(g^*q + q^*g + r^*r))(I \otimes J)u, u\rangle \geq c\theta^2|u|^2 \tag{22}$$

for all sufficiently small θ and all $u \in W$. Set

$$H_1 = i\, tr_W((g^*r + r^*g)(I \otimes J)), H = i\, tr_W((g^*q + q^*g + r^*r)(I \otimes J))\ .$$

(22) implies that for a sufficiently large real number a, $aH_1 + H_2 > 0$. Set

$$f = q + ar\ , \qquad \beta = g^*f + f^*g + r^*r\ ,$$

Then $tr_W(i\beta(I \otimes J)) > 0$ by (22), and $\beta \in S^A$ by (20) and (21). Moreover, $\rho_K\beta\rho_K = \rho_K r^*r\rho_K \geq 0$. Thus β satisfies our conditions. Conversely, assume that there is $\beta \in S^A$ satisfying our conditions. Write $\rho_K\beta\rho_K = r^*r$, where $r \in \mathrm{Hom}\,(\mathrm{Ker}\, g, \mathrm{Ker}\, g)$. Then $\rho_K(\beta - r^*r)\rho_K = 0$, and therefore there is $q \in \mathrm{Hom}\,(W \otimes \boldsymbol{R}^{2n-2}, W \otimes \boldsymbol{R}^{2n-2})$ such that

$$\beta - r^*r = g^*q + q^*g \; .$$

Noting $r^*g = g^*r = 0$ since $\operatorname{Im} g \perp \operatorname{Ker} g$, we see easily that $g_\theta = g + \theta r + \theta^2 q$ satisfies our requirements (20), (21) and (22).

Appendix

Proof of Theorem 1.4. Set $\langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. Expanding $J(x, \xi + z\langle\xi\rangle^{\frac{1}{2}})$ in Taylor's series in z, multiplying by $bt\, g(z)^2$, integrating over $\boldsymbol{R}^n$ in z, and noting that $g(z)$ is an even function, we find that

$$J(x, \xi) = J_1(x, \xi) - (\partial^2 J(x, \xi)/\partial\xi_j\partial\xi_k)a_{jk}\langle\xi\rangle + R(x, \xi) \; , \tag{13}$$

where

$$J_1(x, \xi) = \int J(x, \xi + z\langle\xi\rangle^{\frac{1}{2}})g(z)^2 dz \; , \tag{14}$$

and $R(x, \xi)$ is of order $l - 2$. Set

$$\gamma(\chi, \xi) = \hat{J}(\chi, \xi) \; , \qquad \gamma_1(\chi, \xi) = \hat{J}_1(\chi, \xi) \; , \tag{15}$$

where $\wedge$ indicates Fourier transform in the space variables. Then by applying a change of variables $z = \langle\xi\rangle^{-\frac{1}{2}}(\zeta - \xi)$ to (13),

$$\gamma_1(\chi, \xi) = \int \gamma(\chi, \xi) g(\langle\xi\rangle^{-\frac{1}{2}}(\zeta - \xi))^2 \langle\xi\rangle^{-n/2} d\zeta \; . \tag{16}$$

In view of (13) we are interested in estimating $\int \langle\gamma_1(\chi - \xi)\hat{u}(\xi), \hat{u}(\chi)\rangle d\xi d\chi$ from below. However, instead of γ_1 we first consider

$$\begin{aligned} \gamma_2(\chi, \xi) &= \int \gamma(\chi, \zeta) g(\langle\xi + \chi\rangle^{-\frac{1}{2}}(\zeta - \xi - \chi))\langle\xi + \chi\rangle^{-n/4} \\ &\qquad \cdot g(\langle\xi\rangle^{-\frac{1}{2}}(\zeta - \xi))\langle\xi\rangle^{-n/4} d\zeta \\ &= \int \gamma(\chi, \xi + z\langle\xi\rangle^{\frac{1}{2}}) g(\langle\xi + \chi\rangle^{-\frac{1}{2}}(z\langle\xi\rangle^{\frac{1}{2}} - \chi)) \\ &\qquad \cdot \langle\xi + \chi\rangle^{-n/4} g(z)\langle\xi\rangle^{n/4} dz \; , \end{aligned} \tag{17}$$

and then study the difference $\gamma_1 - \gamma_2$.

From the first defining formula of $\gamma_2(\chi, \xi)$, it follows that

$$\int \langle\gamma_2(\chi - \xi, \xi)\hat{u}(\xi), \hat{u}(\chi)\rangle d\xi d\chi = \int \langle J(x, \zeta)u_\zeta(x), u_\zeta(x)\rangle dx d\zeta \; , \tag{18}$$

where $\hat{u}_\zeta(\xi) = g(\langle\xi\rangle^{-\frac{1}{2}}(\zeta - \xi))\langle\xi\rangle^{-n/4}\hat{u}(\xi)$. Therefore by our assumption,

$$\mathcal{R}\int\langle\gamma_2(\chi-\xi,\xi)\hat{u}(\xi),\hat{u}(\chi)\rangle d\xi d\chi\geq 0\ . \tag{19}$$

By (2) and the second defining formula of $\gamma_2(\chi,\xi)$ in (17),

$$\begin{aligned}\gamma_1(\chi,\xi)-\gamma_2(\chi,\xi)&=\int\gamma(\chi,\xi+z\langle\xi\rangle^{\frac{1}{2}})\\&\cdot\{g(\langle\xi+\chi\rangle^{-\frac{1}{2}}(z\langle\xi\rangle^{\frac{1}{2}}-\chi))\langle\xi+\chi\rangle^{-n/4}\langle\xi\rangle^{n/4}-g(z)\}g(z)dz\ .\end{aligned}\tag{20}$$

Note that

$$\begin{aligned}\langle\xi+\chi\rangle^{-a}\langle\xi\rangle^{a}&=1-a\langle\xi\rangle^{-2}(\xi,\chi)+R_a(\chi,\xi)\ ,\\|R_a(\chi,\xi)|&\leq C_a\langle\xi\rangle^{-2}\langle\chi\rangle^{|a+2|+4}\ ,\end{aligned}$$

where (ξ,χ) is the inner product of ξ and χ. Therefore

$$\begin{aligned}&g(\langle\xi+\chi\rangle^{-\frac{1}{2}}(z\langle\xi\rangle^{\frac{1}{2}}-\chi))\langle\xi+\chi\rangle^{-n/4}\langle\xi\rangle^{n/4}-g(z)\\&\quad=-(n/4)\langle\xi\rangle^{-2}(\xi,\chi)g(z)-(\tfrac{1}{2}\langle\xi\rangle^{-2}(\xi,\chi)z_j+\langle\xi\rangle^{-\frac{1}{2}}\chi_j)\partial g/\partial z_j\\&\quad\quad+\tfrac{1}{2}\langle\xi\rangle^{-1}\chi_j\chi_k\partial^2 g/\partial z_j\partial z_k+S_1(\chi,\xi,z)\ ,\\&\quad\quad|S_1(\chi,\xi,z)|\leq C\langle\xi\rangle^{-3/2}\langle\chi\rangle^k\langle z\rangle^k\end{aligned}$$

for a sufficiently large k. Since

$$\begin{aligned}\gamma(\chi,\xi+z\langle\xi\rangle^{\frac{1}{2}})&=\gamma(\chi,\xi)+z\langle\xi\rangle^{\frac{1}{2}}\partial\gamma(\chi,\xi)/\partial\xi_\nu+S_2(\chi,\xi,z)\ ,\\|S_2(\chi,\xi,z)|&\leq C_N\langle\xi\rangle^{-1}\langle z\rangle^k\langle\chi\rangle^{-N}\ ,\end{aligned}$$

it follows then by (20) that

$$\begin{aligned}\gamma_1(\chi,\xi)-\gamma_2(\chi,\xi)&=\tfrac{1}{2}\gamma(\chi,\xi)\langle\xi\rangle^{-1}\chi_j\chi_k\int(\partial^2 g/\partial z_j\partial z_k)g(z)dz\\&\quad-\gamma(\chi,\xi)\langle\xi\rangle^{-2}(\xi,\chi)\left\{(n/4)+\tfrac{1}{2}\int z_j g(z)(\partial g/\partial z_j)dz\right\}\\&\quad-\chi_j(\partial\gamma(\chi,\xi)/\partial\xi_k)\int z_k g(z)(\partial g/\partial z_j)dz+S(\chi,\xi)\\&=\tfrac{1}{2}\gamma(\chi,\xi)\langle\xi\rangle^{-1}\chi_j\chi_k b_{jk}+\hat{T}(\chi,\xi)+S(\chi,\xi)\ ,\end{aligned}$$

where $|S(\chi,\xi)|\leq C_N\langle\xi\rangle^{l-3/2}\langle\chi\rangle^{-N}$, $T(x,\xi)$ is of order $l-1$, and

$$\mathcal{R}\langle T(x,\xi)u,u\rangle=0 \tag{21}$$

for all u. The above equations together with (13), (18) and (19) therefore give

$$\mathcal{R}\langle J(x,D)u,u\rangle\geq-\mathcal{R}\langle L(x,D)u,u\rangle+\mathcal{R}\langle T(x,D)u,u\rangle+\mathcal{R}\langle S(x,D)u,u\rangle\ .$$

Since $\mathscr{R}\langle T(x, D)u, u\rangle = 0$ we can apply our argument to $\langle T(x, D)u, u\rangle$. Note that we have not used the hermitian assumption of $J(x, \xi)$ except for getting (21). Since $T(x, \xi)$ is of order $l - 1$ we find that $|\mathscr{R}\langle T(x, D)u, u\rangle| \geq +\mathscr{R}\langle S'(x, D)u, u\rangle$ where $S'(x, \xi)$ is of order $\leq l - 2$. This completes the proof of our theorem.

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