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The Number of Cusps of Right-angled Polyhedra in Hyperbolic Spaces

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Abstract. As was pointed out by Nikulin [8] and Vinberg [10], a right-angled polyhedron of finite volume in the hyperbolic n-space $\mathbf{H}^n$ has at least one cusp for $n \geq 5$. We obtain non-trivial lower bounds on the number of cusps of such polyhedra. For example, right-angled polyhedra of finite volume must have at least three cusps for $n = 6$. Our theorem also says that the higher the dimension of a right-angled polyhedron becomes, the more cusps it must have.

1. Introduction

Let P be a convex polyhedron in the hyperbolic n-space $\mathbf{H}^n$ with dihedral angles of the form π/m ($m \in \mathbf{N}$) at all its $(n-2)$-dimensional faces. We call this polyhedron a *Coxeter polyhedron*. In particular, a polyhedron is called *right-angled* if all its dihedral angles are $\pi/2$.

We call a polyhedron *acute-angled* if all its dihedral angles do not exceed $\pi/2$. It is known that any k-dimensional face of an acute-angled polyhedron $P \subset \mathbf{H}^n$ belongs only to $(n-k)$ hyperfaces; any k-dimensional face is represented by the intersection of $(n-k)$ hyperfaces (see [1]). In particular, any ordinary vertex belongs only to n hyperfaces. If P is a right-angled polyhedron, then the number of hyperfaces of P which share one cusp is exactly $2(n-1)$. Any of these hyperfaces is parallel to one other and adjacent to the remaining $2(n-2)$ hyperfaces.

An n-dimensional combinatorial polytope is called *simple* if any of its vertices belongs only to n hyperfaces, and *simple at edges* if any of its edges belongs only to $(n-1)$ hyperfaces. In addition, we call an n-dimensional hyperbolic polytope *almost simple* if it is simple at edges and any of its vertices not at infinity belongs only to n hyperfaces. According to the above, any compact acute-angled polyhedron in $\mathbf{H}^n$ is simple, and any acute-angled polyhedron of finite volume with vertices at infinity added is simple at edges. In particular, any right-angled polyhedron of finite volume in $\mathbf{H}^n$ is almost simple.

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Vinberg proved that there are no compact right-angled polyhedra in $\mathbf{H}^n$ for $n > 4$ (see [10]). On the other hand, Dufour proved the following theorem.

THEOREM 1.1 ([3]). *Right-angled polyhedra of finite volume may exist in* $\mathbf{H}^n$ *only if* $n < 13$.

REMARK 1. Before Dufour proved Theorem 1.1, Potyagailo and Vinberg had already shown the nonexistence of right-angled polyhedra of finite volume in $\mathbf{H}^n$ for $n > 14$ in [9].

These results suggest that when the dimension of a right-angled polyhedron in the hyperbolic space becomes higher, then this polyhedron becomes far from compact. Therefore we may expect that the higher the dimension of a right-angled polyhedron is, the more cusps it has for $4 < n < 13$. Let Q^n be a right-angled polyhedron of finite volume in $\mathbf{H}^n$. Denote the number of cusps of Q^n by $c(Q^n)$. Our main theorem shows that this expectation is actually true.

MAIN THEOREM 1.2. *For* Q^n, *a right-angled polyhedron of finite volume in* $\mathbf{H}^n$, *we have the following lower bounds on the number of cusps* $c(Q^n)$:

$$c(Q^6) \geq 3,\ c(Q^7) \geq 17,\ c(Q^8) \geq 36,\ c(Q^9) \geq 91\,,$$
$$c(Q^{10}) \geq 254,\ c(Q^{11}) \geq 741,\ c(Q^{12}) \geq 2200\,.$$

Potyagailo and Vinberg [9] found some examples of right-angled polyhedra of finite volume in $\mathbf{H}^n$ for $n \leq 8$. However, we do not know whether there exist right-angled polyhedra of finite volume in $\mathbf{H}^n$ for $9 \leq n \leq 12$. According to [4], there is no simple ideal Coxeter polyhedron in $\mathbf{H}^n$ with $n \geq 8$. Furthermore in [6], it was shown that there is no right-angled polyhedron in $\mathbf{H}^n$ with $n \geq 7$.

The key element of the proof of the main theorem is a lower bound on the number of 2-dimensional faces of Q^3, which we prove in Section 3. Before doing so, we introduce some known results on right-angled polyhedra in $\mathbf{H}^n$.

2. Right-angled polyhedra in $\mathbf{H}^n$

Let us denote by $\mathbf{H}^n$ the hyperbolic n-space. There are some ways to describe it. In what follows, we use two of them that we now explain below. Denote the unit open ball by

$$B^n := \{x \in \mathbf{R}^n \mid\, | x |< 1\}\,.$$

The metric space consisting of B^n equipped with a Riemannian metric of the form

$$\left(\frac{2}{1- | x |^2}\right)^2 \sum_{i=1}^{n} dx_i^2$$

is called *the conformal ball model* of $\mathbf{H}^n$. Denote the upper half-space by

$$U^n := \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_n > 0\}\,.$$

The metric space consisting of U^n together with the metric

$$\frac{1}{x_n^2}\sum_{i=1}^{n} dx_i^2$$

is called *the upper half-model* of $\mathbf{H}^n$.

Let P^n be a Coxeter polyhedron of finite volume in the hyperbolic n-space $\mathbf{H}^n$. The boundary of $\mathbf{H}^n$ is the $(n-1)$-sphere $\mathbf{S}^{n-1}$ when we consider the conformal ball model. Because the volume of P^n is finite, the intersection of the Euclidean closure of P^n and $\mathbf{S}^{n-1}$ is a finite set of points. Here, the Euclidean closure of a set $A \subset \mathbf{H}^n$ means the closure of A in $\mathbf{R}^n$ contained in B^n. A cusp point (or a point at infinity, an ideal vertex) of P^n is a point c of $\overline{P^n} \cap \mathbf{S}^{n-1}$. We say that P^n *has a cusp* when there is a cusp point of P^n. Denote the set of k-dimensional faces of P^n by $\Omega_k(P^n)$. The number of k-dimensional faces of P^n is denoted by $a_k(P^n)$, and the number of cusps of P^n is denoted by $c(P^n)$. The expression "the faces $F_1, \ldots, F_k$ intersect" means that the faces $F_1, \ldots, F_k$ have a common point. We say these faces *share a cusp or have a common cusp* when their Euclidean closures have a common point at infinity.

Two hyperplanes of $\mathbf{H}^n$ are called *parallel* if they do not intersect. We call that two hyperfaces F_1 and F_2 of P^n are *parallel* if the hyperplanes containing them are parallel and their Euclidean closures intersect in the boundary of $\mathbf{H}^n$. If two hyperfaces are parallel, then the intersection of their Euclidean closures is exactly one cusp of P^n.

Let Q^n be a right-angled polyhedron of finite volume in $\mathbf{H}^n$. A right-angled polyhedron has some good properties stated as (P1), (P2) and (P3) below.

(P1) Any face of Q^n is also a right-angled polyhedron.

(P2) For any hyperface of Q^n passing through a cusp c, there is a unique other parallel hyperface sharing same cusp.

(P3) The number of hyperfaces of Q^n sharing a cusp of Q^n is exactly $2(n-1)$.

These properties follow from the local combinatorial structure of right-angled polyhedra in $\mathbf{H}^n$. In what follows, Q^n always denotes a right-angled polyhedron of finite volume in $\mathbf{H}^n$.

We denote by $\langle F\rangle$ the Euclidean closure of the hyperplane containing a hyperface F of Q^n.

PROPOSITION 2.1. ([9, p.7]) *Let $F_1, F_2, \ldots$ be hyperfaces of a right-angled polyhedron. Then*

(a) *if $\langle F_1\rangle$ and $\langle F_2\rangle$ intersect, then F_1 and F_2 intersect;*

(b) *if F_1, F_2, F_3 are pairwise mutually adjacent, then they meet at an $(n-3)$-dimensional face;*

(c) *if F_1 and F_2 are parallel and F_3 is adjacent to them, then F_1, F_2 and F_3 meet at a cusp;*

(d) *if F_1 and F_2 are parallel and F_3 and F_4 are adjacent to them, then F_1, F_2, F_3 and F_4 meet at a cusp.*

Potyagailo and Vinberg proved certain inequalities describing the relations between the number of cusps and faces of a right-angled polyhedron in $\mathbf{H}^n$.

PROPOSITION 2.2 ([9, p.7, 8]). *Let Q^n be a right-angled polyhedron of finite volume in $\mathbf{H}^n$. Then the following inequalities hold*:

$$a_1(Q^2) + c(Q^2) \geq 5\,, \qquad a_2(Q^3) \geq 6\,, \qquad a_2(Q^3) + 2c(Q^3) \geq 12\,.$$

The first inequality follows from the following obvious lemma.

LEMMA 2.3. *If Q^2 is compact, then Q^2 has more than four edges.*

On the other hand, Nikulin considered the average number of k-dimensional faces in l-dimensional faces of an acute-angled polyhedron P in $\mathbf{H}^n$, denoted by $a_k^l(P)$:

$$a_k^l(P) = \frac{1}{a_k(P)} \sum_{F \in \Omega^k(P)} a_l(F)\,.$$

One of the main ingredients for proving our main theorem is the following Nikulin's inequality:

THEOREM 2.4. *Let P be an acute-angled polyhedron of finite volume in $\mathbf{H}^n$. Then*

$$a_k^l(P) < \binom{n-l}{n-k} \frac{\binom{[\frac{n}{2}]}{l} + \binom{[\frac{n+1}{2}]}{l}}{\binom{[\frac{n}{2}]}{k} + \binom{[\frac{n+1}{2}]}{k}}$$

holds for $l < k \leq [\frac{n}{2}]$.

This inequality is proved by Nikulin [8] for simple convex polyhedra and a generalization is due to Khovanskij [5] for polyhedra simple at edges.

By Nikulin's inequality and Lemma 2.3, we obtain the following proposition, as explained in [10].

PROPOSITION 2.5. *There are no compact right-angled polyhedra in $\mathbf{H}^n$ for $n > 4$.*

This proposition shows that Q^n must have at least one cusp for $n \geq 5$.

Fix a cusp c. Denote the set of all k-dimensional faces which contain c by $\Omega_k^c(Q^n)$. As we have mentioned in the introduction, the number of elements of this set is $2(n-1)$. In the next section, we focus on the case $n = 3$.

3. A lower bound on the number of 2-dimensional faces of Q^3

In this section, we consider a right-angled polyhedron Q^3 in the upper half-model. In this model, the hyperplane must be a vertical Euclidean plane or an upper hemisphere which intersects the boundary of the upper half-space orthogonally. We assume that Q^3 has a cusp

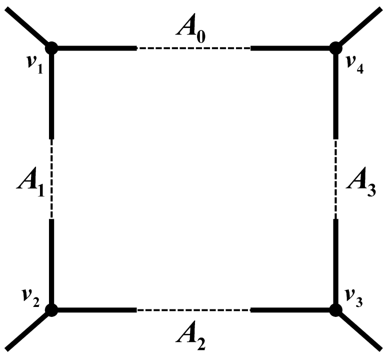

FIGURE 1. Combinatorial structure of Q^3

c. And we also assume that c is the point at infinity of the upper half-space. The set $\Omega_2^c(Q^3)$ has exactly four elements: A_0, A_1, A_2, A_3. In this case, the 2-dimensional faces A_0, A_1, A_2 and A_3 are parts of vertical Euclidean planes. We assume that A_0 and A_2 are parallel, and that A_1 and A_3 are parallel.

The aim of this section is to prove the following lemma.

LEMMA 3.1. *If Q^3 has only one cusp, then Q^3 has at least twelve 2-dimensional faces. Moreover, if Q^3 has exactly twelve 2-dimensional faces, then, after replacing A_0 with A_1 and A_2 with A_3, if necessary, A_0 and A_2 are quadrangles, and A_1 and A_3 are pentagons.*

Assume that $c(Q^3) = 1$, and denote this cusp by c. There are exactly four 2-dimensional faces which share c. Denote these 2-dimensional faces by A_0, A_1, A_2, A_3 as before. Because the faces A_0 and A_1 are adjacent, they have a common edge. This edge starts from a vertex and terminates at c. We denote this vertex by v_1. In the same manner, we denote by v_2 (resp. v_3, v_4) the vertex belonging to A_1 and A_2 (resp. A_2 and A_3, A_3 and A_0) which is an endpoint of an edge terminating at c. Then the local combinatorial structure of Q^3 around c can be depicted as in Fig. 1.

Each vertex v_1, v_2, v_3 and v_4 belongs to two non-compact 2-dimensional faces and one compact 2-dimensional face of Q^3 because Q^3 is almost simple and has only one cusp.

Since Q^3 is a right-angled polyhedron in $\mathbf{H}^3$, it must satisfy the following five conditions:

(1) Any compact 2-dimensional face must have more than four edges.

(2) If two 2-dimensional faces are adjacent, then they have only one common edge.

(3) Any vertex must be shared by exactly three edges.

(4) There are no three 2-dimensional faces which are pairwise adjacent but do not share a vertex.

(5) There are no four 2-dimensional faces which are cyclically adjacent except for A_0, A_1, A_2 and A_3.

Condition (1) comes from Lemma 2.3, and Conditions (2–3) can be seen from the local combinatorial structure of a right-angled polyhedron in $\mathbf{H}^3$. We use the following theorem due to Andreev to obtain Conditions (4–5).

THEOREM 3.2 ([2]). *An acute-angled almost simple polyhedron of finite volume with given dihedral angles, other than a tetrahedron or a triangular prism, exists in* $\mathbf{H}^3$ *if and only if the following conditions are satisfied*:

(a) *if three* 2*-dimensional faces meet at a vertex or a cusp, then the sum of the dihedral angles between them is at least* π (π *for a cusp*);

(b) *if four* 2*-dimensional faces meet at a vertex or a cusp, then all the dihedral angles between them equal* $\frac{\pi}{2}$;

(c) *if three* 2*-dimensional faces are pairwise adjacent but share neither a vertex nor a cusp, then the sum of the dihedral angles between them is less than* π;

(d) *if a* 2*-dimensional face* F_i *is adjacent to* 2*-dimensional faces* F_j *and* F_k, *while* F_j *and* F_k *are not adjacent but have a common cusp which* F_i *does not share, then at least one of the angles formed by* F_i *with* F_j *and with* F_k *is different from* $\frac{\pi}{2}$;

(e) *if four* 2*-dimensional faces are cyclically adjacent but meet at neither a vertex nor a cusp, then at least one of the dihedral angles between them is different from* $\frac{\pi}{2}$.

Since all the dihedral angles of Q^3 are $\frac{\pi}{2}$, by Theorem 3.2 (c), we obtain Condition (4). Moreover, by Theorem 3.2 (b) and (e), we obtain Condition (5).

Before proving Lemma 3.1, we have to prove some sublemmas. Now we prepare suitable notation which will be used in their proofs. Let B_i be a compact 2-dimensional face which is adjacent to A_i and A_{i+1} (integer i can be 0, 1, 2 or 3, and when $i = 3$, we assume that $A_{i+1} = A_0$). One of the endpoints of the edge $A_i \cap A_{i+1}$ is a cusp, and the other is a vertex. Thus any 2-dimensional face which is adjacent to both A_i and A_{i+1} must have this vertex. But, since Q^3 is almost simple, this vertex is shared only by three 2-dimensional faces. That is to say, B_i is the only 2-dimensional face which is adjacent to both A_i and A_{i+1} for $i = 0, 1, 2, 3$. Thus we obtain the following sublemma.

SUBLEMMA 3.3. *No compact* 2*-dimensional face of* Q^3 *is adjacent to both* A_i *and* A_{i+1} *other than* B_i.

On the other hand, by the following sublemma, we know that there is no compact 2-dimensional face of Q^3 which is adjacent to both A_i and A_{i+2} for $i = 0, 1$.

SUBLEMMA 3.4. *Assume that a right-angled polyhedra* Q^3 *has a cusp* c. *Let* A *and* A' *be* 2*-dimensional faces of* Q^3 *which are parallel but sharing this cusp* c. *Then any* 2*-dimensional face which is adjacent to both* A *and* A' *must also have the cusp* c.

PROOF. Since Q^3 is a right-angled polyhedron in $\mathbf{H}^3$, the number of 2-dimensional faces of Q^3 which have c is 4. We denote these faces by A, A', A'' and A'''. We may assume that A'' and A''' (resp. A and A') are parallel. We may also assume that the cusp c

of Q^3 is identified with the point at infinity of the upper half-space. In this case, each of the hyperplanes $\langle A\rangle$, $\langle A'\rangle$, $\langle A''\rangle$ and $\langle A'''\rangle$ has to be a vertical Euclidean plane which intersects the boundary of $\mathbf{H}^3$ orthogonally.

We suppose that there is a 2-dimensional face which is adjacent to both A and A' other than A'' and A'''. We denote this face by B. Since B does not have the cusp c, the hyperplane $\langle B\rangle$ must be an upper hemisphere which orthogonally intersects with $\langle A\rangle$ and $\langle A'\rangle$. Thus both $\langle A\rangle$ and $\langle A'\rangle$ share the north pole of $\langle B\rangle$ in $\mathbf{H}^3$. But A and A' are parallel. Thus $\langle A\rangle$ does not intersect $\langle A'\rangle$ in $\mathbf{H}^3$. Thus there are no 2-dimensional faces which are adjacent to both A and A' other than A'' and A'''. By the same reason, there are no 2-dimensional faces which are adjacent to both A'' and A''' other than A and A'. □

By this sublemma, we know that B_0, B_1, B_2 and B_3 are different. For example, if B_0 and B_1 are the same face, then this face is adjacent to both A_0 and A_2, which contradicts Sublemma 3.4.

From the above, Q^3 has at least eight 2-dimensional faces. Suppose that A_0, A_1, A_2 and A_3 have m vertices in total. Then, other than the edges terminating at the cusp c, faces A_0, A_1, A_2 and A_3 have m edges in total. Since each B_i $(i = 0, 1, 2, 3)$ is adjacent to two of A_0, A_1, A_2 and A_3, there are eight edges shared by B_i $(i = 0, 1, 2, 3)$ and one of A_j $(j = 0, 1, 2, 3)$. Then there are $(m - 8)$ edges left when $m > 8$. Take one of these edges, say e, and fix it. We may assume that e belongs to A_0. Since any pair of faces can share only one edge, e belongs to neither B_0 nor B_3. Therefore there is a new face B_4. Because of Sublemma 3.4, B_4 cannot be adjacent to A_2 which is parallel to A_0. On the other hand, Sublemma 3.3 tells us that B_4 cannot be adjacent to A_1 (resp. A_3), since B_4 is different from B_0 (resp. B_3) by our choice of e. Moreover, by Condition (2), B_4 has a unique common edge with A_0. That is, if there is a 2-dimensional face which is adjacent to A_0 but does not have the edge e, then this face is not B_4. By this observation, Sublemmata 3.3–3.4 and Condition (2), the new compact 2-dimensional faces which are adjacent to each of the $(m - 8)$ edges above are different. Therefore, the number of faces in Q^3 must be greater than or equal to m, the number of vertices of the faces A_0, A_1, A_2 and A_3. The configuration of 2-dimensional faces for the case $m = 9$ is shown in Fig. 2.

We add one more sublemma.

SUBLEMMA 3.5. *The compact 2-dimensional face B_0 (resp. B_1) is not adjacent to B_2 (resp. B_3).*

PROOF. Suppose that B_0 (resp. B_1) is adjacent to B_2 (resp. B_3), then the four 2-dimensional faces: A_0, B_0, B_2 and A_3 (resp. A_1, B_1, B_3 and A_0) are cyclically adjacent. These contradicts Condition (5). Hence B_0 (resp. B_1) is not adjacent to B_2 (resp. B_3). □

In what follows, we present a proof of Lemma 3.1; it is divided into several cases according to the number of vertices of A_0, A_1, A_2 and A_3.

3.1. The number of vertices of A_0, A_1, A_2 and A_3 is eight. In this case, A_0, A_1, A_2 and A_3 are quadrangles. Thus, any compact 2-dimensional face which is adjacent to A_i

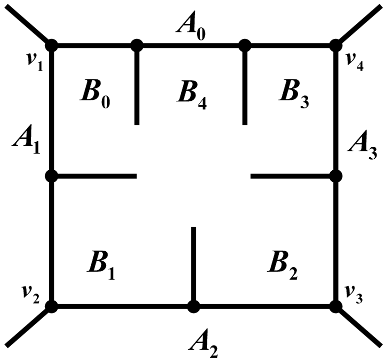

FIGURE 2. The number of vertices of A_0, A_1, A_2 and A_3 is nine.

$(i = 0, 1, 2, 3)$ must be B_0, B_1, B_2 or B_3. For example, the compact 2-dimensional faces which are adjacent to A_0 are only B_0 and B_3 since A_0 has exactly four edges. These four edges are $A_0 \cap A_1$, $A_0 \cap A_3$, $A_0 \cap B_0$ and $A_0 \cap B_3$. Since A_0 has one cusp and three vertices, the vertex of A_0 other than v_1 and v_4 must be $A_0 \cap B_0 \cap B_3$. Thus B_0 is adjacent to B_1. Similarly, we can show that B_1 (resp. B_2, B_3) is adjacent to B_2 (resp. B_3, B_0). On the other hand, Sublemma 3.5 says that B_0 (resp. B_1) is not adjacent to B_2 (resp. B_3). Thus B_0, B_1, B_2 and B_3 are cyclically adjacent. But this does not occur by Condition (5). That is, the present case is impossible.

3.2. The number of vertices of A_0, A_1, A_2 and A_3 is nine. In this case, we may assume that A_0 is a pentagon, and that A_1, A_2 and A_3 are quadrangles. We denote by B_4 the new compact 2-dimensional face which is adjacent to A_0 other than B_0 and B_3. Note that B_4 is adjacent to B_0 and B_3. If B_4 is adjacent to B_1 (resp. B_2), then A_0, B_4, B_1 (resp. B_2) and A_1 (resp. B_2) are cyclically adjacent. But this does not satisfy Condition (5). Thus B_4 is not adjacent to B_1 or B_2. Since B_4 has at least five edges, it is adjacent to at least two 2-dimensional faces besides A_0, B_0 and B_3. Hence Q^3 has at least eleven 2-dimensional faces.

Assume that Q^3 has exactly eleven 2-dimensional faces. In this case, B_4 is adjacent to exactly five 2-dimensional faces: A_0, B_0, B_3 and two other 2-dimensional faces which are compact. One of the last two compact 2-dimensional faces is adjacent to B_0, and the other is adjacent to B_3. We denote by F the first one, and denote by F' the second one; F (resp. F') is adjacent to both B_4 and B_0 (resp. B_4 and B_3). If B_0 is adjacent to F', then B_0, F' and B_4 are pairwise adjacent. Thus, in this case, these faces must share a vertex by Condition (4). But the endpoints of the edge $B_0 \cap B_4$ are $A_0 \cap B_0 \cap B_4$ and $B_0 \cap B_4 \cap F$. That is to say, B_0 is not adjacent to F'. Since A_0 has five edges, B_0 is not adjacent to B_3 by Condition (4). Since B_0 is adjacent to both A_0 and A_1, by Sublemma 3.4, it is adjacent to neither A_2 nor A_3. Moreover, by Sublemma 3.5, B_0 is not adjacent to B_2. Thus B_0 can be adjacent only to A_0,

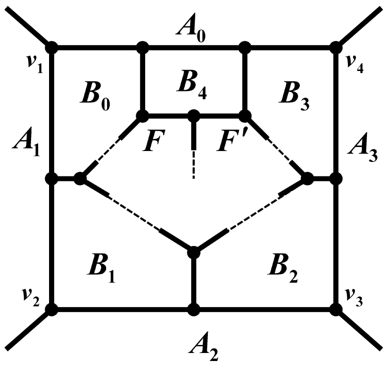

FIGURE 3. $a_2(Q^3) = 11$ in the case 3.2.

A_1, B_1, B_4 and F. On the other hand, by the same reason, B_3 is adjacent only to A_0, A_3, B_2, B_4 and F'. Thus F must be adjacent to B_0, B_1, B_2, B_4 and F' to satisfy Condition (1). But in this case F, B_2, B_3 and B_4 are cyclically adjacent. This is a contradiction to Condition (5). Hence Q^3 has more than eleven 2-dimensional faces.

Now we assume that Q^3 has exactly twelve 2-dimensional faces. Then there are exactly eight compact 2-dimensional faces of Q^3. We denote by F, F' and F'' the compact 2-dimensional faces which are different from B_i $(i = 0, 1, 2, 3, 4)$. We can choose F and F' as above. If F is adjacent to B_2, then F, B_2, B_3 and B_4 are cyclically adjacent. And if F is adjacent to B_3, then F, B_3 and B_4 are pairwise adjacent but do not share a vertex because the endpoints of the edge $B_3 \cap B_4$ are $B_3 \cap B_4 \cap A_0$ and $B_3 \cap B_4 \cap F'$. Thus, by Conditions (4–5), F can be adjacent to neither B_2 nor B_3.

Suppose that B_0 is adjacent to F''. Now we consider the 2-dimensional faces which F'' can be adjacent to. If F is adjacent to B_1, then B_0, B_1 and F are pairwise adjacent. And then they must have a common vertex by Condition (4). It means that one of the endpoints of the edge $B_0 \cap F$ is $B_0 \cap B_1 \cap F$ and the other is $B_0 \cap B_4 \cap F$ by Condition (2). But in this case B_0 has only five edges: $A_0 \cap B_0$, $A_1 \cap B_0$, $B_0 \cap B_1$, $B_0 \cap B_4$, $B_0 \cap F$. That is, B_0 cannot be adjacent to F''. This contradicts our assumption. Thus F is not adjacent to B_1. Thus F can be adjacent only to B_0, B_4, F' and F''. But this violates Condition (1). In the end, B_0 cannot be adjacent to F''. Analogously, we can conclude that B_3 cannot be adjacent to F''. Thus F'' must be adjacent to B_1, B_2, B_4, F and F'. In this case, B_0, B_4, F'', B_1 are cyclically adjacent. This is a contradiction to Condition (5).

Hence, in this case, Q^3 must have more than twelve 2-dimensional faces.

3.3. The number of vertices of A_0, A_1, A_2 and A_3 is ten. Under this assumption, we may assume that one of the following three cases occurs:

(3.3.1) the face A_0 is a hexagon, and the faces A_1, A_2 and A_3 are quadrangles (Fig. 5),

(3.3.2) the faces A_0 and A_1 are pentagons, and the faces A_2 and A_3 are quadrangles

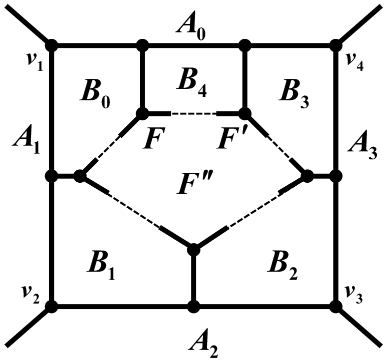

FIGURE 4. $a_2(Q^3) = 12$ in the case 3.2.

(Fig. 6),

(3.3.3) the faces A_0 and A_2 are pentagons, and the faces A_1 and A_3 are quadrangles (Fig. 7).

3.3.1. The face A_0 is a hexagon, and the faces A_1, A_2 and A_3 are quadrangles. We denoted by B_4 the compact 2-dimensional face which is adjacent to both A_0 and B_0. Let B_5 be the compact 2-dimensional face which is adjacent to A_0, B_3 and B_4. If B_4 is adjacent to B_1 (resp. B_2), then the four 2-dimensional faces A_0, A_1, B_1 and B_4 (resp. A_0, A_3, B_2 and B_4) are cyclically adjacent. But this contradicts Condition (5). That is to say, B_4 is adjacent to neither B_1, nor B_2. In addition, since A_0, B_3 and B_4 cannot be pairwise adjacent by Condition (4), B_4 is not adjacent to B_3. Thus there are at least two compact 2-dimensional faces which are adjacent to B_4, other than B_0 and B_5. We denote these faces by F and F'. We suppose that $a_2(Q^3) = 12$. Then the compact 2-dimensional faces are only B_i $(i = 0, 1, 2, 3, 4, 5)$, F and F'. Note that B_5 also has to be adjacent to F and F'. Thus F, F' and B_4 are pairwise adjacent, and F, F' and B_5 are also pairwise adjacent. This means that one of the endpoints of the edge $F \cap F'$ is in B_4, the other is in B_5. But in this case, three compact 2-dimensional faces B_4, B_5 and F (or F') must be pairwise adjacent but do not share a vertex. This is impossible by Condition (4). Hence $a_2(Q^3) > 12$.

3.3.2. The faces A_0 and A_1 are pentagons, and the faces A_2 and A_3 are quadrangles. Let B_6 be the compact 2-dimensional face which is adjacent to A_1, B_0 and B_1. Denote by F the compact 2-dimensional face which is adjacent to B_0 and B_4, and denote by F' the compact 2-dimensional face which is adjacent to B_3 and B_4. Suppose that Q^3 has exactly twelve 2-dimensional faces. By analogy to the case (3.3.1), we can show that B_4 and B_6 are adjacent to F and F'. But in this case there exist four 2-dimensional faces B_4, B_0, B_6 and F (or F'), which are cyclically adjacent. This is a contradiction to Condition (5). Hence Q^3 has more than twelve 2-dimensional faces.

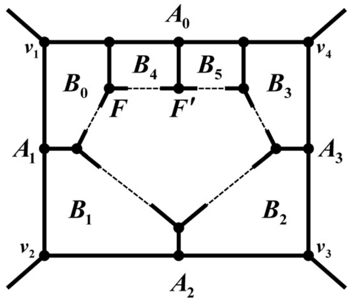

FIGURE 5. A_0 is a hexagon, and A_1, A_2 and A_3 are quadrangle.

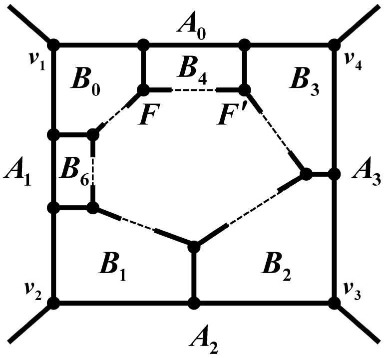

FIGURE 6. A_0 and A_1 are pentagons, and A_2 and A_3 are quadrangles.

3.3.3. The faces A_0 and A_2 are pentagons, and the faces A_1 and A_3 are quadrangles. Let B_7 be the compact 2-dimensional face which is adjacent to A_2, B_1 and B_2. The facets denoted by F and F' are as in the case (3.3.2). By Conditions (4–5), F is different from B_1, B_2, B_3, B_7 and F'. By the same reason, F' is different from B_0, B_1, B_2 and B_7. Thus Q^3 has at least twelve 2-dimensional faces. Note that if Q^3 has exactly twelve 2-dimensional faces, then the combinatorial structure of Q^3 is exactly as shown in Fig. 8. By Andreev's theorem (Theorem 3.2), it is easy to prove that there exists such a polyhedron in $\mathbf{H}^3$.

3.4. The number of vertices in A_0, A_1, A_2 and A_3 is eleven. In this case, there are seven compact 2-dimensional faces which are adjacent to either A_0, A_1, A_2 or A_3. And there are some other compact 2-dimensional faces which are adjacent to those compact faces. Thus Q^3 has at least twelve 2-dimensional faces. If Q^3 has exactly twelve 2-dimensional faces, then there is only one compact 2-dimensional face which is adjacent to the compact faces which are adjacent to either A_0, A_1, A_2 or A_3. But if there is only one such a face, then there

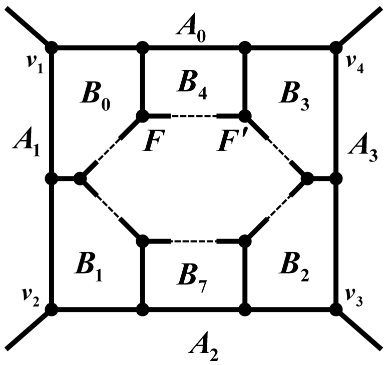

FIGURE 7. A_0 and A_2 are pentagons, and A_1 and A_3 are quadrangles.

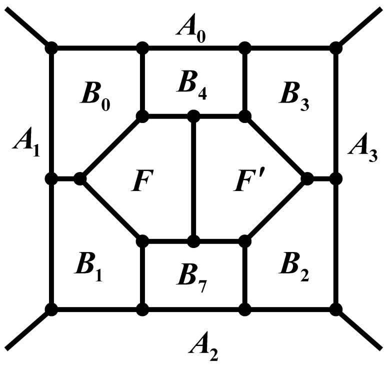

FIGURE 8. Q^3 which has exactly one cusp and twelve 2-dimensional faces.

must be some compact 2-dimensional faces which are adjacent to either A_0, A_1, A_2 or A_3, and which are either triangles or quadrangles. But they do not meet Condition (1). Thus Q^3 has strictly more than twelve 2-dimensional faces.

3.5. The number of vertices in A_0, A_1, A_2 and A_3 is twelve. In this case, there are eight compact 2-dimensional faces which are adjacent to either A_0, A_1, A_2 or A_3. And there are some other compact faces which are adjacent to those compact faces. Thus Q^3 has more than twelve 2-dimensional faces.

3.6. The number of vertices in A_0, A_1, A_2 and A_3 is greater than twelve. In this case, there are more than nine compact 2-dimensional faces which are adjacent to either A_0, A_1, A_2 or A_3. Thus Q^3 has more than twelve 2-dimensional faces.

3.7. Conclusion of the proof of Lemma 3.1 and a corollary. By the cases 3.1–3.6, we have proved Lemma 3.1.

By Proposition 2.2 and Lemma 3.1, we obtain the following corollary.

COROLLARY 3.6. *If* $c(Q^3) \leq 1$, *then* $a_2(Q^3) \geq 12$.

REMARK 2. If $c(Q^3) = 0$ and $a_2 Q^3) = 12$, then we have the compact right-angled dodecahedron. If $c(Q^3) = 1$ and $a_2(Q^3) = 12$, then we have a polyhedron in Fig. 8, which arises by contracting an edge of the dodecahedron above, as described in [7].

4. Proof of Main Theorem: $n = 6$

Assume that Q^6 has exactly one cusp. Then any 3-dimensional face of Q^6 has at most one cusp. Thus any 3-dimensional face of Q^6 has at least twelve 2-dimensional faces by Corollary 3.6. Thus we obtain $a_3^2(Q^6) \geq 12$. But by Nikulin's inequality (Theorem 2.4), we obtain the opposite inequality $a_3^2(Q^6) < 12$. Hence Q^6 has more than one cusp.

Now we assume that Q^6 has exactly two cusps. If each of the 3-dimensional faces of Q^6 has at most one cusp, then each 3-dimensional faces of Q^6 has more than twelve 2-dimensional faces. Thus we obtain $a_3^2(Q^6) \geq 12$. But by Theorem 2.4, we obtain the inequality $a_3^2(Q^6) < 12$. Hence there is a 3-dimensional face which has two cusps. Denote this face by G. We denote one cusp of G by c, and the other cusp by c'. There are three possibilities:

(4.1) there are no 2-dimensional faces of G which have two cusps,

(4.2) there is only one 2-dimensional face which has two cusps,

(4.3) there are two 2-dimensional faces which have two cusps.

Note that there is one edge which starts at c, and terminates at c' in the case (4.3).

Define the 2-dimensional faces A_0, A_1, A_2 and A_3 as before. The case (4.1) (resp. (4.2), (4.3)) is depicted in Fig. 9 (resp. 10, 11). A circle in these figures represents the cusp c'. From now on, we examine each of the above cases.

4.1. There are no 2-dimensional faces of G which have two cusps. The number of 2-dimensional faces of G sharing c is four, and the number of 2-dimensional face sharing c' is also four. There are no 2-dimensional faces which have both cusps. Thus $a_2(G) \geq 8$.

4.2. There is only one 2-dimensional face which has two cusps. We may assume that A_0 has two cusps. Let B_0 be the 2-dimensional face which is adjacent to A_0 and A_1, and let B_1 be the 2-dimensional face which is adjacent to A_1 and A_2. Let B_2 be the 2-dimensional face which is adjacent to A_2 and A_3, and let B_3 be the 2-dimensional face which is adjacent to A_3 and A_0. Because A_0 has c', there is one 2-dimensional face which is parallel to A_0 and sharing the cusp c' with A_0. We denote this face by B_4. Since both B_0 and B_3 are adjacent to A_0, they cannot coincide with B_4. If G has exactly eight 2-dimensional faces, then B_4 must coincide with either B_1 or B_2. Assume that B_4 is B_1. Because B_0 is adjacent to A_0 and A_1, it

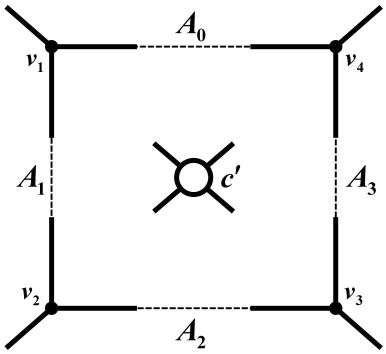

FIGURE 9. The case (4.1)

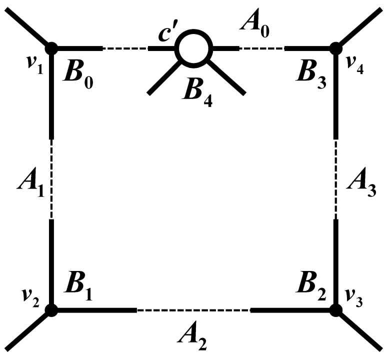

FIGURE 10. The case (4.2)

is adjacent to neither A_2 nor A_3. If B_0 is adjacent to B_2, then B_0, B_2, A_3 and A_0 are cyclically adjacent. But this does not occur because of Theorem 3.2 (e). Thus B_0 is not adjacent to B_2. Eventually, B_0 can be adjacent only to A_0, A_1 and B_4 in this case. That is to say, B_0 must be a triangle. But B_0 has more than three edges because of Proposition 2.2. This means that B_4 cannot coincide with B_1. By the same reason, B_4 cannot coincide with B_2. Hence $a_2(G) \geq 9$.

4.3. There are two 2-dimensional faces which have two cusps. Let E_1 and E_2 be the 2-dimensional faces which share the cusp c', other than A_0 and A_1. Note that exactly four 2-dimensional faces A_0, A_1, E_1 and E_2 have c'. We may assume that E_1 (resp. E_2) and A_0 (resp. A_1) are parallel. Denote by B_2 the 2-dimensional face which is adjacent to A_2 and A_3. By Theorem 3.2 (c), B_2 can be adjacent to neither A_0, nor A_1. That is to say, B_2 is neither E_1 nor E_2. Thus B_2 is compact. Then B_2 has to be adjacent to at least five 2-dimensional faces of G. Then there exists a compact 2-dimensional face which is adjacent to B_2. Denote this face by H.

Assume that $a_2(G) = 8$. In this case, the 2-dimensional faces of G are only A_0, A_1, A_2, A_3, E_1, E_2, B_2 and H. If H is adjacent to both A_2 and A_3, then H must coincide with B_2. Thus H can be adjacent only to one of A_2 and A_3. By Sublemma 3.4, H can be adjacent only to one of A_0 and E_1, and be adjacent to one of A_1 and E_2. Thus H can be adjacent only to at most four 2-dimensional faces of G. This is a contradiction to Condition (1) because H is compact. Hence $a_2(G) \neq 8$.

Assume that $a_2(H) = 9$. In this case, there is another compact 2-dimensional face of G. Denote this face by J.

Suppose that H is adjacent to A_0. Then H is not adjacent to A_1 because A_0, A_1 and H do not have a common vertex. Moreover, H is adjacent to neither A_2 nor E_1 by Sublemma 3.4. Thus, H must be adjacent to A_0, A_3, E_2, B_2 and J because it has at least five edges. Thus the endpoints of the edge $A_3 \cap H$ are $A_0 \cap A_3 \cap H$ and $A_3 \cap B_2 \cap H$. That is to say, A_3 is adjacent only to four 2-dimensional faces A_0, A_2, B_2 and H. Because A_0, E_2 and H are pairwise adjacent, A_0 is adjacent only to four 2-dimensional faces A_1, A_3, E_2 and H. Thus J can be adjacent to A_1, A_2, E_1, E_2, B_2 and H. But, since A_1 and E_2 are parallel, J can be adjacent only to one of them. Thus J must be adjacent to A_2, E_1, B_2 and H to satisfy Condition (1). In this case, B_2, H and J are pairwise adjacent, so they have a common vertex. Thus the four 2-dimensional faces A_2, A_3, H and J are cyclically adjacent but share neither a vertex nor a cusp because the endpoints of the edge $A_2 \cap A_3$ are the cusp c and the vertex $A_2 \cap A_3 \cap B_2$. This is a contradiction to Theorem 3.2 (e). Thus H is not adjacent to A_0.

Analogously, we can prove that H is not adjacent to A_1. Thus H is adjacent to E_1, E_2, B_2, J and one of A_2 and A_3. Since one of the endpoints of the edge $E_1 \cap E_2$ is the cusp c', H is the only 2-dimensional face which is adjacent to both E_1 and E_2. Thus J can be adjacent only to one of E_1 and E_2. In this manner, we consider the 2-dimensional face J. By Sublemma 3.4, J can be adjacent only to one of A_0 and A_2, and only to one of A_1 and A_3. Because there is only one 2-dimensional face which is adjacent to both A_2 and A_3, J cannot be adjacent to both A_2 and A_3. The common edge of A_0 and A_1 starts from one cusp and terminates at the other cusp. Thus, by Theorem 3.2 (c), if there exists a 2-dimensional face which is adjacent to both A_0 and A_1, then it must share a cusp with A_0 and A_1. But the 2-dimensional faces which share a cusp with A_0 and A_1 are nothing but the 2-dimensional faces A_2, A_3, E_1 and E_2. Altogether, there is no 2-dimensional face which is adjacent to both A_0 and A_1. Thus there are only two possibilities. One is that J can be adjacent only to A_0, A_3, E_2 and H, and the other is that J can be adjacent only to A_1, A_2, E_1 and H. But both cases do not meet Condition (1). Thus $a_2(G) \neq 9$, and hence $a_2(G) \geq 10$.

4.4. An estimate for $a_3^2(Q^6)$ and the conclusion of the proof of the Main Theorem. Denote one cusp of Q^6 by c. There are exactly ten 5-dimensional faces of Q^6 which share c. Denote these 5-dimensional faces by P_i ($i = 1, \ldots, 10$). Assume that P_i and P_{11-i} ($1 \leq i \leq 10$) are parallel. It is easy to see that any k-dimensional face having c is the

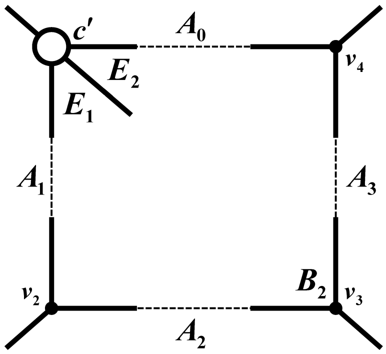

FIGURE 11. The case (4.3)

intersection of $(6-k)$ 5-dimensional faces. Thus there are eighty 3-dimensional faces which have c. Assume that the 3-dimensional face $P_1 \cap P_2 \cap P_3$ has two cusps.

LEMMA 4.1. *If there is a* 3*-dimensional face which has two cusps, then this face and* $P_1 \cap P_2 \cap P_3$ *must satisfy one of the following cases*:

(1) *They are the same* 3*-dimensional face.*

(2) *They have one common* 2*-dimensional face which has two cusps but do not have an edge which joins these two cusps.*

(3) *They have one common edge which starts at one cusp and terminates at the other cusp.*

PROOF. Denote the 3-dimensional face which has two cusps by $P_i \cap P_j \cap P_k$ (i, j and k are different). It is sufficient to prove that $P_i \cap P_j \cap P_k$ is a 3-dimensional face of either P_1, P_2 or P_3. If this 3-dimensional face is a 3-dimensional face of either P_8, P_9 or P_{10}, then it is parallel to $P_1 \cap P_2 \cap P_3$. But in this case, $P_i \cap P_j \cap P_k$ must have exactly one cusp. Thus $P_i \cap P_j \cap P_k$ is not a 3-dimensional face of either P_8, P_9 or P_{10}. Because P_5 and P_6 are parallel, if $P_i \cap P_j \cap P_k$ is a 3-dimensional face of either P_5 or P_6, then it cannot be a 3-dimensional face of the other one. Thus $P_i \cap P_j \cap P_k$ is a 3-dimensional face of either P_1, P_2 or P_3. □

To prove the main theorem, we need to prove the following lemma.

LEMMA 4.2. *Assume that three* 3*-dimensional faces of* Q^6 *have a common cusp, and they do not share the other one. If these* 3*-dimensional faces are pairwise adjacent, then one of them has more than twelve* 2*-dimensional faces.*

PROOF. Let J_1, J_2 and J_3 be the three 3-dimensional faces which have a common cusp, and do not share the other one. By the assumption, J_1, J_2 and J_3 are pairwise adjacent. By Lemma 3.1, each of these faces has at least twelve 2-dimensional faces. Assume that one of these 3-dimensional faces has exactly twelve 2-dimensional faces. In this case, the intersection $J_1 \cap J_2$ is a quadrangle or a pentagon.

Assume that $J_1 \cap J_2$ is a quadrangle. Because the intersection $J_1 \cap J_3$ is a 2-dimensional face of J_1, and is adjacent to $J_1 \cap J_2$, $J_1 \cap J_3$ must be a pentagon by Lemma 3.1. The 2-dimensional face $J_2 \cap J_3$ must be a pentagon because the intersection $J_2 \cap J_3$ is a 2-dimensional face of J_2, and it is adjacent to $J_1 \cap J_2$. The intersections $J_1 \cap J_3$ and $J_2 \cap J_3$ are a 2-dimensional faces of J_3. Note that these two faces are adjacent. Thus J_3 must have more than twelve 2-dimensional faces by Lemma 3.1. But J_3 has only twelve 2-dimensional faces. Hence $J_1 \cap J_2$ is not a quadrangle. Thus $J_1 \cap J_2$ is a pentagon. But as we have seen above, $J_1 \cap J_3$ and $J_2 \cap J_3$ are quadrangles, and thus J_3 must have more than twelve 2-dimensional faces. Hence either J_1, J_2 or J_3 must have more than twelve 2-dimensional faces. □

We consider the case (4.1) again. By Lemma 4.1, there is exactly one 3-dimensional face which has two cusps. We may assume that this 3-dimensional face is the intersection of three 5-dimensional faces P_1, P_2 and P_3. This 3-dimensional face has more than seven 2-dimensional faces by Proposition 2.2. Moreover, by Corollary 3.6, this face is the only 3-dimensional face which has less than twelve 2-dimensional faces. By Theorem 2.4, we obtain the following inequality;

$$a_3^2(Q^6) < 12\,. \tag{4.1}$$

If there are more than three 3-dimensional faces which have more than twelve 2-dimensional faces each, then Q^6 cannot satisfy (4.1). By Lemma 4.2, one of the 3-dimensional faces $P_1 \cap P_2 \cap P_4$, $P_1 \cap P_2 \cap P_5$ and $P_1 \cap P_2 \cap P_8$ has more than twelve 2-dimensional faces. Similarly, one of the 3-dimensional faces $P_1 \cap P_4 \cap P_5$, $P_2 \cap P_4 \cap P_5$ and $P_4 \cap P_5 \cap P_8$ has more than twelve 2-dimensional faces. And one of the 3-dimensional faces $P_1 \cap P_4 \cap P_6$, $P_2 \cap P_4 \cap P_6$ and $P_4 \cap P_6 \cap P_8$ has more than twelve 2-dimensional faces. In addition, one of the 3-dimensional faces $P_1 \cap P_3 \cap P_9$, $P_1 \cap P_4 \cap P_9$ and $P_1 \cap P_5 \cap P_9$ has more than twelve 2-dimensional faces. Thus there are more than three 3-dimensional faces which have more than twelve 2-dimensional faces. Hence, there is no polyhedron Q^6 corresponding to the case (4.1).

We consider the case (4.2): only one 2-dimensional face has two cusps. We can express this 2-dimensional face as the intersection of four 5-dimensional faces P_1, P_2, P_3 and P_4. There are only four 3-dimensional faces $P_1 \cap P_2 \cap P_3$, $P_1 \cap P_2 \cap P_4$, $P_1 \cap P_3 \cap P_4$ and $P_2 \cap P_3 \cap P_4$ which have two cusps. Note that each of these 3-dimensional faces has more than eight 2-dimensional faces. By the same reason as in the case (4.1), if there are at least twelve 3-dimensional faces which have more than twelve 2-dimensional faces each, then Q^6 does not satisfy inequality (4.1). Table 1 is a part of the list of 3-dimensional faces which have exactly one cusp. Any three 3-dimensional faces in the same row are pairwise adjacent. By Lemma 4.2, at least one of the faces in a row has more than twelve 2-dimensional faces. Thus there exist at least twelve 3-dimensional faces which have more than twelve 2-dimensional faces each. Hence Q^6 does not satisfy inequality (4.1).

We consider the case (4.3): exactly one edge connects two cusps. Denote this edge by $P_1 \cap P_2 \cap P_3 \cap P_4 \cap P_5$. Then there are exactly ten 3-dimensional faces $P_1 \cap P_2 \cap P_3$, $P_1 \cap P_2 \cap P_4$, $P_1 \cap P_2 \cap P_5$, $P_1 \cap P_3 \cap P_4$, $P_1 \cap P_3 \cap P_5$, $P_1 \cap P_4 \cap P_5$, $P_2 \cap P_3 \cap P_4$, $P_2 \cap P_3 \cap P_5$, $P_2 \cap P_4 \cap P_5$ and $P_3 \cap P_4 \cap P_5$ which have two cusps. Note that each of these 3-dimensional faces has at least ten 2-dimensional faces. By the same reason in the cases (4.1–4.2), if there are more than twenty 3-dimensional faces which have more than twelve 2-dimensional faces each, then Q^6 does not satisfy inequality (4.1). Table 2 is a part of the list of 3-dimensional faces any three of which in the same row are pairwise adjacent, analogous to Table 1.

By Lemma 4.2, there exist at least twenty 3-dimensional faces which have more than twelve 2-dimensional faces. Thus Q^6 does not satisfy inequality (4.1).

Thus neither of the cases (4.1–4.3) satisfies inequality (4.1). Thus $c(Q^6) \neq 2$. Hence $c(Q^6) \geq 3$.

5. Proof of the Main Theorem: $n = 7$

Assume that Q^7 has exactly m cusps. Fix one cusp of Q^7, and denote it by c. It is clear that there are twelve 6-dimensional faces which share c. Denote these faces by E_i $(1 \leq i \leq 7)$, and assume that E_i and E_j are parallel if $i + j = 13$. Any of the 3-dimensional faces which have c is represented by the intersection of four 6-dimensional faces which have c. If a 3-dimensional face belongs to E_i, then it is not a 3-dimensional face of E_{13-i}. Thus the number of 3-dimensional faces of Q^7 which have c is $\frac{12\times10\times8\times6}{4\times3\times2\times1}$, i.e. two hundreds and forty 3-dimensional faces. By analogy to Lemma 4.1, we can prove the following statement.

LEMMA 5.1. *If there are two different* 3*-dimensional faces of* Q^7 *which share two cusps*, *then these faces either have a common* 2*-dimensional face which has both cusps* (*but no common edge joins the cusps*), *or have a common edge which starts at one cusp and terminates at the other.*

$P_1 \cap P_2 \cap P_6$	$P_1 \cap P_2 \cap P_7$	$P_1 \cap P_2 \cap P_8$
$P_1 \cap P_3 \cap P_6$	$P_1 \cap P_3 \cap P_7$	$P_1 \cap P_3 \cap P_9$
$P_1 \cap P_4 \cap P_6$	$P_1 \cap P_4 \cap P_8$	$P_1 \cap P_4 \cap P_9$
$P_1 \cap P_5 \cap P_7$	$P_1 \cap P_5 \cap P_8$	$P_1 \cap P_5 \cap P_9$
$P_2 \cap P_3 \cap P_6$	$P_2 \cap P_3 \cap P_7$	$P_2 \cap P_3 \cap P_{10}$
$P_2 \cap P_4 \cap P_6$	$P_2 \cap P_4 \cap P_8$	$P_2 \cap P_4 \cap P_{10}$
$P_2 \cap P_5 \cap P_7$	$P_2 \cap P_5 \cap P_8$	$P_2 \cap P_5 \cap P_{10}$
$P_3 \cap P_4 \cap P_6$	$P_3 \cap P_4 \cap P_9$	$P_3 \cap P_4 \cap P_{10}$
$P_3 \cap P_5 \cap P_7$	$P_3 \cap P_5 \cap P_9$	$P_3 \cap P_5 \cap P_{10}$
$P_4 \cap P_5 \cap P_8$	$P_4 \cap P_5 \cap P_9$	$P_4 \cap P_5 \cap P_{10}$
$P_2 \cap P_6 \cap P_{10}$	$P_2 \cap P_7 \cap P_{10}$	$P_2 \cap P_8 \cap P_{10}$
$P_3 \cap P_6 \cap P_{10}$	$P_3 \cap P_7 \cap P_{10}$	$P_3 \cap P_9 \cap P_{10}$

TABLE 1. 3-dimensional faces which satisfy the conditions of Lemma 4.2 in the case (4.2)

$P_1 \cap P_2 \cap P_6$	$P_1 \cap P_2 \cap P_7$	$P_1 \cap P_2 \cap P_8$
$P_1 \cap P_3 \cap P_6$	$P_1 \cap P_3 \cap P_7$	$P_1 \cap P_3 \cap P_9$
$P_1 \cap P_4 \cap P_6$	$P_1 \cap P_4 \cap P_8$	$P_1 \cap P_4 \cap P_9$
$P_1 \cap P_5 \cap P_7$	$P_1 \cap P_5 \cap P_8$	$P_1 \cap P_5 \cap P_9$
$P_2 \cap P_3 \cap P_6$	$P_2 \cap P_3 \cap P_7$	$P_2 \cap P_3 \cap P_{10}$
$P_2 \cap P_4 \cap P_6$	$P_2 \cap P_4 \cap P_8$	$P_2 \cap P_4 \cap P_{10}$
$P_2 \cap P_5 \cap P_7$	$P_2 \cap P_5 \cap P_8$	$P_2 \cap P_5 \cap P_{10}$
$P_3 \cap P_4 \cap P_6$	$P_3 \cap P_4 \cap P_9$	$P_3 \cap P_4 \cap P_{10}$
$P_3 \cap P_5 \cap P_7$	$P_3 \cap P_5 \cap P_9$	$P_3 \cap P_5 \cap P_{10}$
$P_4 \cap P_5 \cap P_8$	$P_4 \cap P_5 \cap P_9$	$P_4 \cap P_5 \cap P_{10}$
$P_2 \cap P_6 \cap P_{10}$	$P_2 \cap P_7 \cap P_{10}$	$P_2 \cap P_8 \cap P_{10}$
$P_3 \cap P_6 \cap P_{10}$	$P_3 \cap P_7 \cap P_{10}$	$P_3 \cap P_9 \cap P_{10}$
$P_4 \cap P_6 \cap P_{10}$	$P_4 \cap P_8 \cap P_{10}$	$P_4 \cap P_9 \cap P_{10}$
$P_5 \cap P_7 \cap P_{10}$	$P_5 \cap P_8 \cap P_{10}$	$P_5 \cap P_9 \cap P_{10}$
$P_1 \cap P_6 \cap P_9$	$P_1 \cap P_7 \cap P_9$	$P_1 \cap P_8 \cap P_9$
$P_6 \cap P_9 \cap P_{10}$	$P_7 \cap P_9 \cap P_{10}$	$P_8 \cap P_9 \cap P_{10}$
$P_1 \cap P_7 \cap P_8$	$P_2 \cap P_7 \cap P_8$	$P_5 \cap P_7 \cap P_8$
$P_6 \cap P_7 \cap P_8$	$P_7 \cap P_8 \cap P_9$	$P_7 \cap P_8 \cap P_{10}$
$P_1 \cap P_6 \cap P_7$	$P_2 \cap P_6 \cap P_7$	$P_3 \cap P_6 \cap P_7$
$P_2 \cap P_6 \cap P_8$	$P_4 \cap P_6 \cap P_8$	$P_6 \cap P_8 \cap P_{10}$

TABLE 2. 3-dimensional faces which satisfy the conditions of Lemma 4.2 in the case (4.3)

Note that any edge of Q^7 is given by the intersection of six 6-dimensional faces of Q^7. Thus if we fix one edge of Q^7, then the number of 3-dimensional faces of Q^7 which have this edge is $\binom{6}{4} = 15$. Thus, by Lemma 5.1, there are no more than fifteen 3-dimensional faces which have two common cusps.

Fix one 3-dimensional face and denote it by F. Assume that F has l cusps. By Lemma 5.1, there are at most $15\binom{m-l}{2}$ 3-dimensional faces which have more than two cusps which do not belong to F. By Proposition 2.2, each of these 3-dimensional faces has at least six 2-dimensional faces. In addition, by Lemma 5.1, there are at most $15l(m-l)$ 3-dimensional faces which have one common cusp with F, and have one cusp which F does not have. By the same reason as above, each of these faces has at least six 2-dimensional faces. By Lemma 5.1, there are at most $14\binom{l}{2}$ 3-dimensional faces which have at least two common cusps with F.

By the second inequality of Proposition 2.2, each of these 3-dimensional faces has at least six 2-dimensional faces. But if a 3-dimensional face has exactly six 2-dimensional faces, by the third inequality of Proposition 2.2, it must have more than two cusps. Thus if one of those 3-dimensional faces has only the cusps that F has, then this 3-dimensional face and F have a common 2-dimensional face which has more than two cusps. Denote this 3-dimensional face by F'. There are at least three 2-dimensional faces of F' which are parallel to $F \cap F'$. In addition, when we look at one cusp of $F \cap F'$, there are two faces which are adjacent to $F \cap F'$. Because F' has exactly six 2-dimensional faces, these two 2-dimensional faces which share a cusp with $F \cap F'$ and are adjacent to $F \cap F'$ have every cusp of $F \cap F'$. The latter is impossible. Thus F' must have more than six 2-dimensional faces in this case. Thus, any 3-dimensional face which shares all the cusps with F has more than six 2-dimensional faces.

Fix one cusp of Q^7, and denote it by c_1. In this case, we note that there are at least $(240 - 15(m-1))$ 3-dimensional faces which have only one cusp c_1 because the number of 3-dimensional faces which have at least two cusps including c_1 is at most $15(m-1)$. Because the number of these 3-dimensional faces which have only the cusp c_1 is positive, m must be smaller than 17. By Corollary 3.6, each of these 3-dimensional faces has at least twelve 2-dimensional faces. Thus,

$$\begin{aligned}
a_3^2(Q^7) &\geq \tfrac{1}{a_3(Q^7)}\Big(6\times15\tbinom{m-l}{2}+6\times15l(m-l)+7\times14\tbinom{l}{2}+12(240-15(m-1))\\
&\quad +12(a_3(Q^7)-(15\tbinom{m-l}{2}+15l(m-l)+14\tbinom{l}{2}+(240-15(m-1))))\Big)\\
&\geq \tfrac{1}{a_3(Q^7)}\Big((9-3)\times15\tbinom{m-l}{2}+(9-3)\times15l(m-l)+(9-2)\times14\tbinom{l}{2}+(9+3)(240-15(m-1))\\
&\quad +9(a_3(Q^7)-(15\tbinom{m-l}{2}+15l(m-l)+14\tbinom{l}{2}+(240-15(m-1))))\Big)\\
&= 9+\tfrac{-3\times15\binom{m-l}{2}-3\times15l(m-l)-2\times14\binom{l}{2}+3(240-15(m-1))}{a_3(Q^7)}\,.
\end{aligned}$$

On the other hand, by Theorem 2.4, we obtain $a_3^2(Q^7) < 9$. In order for Q^7 to satisfy all the above inequalities, the following must hold:

$$-3\times15\binom{m-l}{2}-3\times15l(m-l)-2\times14\binom{l}{2}+3(240-15(m-1))<0\,.$$

Fix another cusp of Q^7, and denote it by c_2. The number of 3-dimensional faces which have exactly one cusp c_2 is at least $240 - 15(m-2)$. By analogy to the above, we obtain:

$$-3\times15\binom{m-l}{2}-3\times15l(m-l)-2\times14\binom{l}{2}+3(240-15(m-1))+3(240-15(m-2))<0\,.$$

By proceeding in this way, because Q^7 has m cusps, we see that

$$\begin{aligned}
-3\times15\binom{m-l}{2}-3\times15l(m-l)-2\times14\binom{l}{2}\ &+3(240-15(m-1))\\
&+3(240-15(m-2))\\
&\cdots\\
&+3(240-15\times1)\ <0\,.
\end{aligned}$$

We simplify the left-hand side of this inequality, and finally obtain

$$17l^2-17l-90m^2+1530m-1440<0\,.$$

The left-hand side is an increasing function of l. Because these exist a face F with $l \geq 2$ cusps, by substituting $l = 2$ in the above inequality, we get

$$90m^2 - 1530m + 1406 > 0 .$$

To satisfy this inequality m must be greater than or equal to 17. A contradiction. Hence $c(Q^7) \geq 17$.

6. Proof of the Main Theorem: $8 \leq n \leq 12$

Our main theorem for $8 \leq n \leq 12$ comes from the following lemma.

LEMMA 6.1. *For $8 \leq n \leq 12$, if any $(n-1)$-dimensional face of Q^n has at least m cusps, then*

$$c(Q^n) \geq 3m - 2n + 1 .$$

PROOF. Let L be an $(n-1)$-dimensional face of Q^n. The number of $(n-1)$-dimensional faces which are parallel to L and share exactly one cusp with L is $c(L)$. Denote these $(n-1)$-dimensional faces by L', L_1, L_2, $\ldots$, $L_{c(L)-1}$. Assume that Q^n has k cusps which are not shared by neither L nor L'. Each L_i $(1 \leq i \leq c(L) - 1)$ shares at least $c(L_i) - 1 - k$ cusps with L'. Because $c(L_i) \geq m$, k is less than or equal to $m - 1$. On the other hand, the number of $(n-1)$-dimensional faces sharing a cusp of L' is $2(n-1)$. Thus, we obtain the following inequality:

$$(c(L_1) - 1 - k) + (c(L_2) - 1 - k) + \cdots + (c(L_{c(L)-1}) - 1 - k) \leq (2(n-1) - 1)(c(L') - 1).$$

Because each L_i $(1 \leq i \leq c(L) - 1)$ and L must have at least m cusps, we have

$$(m - 1 - k)(m - 1) \leq (2(n-1) - 1)(c(L') - 1) . \tag{6.1}$$

By this inequality, we obtain

$$k \geq m - 1 - \frac{(2n-3)(c(L') - 1)}{m - 1} .$$

Thus, we get

$$\begin{aligned} c(Q^n) &\geq c(L) + c(L') - 1 + k \\ &\geq m + c(L') - 1 + m - 1 - \frac{(2n-3)(c(L') - 1)}{m-1} \\ &\geq 2m - 2 + \frac{2n-3}{m-1} + \frac{m - 2(n-1)}{m-1} c(L') . \end{aligned}$$

Because Q^7 has at least seventeen cusps, m is greater than or equal to $2(n-1)$ for $8 \leq n \leq 12$. Since $\frac{m-2(n-1)}{m-1}$ is positive and L' has at least m cusps, we have

$$c(Q^n) \geq 2m - 2 + \frac{2n-3}{m-1} + \frac{m - 2(n-1)}{m-1} c(L')$$

$$\begin{aligned}&\geq 2m-2+\frac{2n-3}{m-1}+\frac{m-2(n-1)}{m-1}m\\&=3m-2n+1\,.\end{aligned}$$

□

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