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On Regular Fréchet-Lie Groups II

Composition Rules of Fourier-Integral Operators on a Riemannian Manifold

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Introduction

In the previous paper [8], we gave a differential geometrical expression of Fourier-integral operators on a closed riemannian manifold N without using local coordinate pathches, which is expressed in the following relatively concrete form: (Cf. (19) for the precise meaning of the notations.)

$$
\begin{aligned}
(1)\qquad & (Fu)(x) \\
&= \sum_{\alpha} \iint \lambda_\alpha a(x; \xi, X) e^{-i\langle \varphi_2(x;\xi) | X\rangle - i|\xi| A_\alpha(X)} (\nu u)^{\cdot}(\varphi_1(x; \xi); X)\, đX\, đ\xi \\
&\quad + (K \circ u)(x) ,
\end{aligned}
$$

where $\varphi=(\varphi_1; \varphi_2)$ is a symplectic transformation of order 1 on $T^*N-\{0\}$. Although our operators such as (1) form much narrower class than what was defined by Hörmander [3] or Guillemin -Sternberg [2], our expression contains less ambiguities, and hence one can give a sort of coordinate system on a "vicinity" of the identity operator of the Fourier-integral operators of order 0 (cf. Theorem 5.8 [8]). Moreover, the above expression seems to be convenient for concrete computation of the fundamental solution of the equation

$$
(2)\qquad \frac{d}{dt} u = \sqrt{-1}\, Pu
$$

for a pseudo-differential operator P of order 1 with a real principal symbol. We shall state the reason in what follows.

Let $G\mathscr{F}^0$ be the group generated by the invertible Fourier-integral operators of order 0, written in the form (1). We regard $G\mathscr{F}^0$ as if it

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were a locally connected topological group. Then, the identity component $G\mathscr{F}_0^0$ is generated by any neighborhood of the identity. In this paper, we shall prove the following:

Theorem A. There is a vicinity $\mathfrak{N}$ of the identity in the space of Fourier-integral operators written in the form (1) such that every element in $\mathfrak{N}$ is invertible and the inverse is written in the form (1).

Theorem B. Let $G\mathscr{F}_0^0$ be the group generated by $\mathfrak{N}$. Then every element in $G\mathscr{F}_0^0$ can be written in the form (1).

Remark that the fundamental solution $e^{\sqrt{-1}\,tP}$ of (2) is contained in $G\mathscr{F}_0^0$, and hence it is written in the form (1). However, the concrete computation must contain the same difficulties as in the case $N=\boldsymbol{R}^n$. The advantage of our expression seems to be in the reduction of the difficulties in computing the fundamental solution to the same level as in the case $N=\boldsymbol{R}^n$, even if N is a closed manifold.

For the proof of above theorems, we must establish the formulae of compositions, and inversions of Fourier-integral operators of order 0. It is in fact the first step of proving that $G\mathscr{F}_0^0$ is a regular Fréchet-Lie group. Although the differentiability of the group operations will be proved in forthcoming papers, it is not hard by the above formulae to see that $G\mathscr{F}_0^0$ is a locally connected topological group.

Now, we would like to recall our situation, and several notations used in the previous paper. Let S^*N be the unit cosphere bundle over N imbedded naturally in the cotangent bundle T^*N, and $\mathscr{D}_\omega(S^*N)$ the group of all contact transformations on S^*N. By Lemma 1.6 in [8], $\mathscr{D}_\omega(S^*N)$ is naturally isomorphic to the group $\mathscr{D}_\Omega^{(1)}$ of all symplectic transformations of order 1 on $T^*N-\{0\}$, where a symplectic transformation φ: $T^*N-\{0\}\rightarrow T^*N-\{0\}$ is called to be order 1, if $\varphi(x;\xi)=(\varphi_1(x;\xi);\varphi_2(x;\xi))$ satisfies

$$(3)\qquad \varphi_1(x;r\xi)=\varphi_1(x;\xi)\,,\qquad \varphi_2(x;r\xi)=r\varphi_2(x;\xi)\,,$$

for every $r>0$, where $(y;\eta)$ means a point in T^*N such that $y\in N$, $\eta\in T_y^*$ (the fibre of T^*N at y). Since $\mathscr{D}_\omega(S^*N)$ is a topological group under the C^∞-topology, so is $\mathscr{D}_\Omega^{(1)}$ through the identification mentioned above.

By $\sum_C^0$ we denote the totality of all C-valued C^∞ functions on T^*N with the following asymptotic expansions:

$$(4)\qquad a(x;\xi)\sim a_0(x;\hat{\xi})+a_{-1}(x;\hat{\xi})r^{-1}+\cdots+a_{-j}(x;\hat{\xi})r^{-j}+\cdots\,,$$

where $r=|\xi|$, $\hat{\xi}=r^{-1}\xi$ and a_{-j}'s are C^∞ functions on S^*N. There is a natural linear mapping α of $\sum_C^0$ into the space $C^\infty(S^*N)^\infty$ of all series of C^∞ functions on S^*N indexed by non-positive integers. It is not hard to see that $\sum_C^0$ is naturally isomorphic to the space of all C^∞ functions on the unit closed disk bundle $\bar{D}^*N$ (cf. (11) of [8]), and the above asymptotic expansion (4) corresponds to Taylor's expansion in the radial direction at $r=1$.

There exists a mapping β: $C^\infty(S^*N)^\infty \to \sum_C^0$ such that $\alpha\beta$=identity by a slight modification of the proof in [4] p. 35. However, it seems for us impossible to choose β to be linear, (and this causes some troubles in making $G\mathscr{F}_0^0$ a Fréchet-Lie group). Remark that for every $a(x;\xi)\in\sum_C^0$, $\beta\alpha(a)-a$ is rapidly decreasing in ξ.

Let $C^\infty(N\times N)$ be the space of all C-valued C^∞ functions on $N\times N$. For each $K(x, y)\in C^\infty(N\times N)$, we defined a smoothing operator $K\circ$ by

$$(5)\qquad (K\circ u)(x)=\int_N K(x, y)u(y)dy\ ,$$

where dy is the volume element of N defined by the riemannian metric. A C^∞ function $\nu(x, y)$ on $N\times N$ will be called a *cut off function* if

(a) $0\leqq\nu(x, y)\leqq 1$, $\nu(x, y)=\nu(y, x)$.

(b) There is a sufficiently small number $\varepsilon>0$, called the *breadth* of ν such that $\nu(x, y)=1$ if the distance $\rho(x, y)\leqq\varepsilon/3$.

(c) $\nu(x, y)=0$ if $\rho(x, y)>2\varepsilon/3$.

Now, if our Fourier-integral operator F written in the form (1) is in a vicinity $\mathfrak{N}$ of the identity, then F can be rewritten in the form:

$$(6)\qquad (Fu)(x)=\int_{T_x^*} a(x;\xi)\widetilde{\nu u}(\varphi(x;\xi))d\xi+(K\circ u)(x)$$

where $\varphi\in\mathscr{D}_\Omega^{(1)}$, $a\in\sum_C^0$, $K\in C^\infty(N\times N)$ and $\widetilde{\nu u}$ is a sort of Fourier transform of $u\in C^\infty(N)$ defined by

$$(7)\qquad \widetilde{\nu u}(y;\eta)=\int_N e^{-i\langle\eta|Y\rangle}\nu(y, z)u(z)dz\ ,\qquad \cdot_y Y=z \text{ (i.e., } \mathrm{Exp}_y\, Y=z)\ .$$

As a matter of course, the expression (6) still contains some ambiguities, but φ and $\alpha(a)$ are uniquely determined by F (cf. Proposition 5.3 in [8]). Hence if we replace a by $\beta\alpha(a)$, then the smoothing term K is uniquely determined also. Therefore one may regard $(\varphi, \alpha(a), K)$ as a sort of a local coordinate system of F.

As it was mentioned above, one of the main purpose of this paper is to give the formulae of compositions and inversions of Fourier-integral operators. Let $F=F(\psi, a, K)$ be the operator given by (6). Although

(ψ, a, K) is not exactly a local coordinate of F, it is convenient to use this as if it were a local coordinate of F. The reason of such a sophisticated manner is based on that $\beta: C^\infty(S^*N)^\infty \to \sum_C^0$ is neither linear nor differentiable. If β is differentiable, then one can replace β by its derivative $(d\beta)_0$ to get a linear splitting. Let $G = F(\psi', b, L)$ be another operator contained in $\mathfrak{N}$. Then, we shall obtain in this paper formulae such as

$$\begin{cases} F(\psi, a, K)F(\psi', b, L) = F(\psi'\psi, c, K') \\ F(\psi, a, K)^{-1} = F(\psi^{-1}, a', L') , \end{cases} \tag{8}$$

where c is given as a function of ψ, ψ', a, b, which will be denoted by $c = (\psi, \psi', a, b)$. Similarly, $K' = K'(\psi, \psi', a, b, K, L)$, $a' = a'(\psi, a)$ and $L' = L'(\psi, a, K)$. In near future, we shall prove that the above functions are smooth in some sense. However, the continuity in the C^∞-topology of these functions is not difficult to prove. Thus, one can get that $G\mathscr{F}_0{}^0$ is a locally connected topological group.

For the proof of Theorem B, the above composition rules (8) are not enough. We have to get a composition rule where ψ is not assumed to be close to the identity. Indeed, we need to compute FG, where $G \in \mathfrak{N}$ and F is a Fourier-integral operator written in the form (1). The essence of the proof of Theorem B is seen in the following:

Proposition A. Let φ_0 be an element of $\mathscr{D}_\omega^{(1)}$. Then, there are a neighborhood $\mathfrak{V}_{\varphi_0}$ of φ_0 and a neighborhood $\mathfrak{U}$ of the identity in $\mathscr{D}_\Omega^{(1)}$ under the C^1 uniform topology such that if F is an operator given by (1) with $\varphi \in \mathfrak{V}_{\varphi_0}$, and if G is an operator given by (6) with $\psi \in \mathfrak{U}$, then FG is an operator written in the same shape as in (1) replacing φ by $\psi\varphi$.

Remark. It is an open question for us whether a composition of two Fourier-integral operators in the form (1) is again expressed by the same form.

§1. Notations, remarks and the summary of the previous paper.

In general, we use the same notations used in [8], but since some of them are not familiar, we repeat these notations here.

N is a compact n-dimensional C^∞ riemannian manifold without boundary. Let g_{ij} be the riemannian metric tensor with respect to a normal chart $(X^1, \cdots, X^n)$ around $x \in N$ and let $g(x) = \det(g_{ij}(x))$. We use

$$đX=\frac{\sqrt{g(x)}}{\sqrt{2\pi}^{\,n}}dX^1\wedge\cdots\wedge dX^n\,,\quad đ\xi=\frac{1}{\sqrt{2\pi}^{\,n}}\frac{1}{\sqrt{g(x)}}d\xi_1\wedge\cdots\wedge d\xi_n$$

as volume forms on T_x, T_x^* respectively, where T_x (resp. T_x^*) is the tangent (resp. cotangent) space of N at x. We use also the notation $đx=(1/\sqrt{2\pi}^{\,n})dx$, where dx is the volume element on N.

Since we use normal charts very often, we have to use exponential mappings Exp_x in the expressions of Fourier-integral operators. Thus, for simplicity of the notations, we use $\cdot_x X$, ${}^x\!\cdot y$ instead of $\mathrm{Exp}_x X$, $\mathrm{Exp}_x^{-1} y$. Moreover, we denote $(y;Y)=\cdot_x(X,Z)$, $(y;\eta)=\cdot_x(X,\zeta)$, if $(y;Y)=(\mathrm{Exp}_x X;(d\,\mathrm{Exp}_x)_X Z)$, $(y;\eta)=(\mathrm{Exp}_x X: (d\,\mathrm{Exp}_x^{-1})_X^*\zeta)$ respectively. (X, Z) and (X,ζ) will be called normal coordinate expressions of $(y;Y)\in TN$, $(y;\eta)\in T^*N$ respectively.

Coordinate transformations between two normal charts will be denoted by $Y=S(x;X,\bar{X})$. Namely, if $y=\cdot_x\bar{X}$, then

$$(9)\qquad \cdot_y S(x;X,\bar{X})=\cdot_x X\,.$$

Since $S(x;\bar{X},\bar{X})\equiv 0$, S can be written in the form (cf. (4), (5) in [8])

$$(10)\qquad S(x;X,\bar{X})=S_1(x;X,\bar{X})(X-\bar{X})\,.$$

Obviously $S(x;X,\bar{X})\in T_{\cdot_x\bar{X}}$. Hence $(\partial S/\partial X)(x;X,\bar{X})$ has an invariant meaning as a linear mapping of T_x into $T_{\cdot_x\bar{X}}$. However, if we fix X and vary $\bar{X}$, then we get a vector field. Note that $S(x;X,\bar{X})$ is defined for X, $\bar{X}$ such that $|X|+|\bar{X}|<r_0$, where r_0 is the injectivity radius of N. For an arbitrarily fixed $\bar{X}$, we define $\tilde{S}(x;X,\hat{X})$ by

$$(11)\qquad (\cdot_x\hat{X};S(x;X,\hat{X}))=\cdot_{\cdot_x\bar{X}}(Y,\tilde{S}(x;X,\hat{X}))\,.$$

$\tilde{S}(x;X,\hat{X})$ is a normal coordinate expression of $S(x;X,\hat{X})$ around $\cdot_x\bar{X}$. We define

$$(12)\qquad \frac{\nabla^r S}{\partial\bar{X}^r}(x;X,\bar{X})=\frac{\partial^r}{\partial\hat{X}^r}\tilde{S}(x;X,\hat{X})|_{\hat{X}=\bar{X}}\,,\qquad r\geqq 0\,.$$

Thus, $(\nabla^r S/\partial\bar{X}^r)(x;X,\bar{X})$ is well-defined as a symmetric r-linear mapping of $T_x\times\cdots\times T_x$ into$T_{\cdot_x\bar{X}}$ (cf. (7)~(8) in [8]). If $r=1$, then $\nabla S/\partial\bar{X}$ is in fact the covariant derivative of S. However $\nabla^r S/\partial\bar{X}^r$ for $r\geqq 2$ is *not* the r-times covariant derivative.

LEMMA 1.1. *In a normal chart $(X^1,\cdots,X^n)$ around x, $\tilde{S}^i$ can be written as*

(13) $$\tilde{S}^i = X^i - \bar{X}^i + \frac{1}{2}\left\{ {i \atop jk} \right\}_{\bar{X}} (X^j - \bar{X}^j)(X^k - \bar{X}^k) + \cdots .$$

Moreover, let $L^r_{\mathrm{sym}}(T_x)$ be the space of all symmetric r-linear mappings of $T_x \times \cdots \times T_x$ into T_x. Then

$$\frac{\nabla S}{\partial \bar{X}}(x; X, 0) = -I + Q(x; X)(X) ,$$

$$\frac{\nabla^2 S}{\partial \bar{X}^2}(x; X, 0) = R(x; X)(X) ,$$

where I = identity and $Q(x; X)$ (resp. $R(x; X)$) *is a $L^1_{\mathrm{sym}}(T_x)$* – (resp. $L^2_{\mathrm{sym}}(T_x)$) *valued quadratic* (resp. *linear*) *form.*

PROOF. Recalling the proof of Lemma 1.5 in [8], we obtain the equation (13). Also, remark that $\left\{ {i \atop jk} \right\}_0 = 0$, and we get the desired result.

REMARK. Obviously,

$$\left. \frac{\partial^2 \tilde{S}^i}{\partial \bar{X}^l \partial \bar{X}^m} \right|_{\bar{X}=0} = -\left(\frac{\partial}{\partial \bar{X}^l} \left\{ {i \atop mk} \right\} \right)_{\bar{X}=0} X^k + \cdots .$$

It is known that (cf. [7] p. 291)

$$\left(\frac{\partial}{\partial \bar{X}^l} \left\{ {i \atop mk} \right\} \right)_{\bar{X}=0} = -\frac{1}{3}(R^i_{mkl} + R^i_{kml}) ,$$

where R^i_{mkl} is the curvature tensor on N expressed by the above normal chart.

For $\varphi(x; \xi) = (\varphi_1(x; \xi); \varphi_2(x; \xi))$, $(\partial\varphi_1/\partial\xi)(x; \xi)$ has an invariant meaning as a linear mapping of T^*_x into $T_{\varphi_1(x;\xi)}$. Now, we want to define other derivatives and higher order derivatives:

Let $(\tilde{\varphi}_1(\cdot_x(X, \tilde{\xi})), \tilde{\varphi}_2(\cdot_x(X, \tilde{\xi})))$ be the normal coordinate expression of φ around $\varphi_1(x; \xi)$. Namely, we set

(14) $$\cdot_{\varphi_1(x;\xi)}(\tilde{\varphi}_1(\cdot_x(X, \tilde{\xi})), \tilde{\varphi}_2(\cdot_x(X, \tilde{\xi}))) = (\varphi_1(\cdot_x(X, \tilde{\xi})); \varphi_2(\cdot_x(X, \tilde{\xi}))) .$$

Obviously, we have

$$\frac{\partial \varphi_1}{\partial \xi}(x; \xi) = \frac{\partial}{\partial \tilde{\xi}} \tilde{\varphi}_1(\cdot_x(0, \tilde{\xi}))_{|\tilde{\xi}=\xi} .$$

We define as follows:

$$(15)\qquad \cdot\begin{cases}\dfrac{\nabla\partial\varphi_1}{\partial\xi^2}(x;\xi)=\dfrac{\partial^2}{\partial\tilde{\xi}^2}\tilde{\varphi}_1(\cdot_x(0,\tilde{\xi}))|_{\tilde{\xi}=\xi}\ ,\\ \dfrac{\nabla^r\varphi_2}{\partial\xi^r}(x;\xi)=\dfrac{\partial^r}{\partial\tilde{\xi}^r}\tilde{\varphi}_2(\cdot_x(0,\tilde{\xi}))|_{\tilde{\xi}=\xi}\ , & r=1,2\\ \dfrac{\nabla^r\varphi_i}{\partial x^r}(x;\xi)=\dfrac{\partial^r}{\partial X^r}\tilde{\varphi}_i(\cdot_x(X,\xi))|_{X=0}\ , & r=1,2,\ i=1,2\\ \dfrac{\nabla^2\varphi_i}{\partial\xi\partial x}(x;\xi)=\dfrac{\partial^2}{\partial\xi\partial X}\tilde{\varphi}_i(\cdot_x(X,\tilde{\xi}))\Big|_{\substack{X=0\\ \tilde{\xi}=\xi}}\ , & i=1,2.\end{cases}$$

Remark that $\nabla\partial\varphi_1/\partial\xi^2$, $\nabla\varphi_2/\partial\xi$ are covariant derivatives but others are not covariant derivatives.

LEMMA 1.2. $\langle\xi|X\rangle=\langle\varphi_2(x;\xi)|(\nabla\varphi_1/\partial X)(x;\xi)X\rangle$

$$\left\langle\varphi_2(x;\xi)\Big|\frac{\partial\varphi_1}{\partial\xi}(x;\xi)(\eta-\xi)\right\rangle=0\ .$$

Proof is given by (25) in [8].

Now, we denote by $\tilde{\Sigma}^0_C$ the space of all C-valued C^∞ functions on $T^*_N\oplus\bar{D}_N$ having the following asymptotic expansions (cf. (11) in [8]):

$$(16)\qquad a(x;\xi,X)\sim a_0(x;\hat{\xi},X)+\cdots+a_{-j}(x;\hat{\xi},X)r^{-j}+\cdots\ ,$$

where $r=|\xi|$, $\hat{\xi}=(1/r)\xi$ and $\bar{D}_N$ is the unit disk bundle in T_N. Let $\mu(r)$ be a C^∞ non-decreasing function on $[0,\infty)$ such that $\mu(r)\equiv1$ on $[0,1]$ and $\mu(r)\equiv r$ on $[2,\infty)$. For any real β, we set $\tilde{\Sigma}^\beta_C=\tilde{\Sigma}^0_C\cdot\mu(r)^\beta$.

Let $\varphi=(\varphi_1;\varphi_2)$ be an element of $\mathscr{D}^{(1)}_\Omega$. For a function $b(x;\xi,Y)\in\tilde{\Sigma}^\beta_C$, we set (cf. (13) in [8])

$$(17)\qquad a(x;\xi,Y)=b(\varphi_1(x;\xi);\varphi_2(x;\xi),Y)\ ,$$

$a(x;\xi,Y)$ is a C^∞ function on the pull-back bundle of T_N by the mapping φ_1: $T^*_N-\{0\}\to N$. We denote by $\tilde{\Sigma}^\beta_\varphi$ the space of all such functions. For a cut off function ν and $u\in C^\infty(N)$, we set

$$(18)\qquad (\nu u)^\cdot(y;Y)=\nu(y,\cdot_yY)u(\cdot_yY)\ .$$

Our Fourier-integral operators given in the previous paper [8] are written in the following form (cf. (15) in [8]):

$$(19)\quad (F(a,\varphi,\nu)u)(x)$$
$$=\sum_\alpha\int_{T^*_x}\int_{T\varphi_1(x;\xi)}\lambda_\alpha(x;\xi)a(x;\xi,X)e^{-i\langle\varphi_2(x;\xi)|X\rangle-i|\xi|A_\alpha(X)}(\nu u)^\cdot(\varphi_1(x;\xi);X)dXd\xi$$
$$+(K\circ u)(x)\ ,$$

where $\varphi \in \mathscr{D}_{\Omega}^{(1)}$, $a \in \tilde{\Sigma}_{\varphi}^{\beta}$, $\{\lambda_{\alpha}(x; \xi)\}$ is a suitable partition of unity on $T_N^* - \{0\}$ by positively homogeneous functions of order 0, $A_{\alpha}(X)$'s are suitable quadratic forms in X depending on φ, and $K\circ$ is a smoothing operator with a smooth kernel $K(x, y) \in C^{\infty}(N \times N)$.

REMARK. Although the amplitude function a is restricted in $\tilde{\Sigma}_{\varphi}^{\beta}$ in [8], it is easy to see the above operator (19) is still well-defined for $a \in \tilde{S}_{1,0,\varphi}^{\beta}$ where $\tilde{S}_{1,0,\varphi}^{\beta}$ is the space defined by the same manner as in (17) from the space $\tilde{S}_{1,0}^{\beta}$ of Hörmander [3]. The whole arguments in [8] except Theorem 5.8 are valid for this class.

Now, note that the group $\mathscr{D}(N)$ of all C^{∞} diffeomorphism of N can be imbedded naturally in $\mathscr{D}_{\Omega}^{(1)}$. The imbedded image $\tilde{\psi}$ of $\psi \in \mathscr{D}(N)$ is given by

$$\tilde{\psi}(x; \xi) = (\psi(x); (d\psi^{-1})_x^* \xi) . \tag{20}$$

If φ in (19) is sufficiently close to the subgroup $\mathscr{D}(N)$, then one can choose $A_{\alpha} \equiv 0$. (Cf. (33) in [8]. See also between (17) and (18) in [8].) Therefore in this case, (19) can be rewritten in the form

$$\begin{aligned} &(F(a, \varphi, \nu)u)(x) \\ &= \int_{T_x^*} \int_N a'(x; \xi, X) e^{-i\langle \varphi_2(x;\xi) | \varphi_1(x;\xi) \cdot z \rangle} \nu(\varphi_1(x; \xi), z) u(z) đ z đ \xi \\ &\quad + (K \circ u)(x) , \qquad z = {}_{\cdot \varphi_1(x;\xi)} X , \end{aligned} \tag{21}$$

where $a' = a(đX/đz)$.

If moreover, φ is sufficiently close to the identity, then one can eliminate the variables X in $a'(x; \xi, X)$ in the expression (21) by replacing a' by some other $a(x; \xi) \in \sum_C^{\beta}$ (cf. §5 in [8]), where $\sum_C^{\beta} = \sum_C^0 \mu(r)^{\beta}$. Therefore, one can rewrite (21) by the same form as in (6):

$$(Fu)(x) = \int_{T_x^*} a(x; \xi) (\widetilde{\nu u})(\varphi(x; \xi)) đ \xi + (K \circ u)(x) , \tag{22}$$

where $(\widetilde{\nu u})(y; \eta)$ is given by (7).

By using the above expression, we can give a unique expression which is stated in the introduction.

§2. Compositions of Fourier-integral operators, I.

In this section, we consider two Fourier-integral operators

$$\begin{aligned} &(Fv)(x) \\ &= \sum_{\alpha} \int_{T_x^*} \int_{T_{\varphi_1(x;\xi)}} \lambda_{\alpha}(x; \xi) a(x; \xi, X) e^{-i\langle \varphi_2(x;\xi) | X \rangle - i|\xi| A_{\alpha}(X)} (\nu v)^{\cdot}(\varphi_1(x; \xi), X) đ X đ \xi , \end{aligned}$$

$$(Gu)(y)=\int_{T_y^*} b(y;\eta)(\widetilde{\nu u})(\psi(y;\eta))d\eta\ ,$$

where $\varphi, \psi \in \mathscr{D}_\Omega^{(1)}$, and ψ is assumed to be sufficiently close to the identity. At first, let us fix its precise meaning. Recall that $\mathscr{D}_\omega(S_N^*)$ is a strong ILH-Lie group (cf. [6], [7]). Hence for any $k\geqq 2\dim N+5$ the completion $\mathscr{D}_\omega^k(S_N^*)$ under the H^k-topology is a topological group and a C^∞-Hilbert manifold. Remark that there is an isomorphism $\wedge$ of $\mathscr{D}_\Omega^{(1)}$ onto $\mathscr{D}_\omega(S_N^*)$ (cf. Lemma 1.6 [8]). Therefore, $\mathscr{D}_\Omega^{(1)}$ has the same properties as $\mathscr{D}_\omega(S_N^*)$ through the above isomorphism. Thus, that ψ is sufficiently close to the identity means, unless otherwise stated, that ψ is contained in a small neighborhood $\mathfrak{U}$ of the identity H^{2n+7}-topology, where $n=\dim N$. In this series of papers, we shall often use the fact that $\mathscr{D}_\omega^k(S_N^*)$, $k\geqq 2n+5$, is a C^∞-Hilbert manifold and a topological group.

Let $\hat{\psi}\in\mathscr{D}_\omega(S_N^*)$. We define $f\in C^\infty(S_N^*)$ by $\hat{\psi}^*\omega=f\omega$, where ω is the naturally defined contact form on S_N^*. From $\hat{\psi}$, $\psi\in\mathscr{D}_\Omega^{(1)}$ is defined as follows:

$$(23)\qquad (\psi_1(y;\eta);\psi_2(y;\eta))=\left(\hat{\psi}_1\Big(y;\frac{\eta}{|\eta|}\Big);\ |\eta|\,f^{-1}\Big(y;\frac{\eta}{|\eta|}\Big)\hat{\psi}_2\Big(y;\frac{\eta}{|\eta|}\Big)\right).$$

Remark that H^{2n+7}-topology is stronger than the C^3-topology by Sobolev's lemma. Thus, if $\hat{\psi}$ is sufficiently close to the identity, then $|f-1|$, $\|df\|$ and $\|d^2f\|$ are sufficiently close to 0. Hence, for an arbitrarily fixed compact subset K in $T_N^*-\{0\}$, the second jet of ψ is sufficiently close to that of the identity mapping on K.

Now, recall how $A_\alpha(X)$'s are defined. There are finite number of points $\{(\bar{x}_\alpha;\bar{\xi}_\alpha)\}$ in S_N^*. Set $(\bar{y}_\alpha;\bar{\eta}_\alpha)=\varphi(\bar{x}_\alpha;\bar{\xi}_\alpha)$. Then the quadratic form $A_\alpha(X)=A_\alpha(\varphi_1(x;\xi);X)$ is defined so that the function $\phi_\alpha(x;\xi|Y_1)$ defined by

$$(24)\qquad \begin{cases}\phi_\alpha(x;\xi|Y_1)=\langle\varphi_2(x;\xi)|X\rangle+|\xi|\,A_\alpha(\varphi_1(x;\xi);X)\\ X=S(\bar{y}_\alpha;Y_1,\bar{Y}_0(x;\xi))\ , \qquad \cdot_{\bar{y}_\alpha}\bar{Y}_0(x;\xi)=\varphi_1(x;\xi)\end{cases}$$

is non-degenerate phase function (cf. Definition 2.2 [8]) on $U'_\alpha\times V'_\alpha$, where U'_α is a conic neighborhood of $(\bar{x}_\alpha;\bar{\xi}_\alpha)$ in $T_N^*-\{0\}$ and V'_α is a neighborhood of 0 in $T_{\bar{y}_\alpha}$ satisfying certain properties stated in the first part of §3 in [8].

Let φ_0 be an arbitrarily fixed element in $\mathscr{D}_\Omega^{(1)}$. Then, it is easy to see that there are a neighborhood $\mathfrak{V}_{\varphi_0}$ of φ_0 and a neighborhood $\mathfrak{U}$ of the identity in the C^1-uniform topology such that for every $\varphi\in\mathfrak{V}_{\varphi_0}$, $\psi\in\mathfrak{U}$,

$$(25)\qquad \begin{cases}\phi'_\alpha(x;\xi\,|\,Y_1)=\langle\psi_2(\varphi(x;\xi))\,|\,Z\rangle+|\xi|\,A_\alpha(\psi_1(\varphi(x;\xi));Z)\\ Z=S(\bar{y}_\alpha;\,Y_1,\,\bar{Y}_0(x;\xi))\,,\qquad \cdot_{\bar{y}_\alpha}\bar{Y}_0(x;\xi)=\psi_1(\varphi(x;\xi))\end{cases}$$

is a non-degenerate phase function on $U'_\alpha\times V'_\alpha$ for each α, where U'_α, V'_α are same neighborhoods as in (24) replaced φ by φ_0.

Now, the purpose of §§2-3 is to compute $(FGu)(x)$. So, we set $v=Gu$, $y=\cdot_{\varphi_1(x;\xi)}X$. Note that

$$(\nu Gu)^{\cdot}(\varphi_1(x;\xi);X)=\nu(\varphi_1(x;\xi),y)(Gu)(y)\,,$$

and

$$(26)\quad (Gu)(y)$$
$$=\int_{T_y^*}\int_N b(y;\eta)e^{-i\langle\psi_2(y;\eta)|Y\rangle}\nu(\psi_1(y;\eta),z)u(z)dzd\eta\,,\qquad (z=\cdot_{\psi_1(y;\eta)}Y)$$
$$=\int_{T_y^*}\int_{T\psi_1(y;\eta)} b(y;\eta)\frac{dz}{dY}e^{-i\langle\psi_2(y;\eta)|Y\rangle}\nu(\psi_1(y;\eta),\,\cdot_{\psi_1(y;\eta)}Y)u(\cdot_{\psi_1(y;\eta)}Y)dYd\eta\,.$$

Thus, we get

$$(27)\qquad (FGu)(x)$$
$$=\sum_\alpha\int_{T_x^*}\int_{T\varphi_1(x;\xi)}\int_{T_y^*}\int_{T\psi_1(y;\eta)} c_\alpha e^{-i\phi_\alpha}u(\cdot_{\psi_1(\cdot_{\varphi_1(x;\xi)}X;\eta)}Y)dYd\eta dXd\xi\,,$$

where

$$(28)\qquad \begin{cases} y=\cdot_{\varphi_1(x;\xi)}X\\ \phi_\alpha=\langle\varphi_2(x;\xi)\,|\,X\rangle+|\xi|\,A_\alpha(X)+\langle\psi_2(y;\eta)\,|\,Y\rangle\\ c_\alpha=(\lambda_\alpha a)(x;\xi,X)\nu(\varphi_1(x;\xi),\,\cdot_{\varphi_1(x;\xi)}X)b(y;\eta)\dfrac{dz}{dY}\nu(\psi_1(y;\eta),\,\cdot_{\psi_1(y;\eta)}Y)\,.\end{cases}$$

Remark that the operators considered here are defined precisely by using osillatory integrals (cf. [8]). Therefore, one can change the order of integrations by using standard Fubini's theorem.

Let ε be the breadth of the cut off function ν. Then, in (28) one may assume that $|X|$, $|Y|$ are less then $2\varepsilon/3$ because otherwise $c_\alpha=0$. Therefore, one may regard ϕ_α as a globally defined function by multiplying a suitable cut off function. In what follows we treat $A_\alpha=A_\alpha(\varphi_1(x;\xi);X)$ also as a globally defined quadratic form such that $A_\alpha\equiv 0$ if $(x;\xi)\notin U'_\alpha$. Now, recall the definition of δ_0 (cf. (27) in [8]). This was a positive number related to the Lebesgue number of $\{U'_\alpha\}$. It is not hard to see that if φ_0 is sufficiently close to the identity, then δ_0 can be chosen to be sufficiently close to the half of the injectivity radius r_0 of N.

Suppose $\varepsilon < r_0/12$. Since ψ is sufficiently close to the identity, we may suppose

$$\rho(\psi_1(\varphi(x;\xi)),\ \psi_1(\cdot_{\varphi_1(x;\xi)}X;\eta)) < 3\varepsilon \tag{29}$$

for every $(x;\xi) \in T_N^* - \{0\}$, X such that $|X| < 2\varepsilon/3$ and $\eta \in T_\nu^* - \{0\}$, where ρ is the distance function. Hence, we set

$$\psi_1(\cdot_{\varphi_1(x;\xi)}X;\eta) = \cdot_{\psi_1(\varphi(x;\xi))}Z_1\ , \qquad |Z_1| < 3\varepsilon\ . \tag{30}$$

Z_1 is a C^∞ function of $(x;\xi)$ and X, η such that $Z_1 = 0$ if $X = 0$ and $\eta = \varphi_2(x;\xi)$. Obviously, if $\psi = \mathrm{id}$, then $Z_1 = X$.

Using these notations, we have

$$\cdot_{\psi_1(\cdot_{\varphi_1(x;\xi)}X;\eta)}Y = \dot{\cdot}_{\psi_1(\varphi(x;\xi))Z_1}Y\ . \tag{31}$$

Thus, putting $\dot{\cdot}_{\psi_1(\varphi(x;\xi))Z_1}Y = \cdot_{\psi_1(\varphi(x;\xi))}Z$, we get $|Z| < 11\varepsilon/3$ and

$$Y = S(\psi_1(\varphi(x;\xi));\ Z,\ Z_1)\ . \tag{32}$$

If $\psi = \mathrm{id}$, then $Y = S(\varphi_1(x;\xi);\ Z,\ X)$.

Therefore, (27) is rewritten in the form

$$(FGu)(x) = \sum_\alpha \iiiint c'_\alpha e^{-i\phi_\alpha} u(\cdot_{\psi_1(\varphi(x;\xi))}Z)\, đZ\, đ\eta\, dX\, đ\xi\ , \tag{33}$$

where

$$\begin{aligned}\phi_\alpha &= \phi_\alpha(x;\xi\,|\,Z,\eta,X) \\ &= \langle\varphi_2(x;\xi)\,|\,X\rangle + |\xi|\,A_\alpha(\varphi_1(x;\xi);\ X) \\ &\quad + \langle\psi_2(\cdot_{\varphi_1(x;\xi)}X;\eta)\,|\,S(\psi_1(\varphi(x;\xi));\ Z,\ Z_1)\rangle\ .\end{aligned} \tag{34}$$

If $\psi = \mathrm{id}$, then $\phi_\alpha = \langle\varphi_2(x;\xi)\,|\,X\rangle + |\xi|\,A_\alpha(\varphi_1(x;\xi);\ X) + \langle\eta\,|\,S(\varphi_1(x;\xi); Z,\ X)\rangle$. In this section, we are mainly concerned with critical points of the above function ϕ_α for each fixed $(x;\xi\,|\,Z)$. Note that

$$\phi_\alpha(x; r\xi\,|\,Z, r\eta, X) = r\phi_\alpha(x;\xi\,|\,Z,\eta,X)\ , \qquad r > 0\ . \tag{35}$$

Hence, if $X = \tau_1(x;\xi\,|\,Z)$, $\eta = \tau_2(x;\xi\,|\,Z)$ is a critical point of $\phi_\alpha(x;\xi\,|\,Z, *, *)$, then

$$X = \tau_1(x;\xi\,|\,Z)\ , \qquad \eta = r\tau_2(x;\xi\,|\,Z)\ , \qquad r > 0\ ,$$

is a critical point of $\phi_\alpha(x; r\xi\,|\,Z, *, *)$.

REMARK. In the next section, we shall prove that the critical point $(\tau_1(x;\xi\,|\,Z);\ \tau_2(x;\xi\,|\,Z))$ is unique for every $(x;\xi\,|\,Z)$. In this section, we shall only show the existence of a critical point and a property of a critical value.

Now, we use the following normal coordinate expressions

$$\begin{cases} {}_{\cdot\varphi_1(x;\xi)}(X, \tilde{\eta}) = ({}_{\cdot\varphi_1(x;\xi)}X; \eta) \\ {}_{\cdot\varphi_1(x;\xi)}(X, \tilde{S}) = ({}_{\cdot\varphi_1(x;\xi)}X; S) . \end{cases}$$

LEMMA 2.1. *If* $\psi = \mathrm{id}$, *then* $\tau_1(x; \xi | Z) = Z$, $\tau_2(x; \xi | Z) = \varphi_2(x; \xi) + 2|\xi| A'_\alpha(Z)$, *where* $A'_\alpha(Z)$ *is the derivative* $(\partial/\partial Z) A_\alpha(\varphi_1(x; \xi); Z)$. *Moreover,*

$$\begin{aligned} \phi_\alpha = {} & \langle \varphi_2(x; \xi) | Z \rangle + |\xi| A_\alpha(Z) \\ & + \langle \tilde{\eta} - \varphi_2(x; \xi) - 2|\xi| A'_\alpha(Z) | \tilde{S}(\varphi_1(x; \xi); Z, X) \rangle \\ & + |\xi| A_\alpha(X-Z) + \langle \varphi_2(x; \xi) + 2|\xi| A'_\alpha(Z) | q(x; \xi | Z, X-Z) \rangle , \end{aligned}$$

where $q(x; \xi | Z, X-Z) = O(|X-Z|^2)$, *whenever* $|Z| < 11\varepsilon/3$, $|X| < 2\varepsilon/3$.

PROOF. By (34) and the remark there, we see

$$\frac{\partial \phi_\alpha}{\partial X} = \langle \varphi_2(x; \xi) | * \rangle + 2|\xi| A'_\alpha(X)(*) + \left\langle \eta \left| \frac{\nabla S}{\partial Z_1}(\varphi_1(x; \xi); Z, X)(*) \right. \right\rangle$$

$$\frac{\partial \phi_\alpha}{\partial \eta} = \langle * | S(\varphi_1(x; \xi); Z, X) \rangle .$$

Hence, solving $\partial\phi_\alpha/\partial\eta = 0$, we get $X = Z$. Substitute this to the first equality. Since $(\nabla S/\partial Z_1)(\varphi_1(x; \xi); Z, Z) = -I$ (cf. Lemma 2.4 and (5) in [8]), $\partial\phi_\alpha/\partial X = 0$ implies $\eta - \varphi_2(x; \xi) = 2|\xi| A'_\alpha(Z)$. To get the second assertion, remark that

$$\langle \varphi_2(x; \xi) | X \rangle = \langle \varphi_2(x; \xi) | Z \rangle + \langle \varphi_2(x; \xi) | X - Z \rangle .$$

$$A_\alpha(X) = A_\alpha(Z) + 2A'_\alpha(Z)(X-Z) + A_\alpha(X-Z) .$$

Setting

$$\tilde{S}(\varphi_1(x; \xi); Z, X) = Z - X + q(x; \xi | Z, X-Z) \tag{36}$$

by using (13), we get the desired result.

Since $S(\varphi_1(x; \xi); Z, Z) \equiv 0$, we may set (cf. (4) in [8])

$$\tilde{S}(\varphi_1(x; \xi); Z, X) = \tilde{S}_1(\varphi_1(x; \xi); Z, X)(Z-X) \qquad \text{(cf. (10))} .$$

Moreover, one can set

$$\begin{cases} A_\alpha(X-Z) = \dfrac{1}{2} A'_\alpha(X-Z)(X-Z) \\ q(x; \xi | Z, X-Z) = q_1(x; \xi | Z, X-Z)(X-Z) . \end{cases} \tag{37}$$

Hence, ϕ_α in the above lemma can be rewritten as

$$\phi_\alpha=\langle\varphi_2(x;\xi)|Z\rangle+|\xi|A_\alpha(Z)-\langle\eta''|X-Z\rangle\ , \tag{38}$$

where

$$\begin{aligned}\eta''=&(\tilde{\eta}-\varphi_2(x;\xi)-2|\xi|A'_\alpha(Z))\tilde{S}_1(\varphi_1(x;\xi);Z,X)\\&-\frac{1}{2}|\xi|A'_\alpha(X-Z)-(\varphi_2(x;\xi)+2|\xi|A'_\alpha(Z))q_1(x;\xi|Z,X-Z)\ .\end{aligned} \tag{39}$$

Now, we shall relax the condition $\psi=\mathrm{id}$, and we want to get a critical value of ϕ_α. For this purpose, Taylor's expansion at $(X,\eta)=(Z,\varphi_2(x;\xi)+2|\xi|A'_\alpha(Z))$ will be discussed at the first stage. However, strictly speaking, $\varphi_2(x;\xi)+2|\xi|A'_\alpha(Z)$ does not make sense, because the base points of Z and $\varphi_2(x;\xi)$ are different, if $\psi\neq\mathrm{id}$. Thus, we have to consider a normal coordinate expression of ϕ_α around $z=\varphi_1(x;\xi)$.
Set as follows:

$$\begin{cases}\cdot_z(\tilde{\psi}_1(\cdot_z(X,\tilde{\eta})),\tilde{\psi}_2(\cdot_z(X,\tilde{\eta})))=(\psi_1(\cdot_z(X,\tilde{\eta}));\psi_2(\cdot_z(X,\tilde{\eta})))\\ \cdot_z(\tilde{\psi}_1(\varphi(x;\xi)),\tilde{Z})=(\psi_1(\varphi(x;\xi));Z)\\ \cdot_z(\tilde{\psi}_1(\varphi(x;\xi)),\tilde{Z}_1)=(\psi_1(\varphi(x;\xi));Z_1)\\ \cdot_z(\tilde{W}(y;Y),\tilde{S}(y;Z,Y))=(\cdot_yY;S(y;Z,Y))\ ,\end{cases} \tag{40}$$

where $\cdot_z(X,\tilde{\eta})=(\cdot_zX;\eta)$.

Remark that (34) can be written as

$$\begin{aligned}\phi_\alpha&=\phi_\alpha(x;\xi|\tilde{Z},\tilde{\eta},X)\\&=\langle\varphi_2(x;\xi)|X\rangle+|\xi|A_\alpha(X)\\&\quad+\langle\tilde{\psi}_2(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta}))|\tilde{S}(\psi_1(\varphi(x;\xi));Z,Z_1)\rangle\ .\end{aligned} \tag{41}$$

Now, set

$$\phi_\alpha=g_0(x;\xi|\tilde{Z})+g_1(x;\xi|\tilde{Z})X'+h_1(x;\xi|\tilde{Z})\tilde{\eta}'+\rho(x;\xi|Z,\tilde{\eta}',X')\ , \tag{42}$$

where $X'=X-\tilde{Z}$. $\tilde{\eta}'=\tilde{\eta}-\varphi_2(x;\xi)-2|\xi|A'_\alpha(\tilde{Z})$ and ρ is the remainder term.

LEMMA 2.2. $\tilde{Z}_1=\tilde{Z}_1(x;\xi|\tilde{\eta},X)=\tilde{S}(z;\tilde{\psi}_1(\cdot_z(X,\tilde{\eta})),\tilde{\psi}_1(\varphi(x;\xi)))$, *where* $z=\varphi_1(x;\xi)$. *If we set* $\mathfrak{X}_0=\tilde{Z}_1(x;\xi|\varphi_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z}),\tilde{Z})$, *then there is a matrix* $M=M(x;\xi)$ *such that* $\mathfrak{X}_0(x;\xi|Z)=M\tilde{Z}+O(|\tilde{Z}|^2)$, *whenever* $|Z|<11\varepsilon/3$, *and that* $M(x;\xi)$ *is sufficiently close to the identity matrix, provided that* ψ *is sufficiently close to the identity.*

PROOF. By definition, $\cdot_{\psi_1(\varphi(x;\xi))}Z_1=\cdot_z\tilde{\psi}_1(\cdot_z(X,\tilde{\eta}))$. Remark that

$\psi_1(\varphi(x;\xi))={}_{\cdot z}\tilde{\psi}_1(\varphi(x;\xi))$. Hence

$$\cdot{}_{\cdot z}\tilde{\psi}_{1(\varphi(x;\xi))}Z_1={}_{\cdot z}\tilde{\psi}_1({}_{\cdot z}(X,\tilde{\eta}))\ .$$

Therefore

$$Z_1=S(\varphi_1(x;\xi);\tilde{\psi}_1(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta})),\tilde{\psi}_1(\varphi(x;\xi)))\ . \tag{43}$$

Thus, taking the normal coordinate expression at z, we get the first equality for $\tilde{Z}_1$.

Remark that $\tilde{\lambda}_0=\tilde{S}(z;\tilde{\psi}_1(\cdot_z(\tilde{Z},\varphi_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z}))),\tilde{\psi}_1(\varphi(x;\xi)))$. Since $\cdot_z(\tilde{Z},\varphi_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z}))=\varphi(x;\xi)$ if $\tilde{Z}=0$, we have $\tilde{\lambda}_0|_{\tilde{Z}=0}=0$. Moreover, it is not hard to see that

$$\frac{\partial\tilde{\lambda}_0}{\partial\tilde{Z}}\Big|_{\tilde{Z}=0}=\frac{\partial\tilde{\psi}_1}{\partial X}(\varphi(x;\xi))+\frac{\partial\tilde{\psi}_1}{\partial\tilde{\eta}}(\varphi(x;\xi))(2|\xi|A'_\alpha(*)) \tag{44}$$

by using Lemma 1.1. Since ψ is sufficiently close to the identity, we see that

$$\left\|\frac{\partial\tilde{\psi}_1}{\partial X}-I\right\|\ ,\qquad |\xi|\left\|\frac{\partial\tilde{\psi}_1}{\partial\tilde{\eta}}\right\|$$

are sufficiently close to 0 uniformly on $T^*_N-\{0\}$. Thus, we get the second assertion.

LEMMA 2.3. *Notations being as above, $g_0(x;\xi|\tilde{Z})$ is given as follows:*

$$g_0(x;\xi|\tilde{Z})=\langle\psi_2(\varphi(x;\xi))|Z\rangle+|\xi|A_\alpha(\psi_1(\varphi(x;\xi));Z)+a'_0(x;\xi|Z)Z^2\ ,$$

where $a'_0(x;\xi|Z)$ is a quadratic form satisfying the following: For any $\delta>0$ there is a neighborhood $\mathfrak{U}$ of the identity in $\mathscr{D}_\omega(S^*_N)$ such that if $\hat{\psi}\in\mathfrak{U}$, then $\|a'_0(x;\xi|Z)\|<\delta|\xi|$ for every $(x;\xi)\in T^*_N-\{0\}$ and Z such that $|Z|<11\varepsilon/3$, provided that ε is sufficiently small.

PROOF. By (41), we have

$$\begin{aligned}g_0(x;\xi|\tilde{Z})=&\langle\varphi_2(x;\xi)|\tilde{Z}\rangle+|\xi|A_\alpha(\tilde{Z})\\&+\langle\tilde{\psi}_2(\cdot_{\varphi_1(x;\xi)}(\tilde{Z},\tilde{\varphi}_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z})))|\tilde{S}(\psi_1(\varphi(x;\xi));Z,\lambda_0)\rangle\end{aligned}$$

where $\lambda_0=Z_1(x;\xi|\varphi_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z}),\tilde{Z})$. Note that

$$\begin{aligned}&\tilde{\psi}_2(\cdot_{\varphi_1(x;\xi)}(\tilde{Z},\tilde{\varphi}_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z})))\\&\qquad=\tilde{\psi}_2(\varphi(x;\xi))+c(x;\xi|\tilde{Z})\tilde{Z}\ ,\end{aligned} \tag{45}$$

where $c(x;\xi|\tilde{Z})$ is a linear form such that $|\xi|^{-1}\|c(x;\xi|\tilde{Z})\|$ is bounded, if

$|Z|<11\varepsilon/3$. By (13), one may set

$$\tilde{S}(\psi_1(\varphi(x;\xi)); Z, \lambda_0) \\ =\tilde{Z}-\tilde{\lambda}_0+\tilde{Q}(x;\xi|\tilde{Z},\tilde{\lambda}_0)(\tilde{Z}-\tilde{\lambda}_0)^2 , \tag{46}$$

where $\tilde{Q}$ is a quadratic form such that $\|\tilde{Q}(x;\xi|\tilde{Z},\tilde{\lambda}_0)\|$ is bounded if $|Z|<11\varepsilon/3$. On the other hand, by Lemma 2.2

$$\tilde{\lambda}_0=\frac{\partial\tilde{\psi}_1}{\partial\tilde{Z}}(\varphi(x;\xi))\tilde{Z}+\frac{\partial\tilde{\psi}_1}{\partial\tilde{\eta}}(\varphi(x;\xi))2|\xi|A'_\alpha(\tilde{Z})+b(x;\xi|\tilde{Z})\tilde{Z}^2 \tag{47}$$

where $b(x;\xi|\tilde{Z})$ is a quadratic form, satisfying that for any $\delta>0$, there is a neighborhood $\mathfrak{U}$ of the identity such that if $\hat{\psi}\in\mathfrak{U}$, then $\|b(x;\xi|\tilde{Z})\|<\delta$. Moreover, one may assume $\|M(x;\xi)-I\|<\delta$ (cf. Lemma 2.2).

By Lemma 1.2, we see that

$$\left\langle \psi_2(\varphi(x;\xi))\middle|\frac{\partial\tilde{\psi}_1}{\partial\tilde{Z}}(\varphi(x;\xi))\tilde{Z}+\frac{\partial\tilde{\psi}_1}{\partial\tilde{\eta}}(\varphi(x;\xi))2|\xi|A'_\alpha(\tilde{Z})\right\rangle=\langle\varphi_2(x;\xi)|\tilde{Z}\rangle . \tag{48}$$

Hence

$$\begin{aligned} g_0(x;\xi|\tilde{Z})=&\langle\tilde{\psi}_2(\varphi(x;\xi))|\tilde{Z}\rangle+|\xi|A_\alpha(\tilde{Z})-\langle\tilde{\psi}_2(\varphi(x;\xi))|b(x;\xi|\tilde{Z})\tilde{Z}^2\rangle \\ &+\langle\tilde{\psi}_2(\varphi(x;\xi))|\tilde{Q}(x;\xi|\tilde{Z})(\tilde{Z}-\tilde{\lambda}_0)^2\rangle \\ &+\langle c(x;\xi|\tilde{Z})\tilde{Z}|\tilde{Z}-\tilde{\lambda}_0+\tilde{Q}(x;\xi|\tilde{Z})(\tilde{Z}-\tilde{\lambda}_0(\tilde{Z}))^2\rangle . \end{aligned}$$

Remark that $\langle\tilde{\psi}_2(\varphi(x;\xi))|\tilde{Z}\rangle=\langle\psi_2(\varphi(x;\xi))|Z\rangle$, and

$$|A_\alpha(\varphi_1(x;\xi);\tilde{Z})-A_\alpha(\psi_1(\varphi(x;\xi));Z)|$$

is sufficiently close to 0 if $|Z|<11\varepsilon/3$ as quadratic forms. Therefore by the estimates in (45), (46), (47), we get desired result.

LEMMA 2.4. *Notations being as in* (42), *we have*

$$g_1(x;r\xi|\tilde{Z})=rg_1(x;\xi|\tilde{Z}) , \qquad h_1(x;r\xi|\tilde{Z})=h_1(x;\xi|\tilde{Z}) , \qquad r>0 ,$$

and $g_1(x;\xi|0)=0$, $h_1(x;\xi|0)=0$. *Moreovr, for any* δ *there is a neighborhood* $\mathfrak{U}$ *of the identity such that if* $\hat{\psi}\in\mathfrak{U}$ *then*

$$\|g_1(x;\xi|\tilde{Z})\|<(\delta+2\|A'_\alpha\|)|\xi||\tilde{Z}| , \qquad \|h_1(x;\xi|\tilde{Z})\|<\delta|Z| ,$$

whenever $|Z|<11\varepsilon/3$, *provided that* ε *is sufficiently small.*

PROOF. Remark that $\tilde{S}(\psi_1(\varphi(x;\xi)); Z, \lambda_0)$ can be regarded also as a function of $\tilde{Z}$, $\tilde{\lambda}_0$ instead of Z, λ_0. We denote this by $\tilde{S}(\psi_1(\varphi(x;\xi)); \tilde{Z}, \tilde{\lambda}_0)$. By using (41), we see, by putting $*=\varphi_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z})$ that

$$(49)\qquad g_1(x;\xi\,|\,\tilde{Z})X' = \langle \varphi_2(x;\xi)\,|\,X'\rangle + 2|\xi|A'_\alpha(\tilde{Z})(X')$$
$$+\left\langle \frac{\partial\tilde{\psi}_2}{\partial X}(\cdot_{\varphi_1(x;\xi)}(\tilde{Z},*))X'\,\middle|\,\tilde{S}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{\lambda}_0)\right\rangle$$
$$+\left\langle \tilde{\psi}_2(\cdot_{\varphi_1(x;\xi)}(\tilde{Z},*))\,\middle|\,\frac{\partial\tilde{S}}{\partial\tilde{Z}_1}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{\lambda}_0)\frac{\partial\tilde{Z}_1}{\partial X}(x;\xi\,|\,*,\tilde{Z})X'\right\rangle .$$

Recall Lemma 2.2, and we see

$$(50)\qquad \frac{\partial\tilde{Z}_1}{\partial X} = \frac{\partial\tilde{S}}{\partial Z}(z;\tilde{\psi}_1(\cdot_z(\tilde{Z},*)),\tilde{\psi}_1(\varphi(x;\xi)))\frac{\partial\tilde{\psi}_1}{\partial X}(\cdot_z(\tilde{Z},*)) ,\qquad z=\varphi_1(x;\xi) .$$

Recall again $*=\varphi_2(x;\xi)+2|\xi|A'_\alpha(\tilde{Z})$. Therefore

$$(51)\qquad \left.\frac{\partial\tilde{Z}_1}{\partial X}\right|_{\tilde{Z}=0} = \frac{\partial\tilde{\psi}_1}{\partial X}(\varphi(x;\xi)) .$$

Remark that

$$(52)\qquad \frac{\partial\tilde{S}}{\partial\tilde{Z}_1}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{\lambda}_0) = -I + O(|\tilde{Z}-\tilde{\lambda}_0|) .$$

Therefore, using (45), (51) and Lemma 1.2, we see that the term $\langle\varphi_2(x;\xi)\,|\,X'\rangle$ is cancelled out, and hence we get $g_1(x;\xi\,|\,0)=0$. We get the first inequality by a direct estimation.

On the other hand, we have

$$(53)\qquad h_1(x;\xi\,|\,\tilde{Z})\tilde{\eta}'$$
$$=\left\langle \frac{\partial\tilde{\psi}_2}{\partial\tilde{\eta}}(\cdot_{\varphi_1(x;\xi)}(\tilde{Z},*))\tilde{\eta}'\,\middle|\,\tilde{S}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{\lambda}_0)\right.$$
$$+\left\langle \psi_2(\cdot_{\varphi_1(x;\xi)}(\tilde{Z},*))\,\middle|\,\frac{\partial\tilde{S}}{\partial\tilde{Z}_1}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{\lambda}_0)\frac{\partial\tilde{Z}_1}{\partial\tilde{\eta}}(x;\xi\,|\,*,\tilde{Z})\tilde{\eta}'\right\rangle .$$

Remark that

$$(54)\qquad \left.\frac{\partial\tilde{Z}_1}{\partial\tilde{\eta}}\right|_{\tilde{Z}=0} = \frac{\partial\tilde{\psi}_1}{\partial\tilde{\eta}}(\varphi(x;\xi)) .$$

Then by (52), (45) and Lemma 1.2, the constant term in $\tilde{Z}$ in the second term of (53) vanishes. Thus, using (46), (47) we get the desired estimate. The first assertion is easy to prove.

Now, we consider the equations

$$(55)\qquad \begin{cases} \dfrac{\partial \phi_\alpha}{\partial X'} = g_1(x;\xi\,|\,\tilde{Z}) + \dfrac{\partial \rho}{\partial X'}(x;\xi\,|\,\tilde{Z},\tilde{\eta}',X') = 0 \\ \dfrac{\partial \phi_\alpha}{\partial \tilde{\eta}'} = h_1(x;\xi\,|\,\tilde{Z}) + \dfrac{\partial \rho}{\partial \tilde{\eta}'}(x;\xi\,|\,\tilde{Z},\tilde{\eta}',X') = 0 \end{cases} \qquad \text{(cf. (42))} .$$

Lemma 2.5. *There is a neighborhood $\mathfrak{U}$ of the identity in $\mathscr{D}_\omega(S_N^*)$ such that if $\hat{\psi} \in \mathscr{D}_\omega(S_N^*)$, then the Jacobian*

$$\frac{\partial(\partial\rho/\partial X', \partial\rho/\partial\tilde{\eta}')}{\partial(X,\tilde{\eta}')}(x;\xi\,|\,\tilde{Z},0,0)$$

does not vanish for every $(x;\xi) \in T_N^* - \{0\}$ and $Z \in T_{\psi_1(\varphi(x;\xi))}$ such that $|Z| < 11\varepsilon/3$.

Proof. Set

$$(56)\qquad \rho = \rho_{11}(x;\xi\,|\,\tilde{Z})\tilde{\eta}'^2 + 2\rho_{12}(x;\xi\,|\,\tilde{Z})\tilde{\eta}'X' + \rho_{22}(x;\xi\,|\,\tilde{Z})X'^2 + \cdots .$$

Then, the above Jacobian is given by

$$J = \det\begin{pmatrix} \rho_{11}, & \rho_{12} \\ \rho_{12}, & \rho_{22} \end{pmatrix},$$

and it is positively homogeneous of order 0 with respect to ξ. Hence we may assume that $(x;\xi)$ is contained in the compact subset S_N^*. Remark that ρ_{ij} depends continuously on $\hat{\psi}$ in the C^3-topology. If $\hat{\psi} = \mathrm{id}$, then by Lemma 2.1 we see that $\rho_{11} = 0$, $\rho_{12} = -I$ and

$$\rho_{22} = 2|\xi|A'_\alpha + \left\langle \tau_2 \middle| \begin{Bmatrix} * \\ * \ \ * \end{Bmatrix}_{\tilde{Z}} \right\rangle \qquad \text{(cf. (13))} .$$

Therefore, $J = (-1)^{\dim N}$ if $\psi = \mathrm{id}$, and hence we get the desired result.

Remark that $g_1(x;\xi\,|\,0) = 0$, $h_1(x;\xi\,|\,0) = 0$ by Lemma 2.4. Therefore, if $\tilde{Z} = 0$, then $(X',\eta') = (0,0)$ is a solution of (55). Thus, by using the implicit function theorem, we obtain the following:

Lemma 2.6. *There is a critical point $(\tau_1'(x;\xi\,|\,Z), \tau_2'(x;\xi\,|\,Z))$ of ϕ_α such that $\tau_i'(x;\xi\,|\,0) = 0$, $i = 1, 2$. Moreover, for every $\delta > 0$, there is a neighborhood $\mathfrak{U}$ of the identity such that if $\hat{\psi} \in \mathfrak{U}$ then*

$$|\tau_1'(x;\xi\,|\,\tilde{Z})| < \delta|Z| , \qquad |\tau_2'(x;\xi\,|\,\tilde{Z})| < \delta|\xi||Z| ,$$

for $|Z| < 11\varepsilon/3$, provided that ε is sufficiently small.

Proof. For each fixed $(x;\xi) \in S_N^*$, there is a neighborhood V of 0 in $T_{\psi_1(\varphi(x;\xi))}$ such that for every $Z \in V$ there is a solution (τ_1', τ_2') of (55) by using the implicit function theorem. If $Z=0$, then (τ_1', τ_2') must be $(0, 0)$ by the above argument. Remark that S_N^* is compact. Then, we obtain the first assertion by using (35).

To obtain the inequalities, remark that $\tau_1' \equiv 0$, $\tau_2' \equiv 0$ if $\psi = \mathrm{id}$. Since $\partial\rho/\partial X'$, $\partial\rho/\partial\tilde{\eta}'$ and their derivatives depend continuously on $\hat{\psi}$ in the C^3-topology, we see that $\partial\tau_i'/\partial\tilde{Z}$ depends continuously on $\tilde{Z}$ and $\hat{\psi}$. Hence we get the desired inequality.

COROLLARY 2.7. *The critical value of ϕ_α at (τ_1', τ_2') can be written in the form:*

$$\phi_\alpha(x; \xi \,|\, \tilde{Z}, \tau_2', \tau_1') = \langle \psi_2(\varphi(x; \xi)) \,|\, Z \rangle + |\xi| A_\alpha(\psi_1(\varphi(x; \xi)); Z) \\ + a_0(x; \xi \,|\, Z) Z^2 ,$$

where $a_0(x; \xi; Z)$ is a quadratic form satisfying the following: For any $\delta > 0$, there is a neighborhood of the identity $\mathfrak{U}$ in $\mathscr{D}_\omega(S_N^*)$ such that if $\hat{\psi} \in \mathfrak{U}$, then $\|a_0(x; \xi; Z)\| < \delta |\xi|$ for every $(x; \xi) \in T_N^* - \{0\}$ and Z such that $|Z| < 11\varepsilon/3$, provided that ε is sufficiently small.

PROOF. Recall that $\phi_\alpha = g_0 + g_1 X' + h_1 \eta' + \rho$. Hence the critical value of ϕ_α is given by

$$g_0(x; \xi \,|\, \tilde{Z}) + g_1(x; \xi \,|\, \tilde{Z}) \tau_1'(x; \xi \,|\, \tilde{Z}) + h_1(x; \xi \,|\, \tilde{Z}) \tau_2'(x; \xi \,|\, \tilde{Z}) \\ + \rho(x; \xi \,|\, \tilde{Z}, \tau_2', \tau_1') .$$

Thus, by Lemmas 2.3–4 and Lemma 2.6 we get the desired estimate.

§3. Compositions of Fourier-integral operators, II.

In the previous section, we have chosen a critical point (τ_1', τ_2') of ϕ_α. Using Corollary 2.7, ϕ_α can be written in the form

$$\phi_\alpha = \langle \psi_2(\varphi(x; \xi)) \,|\, Z \rangle + |\xi| A_\alpha(\psi_1(\varphi(x; \xi)); Z) + a_0(x; \xi \,|\, Z) Z^2 \\ + \tilde{h}(x; \xi \,|\, \tilde{Z}, \tilde{\eta}' - \tau_2', X' - \tau_1')(\tilde{\eta}' - \tau_2', X' - \tau_1') \tag{57}$$

where $\tilde{h}$ is a quadratic form. Set $\tilde{\eta}'' = \tilde{\eta}' - \tau_2'$, $X'' = X' - \tau_1'$, and rewrite (57) as follows:

$$\phi_\alpha = \tilde{\phi}_\alpha + \tilde{h}(x; \xi \,|\, \tilde{Z}, \tilde{\eta}'', X'')(\tilde{\eta}'', X'') , \tag{58}$$

where

$$\tilde{\phi}_\alpha = \langle \psi_2(\varphi(x; \xi)) \,|\, Z \rangle + |\xi| A_\alpha(\psi_1(\varphi(x; \xi)); Z) + a_0(x; \xi \,|\, Z) Z^2 . \tag{59}$$

Then, (33) can be rewritten as

(60) $$(FGu)(x)=\sum_{\alpha}\iint c''_{\alpha}e^{-i\tilde{\phi}_{\alpha}}u(\cdot\psi_{1(\varphi(x;\xi))}Z)đZđ\xi ,$$

where

(61) $$c''_{\alpha}=\iint c'_{\alpha}e^{-i\tilde{h}}dX''d\tilde{\eta}'' .$$

The main purpose of this section is to show the following:

PROPOSITION 3.1. *By a suitable change of coordinate system*

$$\tilde{\eta}''=\tilde{\eta}''(\tilde{\eta},\bar{X}) ,\qquad X''=X''(\tilde{\eta},\bar{X}) ,$$

$\tilde{h}$ can be changed into $-\langle\tilde{\eta}|\bar{X}\rangle$. Especially, $\phi_{\alpha}(x;\xi;Z|\tilde{\eta}',X')$ has only one critical point (τ_1,τ_2) for every $(x;\xi;Z)$.

The above proposition will be proved in several lemmas below.

By Taylor's theorem for the variables X'', $\tilde{h}$ in (58) can be written as follows:

(62) $$\tilde{h}=\tilde{h}|_{X''=0}+\frac{\partial\tilde{h}}{\partial X''}\bigg|_{X''=0}X''+R(x;\xi|\tilde{Z},\tilde{\eta}'',X'')X''^2 .$$

If we set

$$\tilde{\tau}_1=\tilde{Z}+\tau'_1 ,\qquad \tilde{\tau}_2=\varphi_2(x;\xi)+2|\xi|A'_{\alpha}(\tilde{Z})+\tau'_2 ,$$

then obviously $X''=X-\tilde{\tau}_1$ and $\tilde{\eta}''=\tilde{\eta}-\tilde{\tau}_2$ (cf. (42)). For the simplicity of notations, we denote

(63) $$\tilde{h}|_{X''=0}=P(x;\xi|\tilde{Z},\tilde{\eta}) ,\qquad \frac{\partial\tilde{h}}{\partial X''}\bigg|_{X''=0}=Q(x;\xi|\tilde{Z},\tilde{\eta}) .$$

Remark that P and Q are positively homogeneous of degree 1 (cf. (41)), and these are not differentiable at $\tilde{\eta}=0$ in general.

LEMMA 3.2. *Let $f(\tilde{\eta})$ be a C^{∞}-function defined on $\boldsymbol{R}^n-\{0\}$. Suppose f is positively homogeneous of degree* 1. *Then for any $\tilde{\tau}_2\in\boldsymbol{R}^n-\{0\}$,*

$$f(\tilde{\eta})=f(\tilde{\tau}_2)+\int_0^1\frac{\partial f}{\partial\tilde{\eta}}(\tilde{\tau}_2+t(\tilde{\eta}-\tilde{\tau}_2))dt(\tilde{\eta}-\tilde{\tau}_2)$$

for every $\tilde{\eta}\in\boldsymbol{R}^n-\{0\}$.

PROOF. The above fact is well-known Taylor's theorem, if f is C^1

on the whole space. However, the above expansion does not valid if

$$\tilde{\eta}=\lambda\tilde{\tau}_2\,,\qquad \lambda<0\,.$$

Even if this is the case, the right hand side of the above expression makes sense because $\partial f/\partial\tilde{\eta}$ is of positively homogeneous of degree 0. Therefore applying Taylor's theorem for $\tilde{\eta}+\varepsilon$ and $\tilde{\tau}_2+\varepsilon$ and take $\lim_{\varepsilon\to 0}$, we get the desired result.

By the above lemma, we can set

$$\tilde{h}=\int_0^1\frac{\partial P}{\partial\tilde{\eta}}(x;\xi\,|\,\tilde{Z},\tilde{\tau}_2+t\tilde{\eta}'')dt\tilde{\eta}''+\int_0^1\frac{\partial Q}{\partial\tilde{\eta}}(x;\xi\,|\,\tilde{Z},\tilde{\tau}_2+t\tilde{\eta}'')dt\tilde{\eta}''X'' \\ +R(x;\xi\,|\,\tilde{Z},\tilde{\eta}'',X'')X''^2\,. \tag{64}$$

LEMMA 3.3. *Notations and assumptions being as above, given $\delta>0$ and $K>0$, there is a neighborhood $\mathfrak{U}$ of the identity in $\mathscr{D}_\omega(S_N^*)$ such that if $\hat{\psi}\in\mathfrak{U}$, then*

$$\left|\int_0^1\frac{\partial P}{\partial\tilde{\eta}}(x;\xi\,|\,\tilde{Z},\tilde{\tau}_2+t\tilde{\eta}'')dt\tilde{\eta}''\right|<\delta\,|\tilde{\eta}''|\,,$$

$$\left|\int_0^1\frac{\partial Q}{\partial\tilde{\eta}}(x;\xi\,|\,\tilde{Z},\tilde{\tau}_2+t\tilde{\eta}'')dt\tilde{\eta}''X''+\langle\tilde{\eta}''\,|\,X''\rangle\right|<\delta\,|\tilde{\eta}''|\,|X''|\,,$$

$$|R(x;\xi\,|\,\tilde{Z},\tilde{\eta}'',X'')X''^2|<K(|\xi|+|\eta|)\,|X''|^2\,.$$

PROOF. Recall (41). Since $\tilde{S}(*;Z,Z_1)$ is also a function of $\tilde{Z}$, $\tilde{Z}_1$, we have only to compute

$$\langle\tilde{\psi}_2(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta}))\,|\,\tilde{S}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{S}(\varphi_1(x;\xi);\tilde{\psi}_1(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta})),\tilde{\psi}_1(\varphi(x;\xi)))))\rangle\,.$$

If we set $X=\tilde{\tau}_1$, then we get P. By Lemma 2.2 and Lemma 2.6, we see

$$\tilde{S}(\varphi_1(x;\xi);\tilde{\psi}_1(\cdot_{\varphi_1(x;\xi)}(\tilde{\tau}_1,\tilde{\eta})),\tilde{\psi}_1(\varphi(x;\xi)))$$

is sufficiently close to $\tilde{Z}$ whenever ε and hence $|\tilde{Z}|$ is sufficiently small. By Lemma 2.2, $\tilde{S}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{Z}_1)$ is sufficiently close to 0. Now, since

$$\frac{\partial P}{\partial\tilde{\eta}}=\left\langle\frac{\partial\tilde{\psi}_2}{\partial\tilde{\eta}}\,\middle|\,\tilde{S}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{Z}_1)\right\rangle+\left\langle\tilde{\psi}_2\,\middle|\,\frac{\partial\tilde{S}}{\partial\tilde{Z}_1}\cdot\frac{\partial\tilde{S}}{\partial Z}\frac{\partial\tilde{\psi}_1}{\partial\tilde{\eta}}\right\rangle,$$

and $\|\partial\tilde{\psi}_1/\partial\tilde{\eta}\|$ is sufficiently close to 0, we get the first inequality by remarking that $\|\tilde{\psi}_2(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta}))\|\cdot\|(\partial\tilde{\psi}_1/\partial\tilde{\eta})(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta}))\|$ is positively homogeneous of degree 0 with respect to $\tilde{\eta}$.

To get the second inequality, we remark $\|\partial\tilde{\psi}_2/\partial X\|\cdot\|\partial\tilde{\psi}_1/\partial\tilde{\eta}\|$ and $\|\partial\tilde{\psi}_1/\partial X-I\|$, $\|\tilde{\psi}_2\|\cdot\|\partial^2\tilde{\psi}_1/\partial\tilde{\eta}\partial X\|$, $\|\partial^2\tilde{\psi}_2/\partial\tilde{\eta}\partial X\|$ are sufficiently close to 0 uniformly. Hence we have only to estimate

$$\left\langle\frac{\partial\tilde{\psi}_2}{\partial\tilde{\eta}}(\cdot_{\varphi_1(x;\xi)}(X,\tilde{\eta}))\,\middle|\,\frac{\partial\tilde{S}}{\partial\tilde{Z}_1}(\psi_1(\varphi(x;\xi));\tilde{Z},\tilde{Z}_1)\frac{\partial\tilde{Z}_1}{\partial X}\right\rangle .$$

Since $\partial\tilde{S}/\partial\tilde{Z}_1$, $\partial\tilde{S}/\partial\tilde{Z}$ are sufficiently close to $-I$, I respectively, whenever $|Z|$ is sufficiently close to 0, we obtain easily the second inequality.

Recall that R is given by

$$R=\int_0^1(1-t)\frac{\partial^2 h}{\partial X^2}(x;\xi\,|\,\tilde{Z},\tilde{\eta},\tilde{\tau}_1+tX'')dt .$$

Hence remarking (35) and the fact that h is positively homogeneous of degree 1 with respect to $(\xi,\tilde{\eta})$, we see that $\partial^2h/\partial X^2$ inside of the integration sign is of positively homogeneous of degree 1 in $(\xi,\tilde{\eta})$. Thus, we get the third inequality.

By the similar reasoning, we also see that

$$\left\|\frac{\partial P}{\partial\tilde{\eta}}\right\|<\delta ,\qquad \left\|\frac{\partial^2 P}{\partial\tilde{\eta}^2}\right\|<\delta|\tilde{\eta}|^{-1} .$$

LEMMA 3.4. *Let $f(\tilde{\eta})$ be a C^∞-function on $\boldsymbol{R}^n-\{0\}$, positively homogeneous of degree 1. Suppose $\|\partial f/\partial\tilde{\eta}\|<\delta$, $\|\partial^2f/\partial\tilde{\eta}^2\|<\delta|\tilde{\eta}|^{-1}$. Then, there is $K>0$ such that for every $\theta=(\theta_1,\cdots,\theta_n)\in\boldsymbol{R}^n-\{0\}$,*

$$f(\tilde{\eta})=f(\theta)+\frac{\partial f}{\partial\tilde{\eta}_i}(\theta)(\tilde{\eta}_i-\theta_i)+g^{ij}(\tilde{\eta},\theta)(\tilde{\eta}_i-\theta_i)(\tilde{\eta}_j-\theta_j) ,$$

where $|g^{ij}(\tilde{\eta},\theta)|<K\delta(|\tilde{\eta}|+|\theta|)^{-1}$ and $|\ \ |$ is the norm given by an arbitrarily fixed riemannian inner product on $\boldsymbol{R}^n$.

PROOF. By a suitable choice of an orthonormal coordinate system, we may assume that $\theta=(\theta_1,0,\cdots,0)$, $\theta_1>0$.

(i) At first, we consider on the domain $D_0=\{\tilde{\eta}\in\boldsymbol{R}^n;0<\tilde{\eta}_1,|\tilde{\eta}|<3\theta_1\}$. By Taylor's theorem, we can set

$$g^{ij}(\tilde{\eta},\theta)=\int_0^1(1-t)\frac{\partial^2 f}{\partial\tilde{\eta}_i\partial\tilde{\eta}_j}(\theta+t(\tilde{\eta}-\theta))dt . \tag{65}$$

Hence

$$|g^{ij}(\eta,\theta)|<\delta\int_0^1\frac{(1-t)dt}{|\theta+t(\tilde{\eta}-\theta)|} . \tag{66}$$

Set $\tilde{\eta}=a\theta+\omega$, $\langle\omega|\theta\rangle=0$, and we have by using $0<a<3$ that

$$\frac{|\theta+t(\tilde{\eta}-\theta)|}{1-t}\geqq|\theta|>\frac{1}{15}(|\theta|+|\tilde{\eta}|)\,. \tag{67}$$

Hence

$$|g^{ij}(\tilde{\eta},\theta)|<15\delta\frac{1}{|\theta|+|\tilde{\eta}|}\,. \tag{68}$$

(ii) Next, we consider on the domain

$$D_1=\{\tilde{\eta}\in\boldsymbol{R}^n-\{0\};\,|\omega|<\sqrt{n+1}\,|\tilde{\eta}_1|\}\cap\{\tilde{\eta}\in\boldsymbol{R}^n-\{0\};\,\tilde{\eta}_1>2\theta_1 \text{ or } \tilde{\eta}_1<0\}\,.$$

By Lemma 3.2, we can set

$$f(\tilde{\eta})=f(\theta)+\frac{\partial f}{\partial\tilde{\eta}}(\tilde{\eta}-\theta)+\int_0^1\left(\frac{\partial f}{\partial\tilde{\eta}}(\theta+t(\tilde{\eta}-\theta))-\frac{\partial f}{\partial\tilde{\eta}}(\theta)\right)dt(\tilde{\eta}-\theta)\,. \tag{69}$$

Denoting the last term by $r^j(\tilde{\eta},\theta)(\tilde{\eta}_j-\theta_j)$, we see that

$$|r^j(\tilde{\eta},\theta)|<2\delta\,.$$

Set $g^{ij}(\tilde{\eta},\theta)=0$ for $i\geqq 2$ and set

$$g^{1j}(\tilde{\eta},\theta)=r^j(\tilde{\eta},\theta)/(\tilde{\eta}_1-\theta_1)\,.$$

On the domain D_1, we have easily $|\tilde{\eta}_1|+|\theta_1|\leqq 3|\tilde{\eta}_1-\theta_1|$, and hence

$$|g^{1j}(\eta,\theta)|<6\delta\frac{|\tilde{\eta}|+|\theta|}{|\tilde{\eta}_1|+|\theta_1|}\frac{1}{|\tilde{\eta}|+|\theta|}\leqq\frac{12\sqrt{n+1}\,\delta}{|\tilde{\eta}|+|\theta|}\,.$$

(iii) Let $\tilde{\eta}=a\theta+\omega$, $\langle\tilde{\eta}|\omega\rangle=0$. Consider the domain

$$D_1'=\{\tilde{\eta};\,|a|<3,\,|\tilde{\eta}_1|<|\omega|<3\sqrt{n}\,|\theta|\}\,.$$

In this domain, we use again (65) and by (66), we get

$$\begin{aligned}|g^{ij}|&<\sqrt{2}\,\delta\int_0^1\frac{dt}{\left|1-\frac{t}{1-t}|a|\right||\theta|+\frac{t}{1-t}|\omega|}\\&<\sqrt{2}\,\delta\int_0^1\frac{dt}{\left|1-\frac{t}{1-t}|a|\right|+\frac{t}{1-t}|a|}\cdot\frac{1}{|\theta|}\leqq\sqrt{2}\,\delta\frac{1}{|\theta|}\,.\end{aligned} \tag{70}$$

Using $|a|<3$ and $|\omega|<3\sqrt{n}\,|\theta|$, we get easily that $|\theta|\geqq(3\sqrt{n+1}+1)^{-1}(|\theta|+|\tilde{\eta}|)$, and hence on D_1'

(71) $$|g^{ij}(\tilde{\eta},\theta)|<(3\sqrt{n+1}+1)\sqrt{2}\frac{\delta}{|\tilde{\eta}|+|\theta|}.$$

(iv) Remark that $D_0\cup D_1\cup D_1'$ covers a neighborhood of 0. For every i, $2\leqq i\leqq n$, we set

$$D_i=\{\tilde{\eta};\ |\theta|<|\tilde{\eta}|<\sqrt{n}\,|\tilde{\eta}_i|\}.$$

On this domain we use (69). So, set $g^{aj}=0$ for $a\neq i$, and

$$g^{ij}(\tilde{\eta},\theta)=r^j(\tilde{\eta},\theta)/\tilde{\eta}_i.$$

Since $(|\theta|+|\tilde{\eta}|)/|\tilde{\eta}_i|<2\sqrt{n}$, we have

$$|g^{ij}(\tilde{\eta},\theta)|<4\sqrt{n}\,\delta\frac{1}{|\theta|+|\tilde{\eta}|}.$$

(v) Remark that $D_0\cup D_1\cup D_1'\cup(\cup_{i\geqq 2}D_i)=\boldsymbol{R}^n-\{0\}$. Hence we get the desired result by using an appropriate partition of unity.

Recall $\tilde{\eta}''=\tilde{\eta}-\tilde{\tau}_2=\tilde{\eta}-\varphi_2(x;\xi)-2|\xi|A_\alpha'(\tilde{Z})-\tau_2'(x;\xi|\tilde{Z})$. Applying the above result to our function $P(x;\xi|\tilde{Z},\tilde{\eta})$ at $\theta=\tilde{\tau}_2(x;\xi|\tilde{Z})$. Then by using Lemma 2.6 and Lemma 3.3, we obtain the following:

COROLLARY 3.5. *For any $\delta>0$, there is a neighborhood $\mathfrak{U}$ of the identity of $\mathscr{D}_\omega(S_N^*)$ such that if $\hat{\varphi}\in\mathfrak{U}$, and $|Z|<11\varepsilon/3$, then $\tilde{h}$ in* (58) *can be written as*

$$\tilde{h}=\bar{P}(x;\xi|Z,\tilde{\eta}'')\tilde{\eta}''^2+Q(x;\xi|\tilde{Z},\tilde{\eta}'')\tilde{\eta}''X''+R(x;\xi|\tilde{Z},\tilde{\eta}''\cdot X'')X''^2,$$

where $\|\bar{P}\|<\delta/(|\xi|+|\tilde{\eta}|)$, $\|Q+\langle *|*\rangle\|<\delta$, provided that ε is sufficiently small. Moreover, there is a positive constant K such that $\|R\|<K(|\xi|+|\tilde{\eta}|)$.

PROOF OF PROPOSITION 3.1. Now, write $\tilde{h}$ in the above corollary as follows:

$$\tilde{h}=\bar{P}^{ij}\tilde{\eta}_i''\tilde{\eta}_j''+\bar{Q}_k^i\tilde{\eta}_i''X''^k+R_{kl}X''^kX''^l.$$

Set $Y^i=\bar{Q}_k^iX''^k$. Since $(\bar{Q}_k^i)$ is sufficiently close to $(-\delta_k^i)$ and hence an invertible matrix, one may rewrite $\tilde{h}$ as follows:

(72) $$\tilde{h}=\bar{P}^{ij}\tilde{\eta}_i''\tilde{\eta}_j''+\tilde{\eta}_i''Y^i+\bar{R}_{kl}Y^kY^l.$$

Now, we want to find out functions $f_k^i(x;\xi|\tilde{Z},\tilde{\eta}'',Y)$, $a_{ij}(x;\xi|\tilde{Z},\tilde{\eta}'',Y)$ such that

(73) $$\begin{aligned}\tilde{h}&=-(\tilde{\eta}_i''+a_{ij}Y^j)(f_k^iY^k-\bar{P}^{il}\tilde{\eta}_l'')\\&=\bar{P}^{il}\tilde{\eta}_i''\tilde{\eta}_l''+(-f_j^l+\bar{P}^{il}a_{ij})\tilde{\eta}_l''Y^j-f_k^ia_{ij}Y^jY^k.\end{aligned}$$

Therefore, f^i_j, a_{ij} must satisfy

$$(74)\qquad \begin{cases} -f^l_j+\bar{P}^{li}a_{ij}=+\delta^l_j \\ f^i_k a_{ij}=-\bar{R}_{kj}\ . \end{cases}$$

Assume for a while that (f^i_k) is invertible. Then $a_{ij}=-(f^{-1})^k_i\bar{R}_{kj}$. Therefore, we have only to solve

$$f^l_j+\bar{P}^{li}(f^{-1})^k_i\bar{R}_{kj}=-\delta^l_j$$

within the invertible matrices. Remark that $\bar{P}^{li}\bar{R}_{kj}$ is sufficiently close to 0 uniformly because of Corollary 3.5. Hence by the implicit function theorem, we can find (f^l_j) which is uniformly close to the identity matrix.

Now, set $\bar{\eta}_i=\tilde{\eta}''_i+a_{ij}Y^j$, $X^i=f^i_kY^k-\bar{P}^{il}\tilde{\eta}''_l$ and we get the desired result.

Recall how c_α was defined in (27). c_α was a C^∞ function of $(x;\xi, X, \eta, Y)$. By (31), Y is given as a function of $(x;\xi, Z|X)$. Hence rewriting c_α as a function of $(x;\xi|Z, \eta, X)$, c'_α is given by

$$c'_\alpha=c_\alpha\frac{dY}{dZ}\ .$$

For each fixed $(x;\xi|Z)$, c'_α can be regarded as a function of $\tilde{\eta}$, X by taking a normal coordinate expression. Hence, it is not hard to see that c''_α given in (61) is well-defined and $c''_\alpha\in\sum^0_{\psi\varphi}$. By the above argument, we can conclude that FG is given by (60).

Now, recall (59) and (25). By Corollary 2.7, one may assume that

$$\tilde{\phi}''_\alpha(x;\xi|Y_1)=\langle\psi_2(\varphi(x;\xi))|Z\rangle+|\xi|A_\alpha(\psi_1(\varphi(x;\xi));Z)+a_0(x;\xi|Z)Z^2\ ,$$

$$Z=S(\bar{y}_\alpha;Y_1,\bar{Y}_0(x;\xi))\ ,\qquad {}_{\bar{y}_\alpha}\bar{Y}_0(x;\xi)=\psi_1(\varphi(x;\xi))\ ,$$

is still a non-degenerate phase function on $U'_\alpha\times V'_\alpha$ for each α. By the same method as used in [3] pp. 139–140, Lemma 3.5 [8] and [9], we obtain the following:

LEMMA 3.6. *There is a C^∞ fiber preserving diffeomorphism Φ such that $\tilde{\phi}''_\alpha(x;\Phi_2(x;\xi)|Y_1)=\phi'_\alpha(x;\xi|Y_1)$, where $\Phi(x;Y_1,\xi)=(x;\Phi_2(x;Y_1,\xi))$ and ϕ'_α is given by* (25). *Moreover, Φ depends continuously on φ, ψ under the C^∞ topology.*

PROOF. Remark that $\partial\phi'_\alpha/\partial\xi=0$ implies $Y_1=\bar{Y}_0(x;\xi)$ and that $\partial^2\phi'_\alpha/\partial Y_1\partial\xi$ is non-singular on $U'_\alpha\times V'_\alpha$ (cf. (24), (25)). Note that the above fact holds also for $\tilde{\phi}''_\alpha$. Hence by the same manner as in [3] pp. 139–140, we get the desired result. The continuity can be obtained easily, for Φ is obtained by using an implicit function theorem.

Recall the definition of c in (27), and Z, Z_1 in (29) and (31). Then, we see that c'_α in (33) vanishes identically if $|Z|>11\varepsilon/3$. Recall also the definition c''_α in (61). By the argument between (28) and (29) in [8], we see that (60) can be written by

$$(FGu)(x)=\sum_\alpha \iint c''_\alpha(x;\ \xi\,|\,Z)e^{-i\phi'_\alpha(\Phi^{-1}(x;Y_1,\xi)|Y_1)}u(\cdot_{\bar{y}_\alpha}Y_1)\frac{đZ}{đY_1}đY_1 đ\xi\ ,$$

where $Z=S(\bar{y}_\alpha;\ Y_1,\ \bar{Y}_0(x;\ \xi))$, $\cdot_{\bar{y}_\alpha}\bar{Y}_0(x;\ \xi)=\psi_1(\varphi(x;\ \xi))$. Thus, by changing variables, we see

$$\begin{aligned}(75)\qquad &(FGu)(x)\\ &=\sum_\alpha \iint \bar{c}_\alpha(x;\ \xi\,|\,Z)e^{-i\langle\psi_2(\varphi(x;\xi))|Z\rangle-i|\xi|A_\alpha(\psi_1(\varphi(x;\xi));Z)}\\ &\quad\times(\nu_1 u)^{\cdot}(\psi_1(\varphi(x;\ \xi));\ Z)đZđ\xi\ ,\end{aligned}$$

where

$$\bar{c}_\alpha=c''_\alpha(\Phi(x;\ Y_1,\ \xi)\,|\,Z)\frac{đZ}{đY_1}(\Phi(x;\ Y_1,\ \xi)\,|\,Z)\frac{đY_1}{đZ}(x;\ \xi\,|\,Z)\frac{đ\Phi_2}{đ\xi}(x;\ Y_1,\ \xi)\ ,$$

and ν_1 is a cut off function of the breadth $11\varepsilon(<\delta_0)$. It is not hard to see that $\bar{c}_\alpha\in\tilde{\Sigma}^0_{\psi\varphi}$. Thus, by replacing the cut off function ν_1 by ν, we obtain the following by using Lemma 3.3 in [8]:

$$\begin{aligned}(76)\qquad &(FGu)(x)\\ &=\sum_\alpha \iint \bar{c}_\alpha e^{-i\langle\psi_2(\varphi(x;\xi))|Z\rangle-i|\xi|A_\alpha(\psi_1(\varphi(x;\xi));Z)}\\ &\quad\times(\nu u)^{\cdot}(\psi_1(\varphi(x;\ \xi));\ Z)đZđ\xi+(K\circ u)(x)\ ,\end{aligned}$$

where K is a smoothing operator. The above equality is the desired composition rule. (Proposition A in the introduction is thereby proved.)

Now, suppose furthermore that φ and ψ are sufficiently close to the identity. Then, we can set $A_\alpha\equiv 0$, and hence we can eliminate the suffix α and $\sum_\alpha$ in the all computations in §§2–3. Moreover, one may assume that $(Fv)(x)$ in the first part of §2 can be written in the form

$$(77)\qquad (Fv)(x)=\int a(x;\ \xi)(\widetilde{\nu v})(\varphi(x;\ \xi))đ\xi\ ,$$

where $a\in\sum^0_C$. Recall that $\sum^0_C$ is a locally convex Fréchet space through the identification with $C^\infty(\bar{D}^*_N)$ (cf. between (10) and (11) in [8]).

Now, recalling how the amplitude function $\bar{c}$ of FG was computed in (28), (33), (61). Then, we obtain that $\bar{c}=c(x;\xi;Z)$ depends continuously on a, b, φ and ψ. Since one may assume that $\psi\varphi$ is still sufficiently close to the identity. FG can be rewritten in the form

$$(78)\qquad (FGu)(x)=\int c(x;\xi)(\widetilde{\nu u})(\psi\varphi(x;\xi))d\xi+(K\circ u)(x)\ ,$$

by using Proposition 4.1 in [8] (cf. see also [9]). Recall the argument in §4 in [8], and we obtain the following:

LEMMA 3.7. *Notations and assumptions being as above, $c(x;\xi)$ and $K(x, y)$ depend continuously on a, b, φ and ψ.*

PROOF. By the above argument, we see that FGu can be written in the form

$$(FGu)(x)=\int c(x;\xi)(\widetilde{\nu_1 u})(\psi\varphi(x;\xi))d\xi\ ,$$

and $c\in\sum_C^0$ depends continuously on $a, b\in\sum_C^0$ and $\varphi, \psi\in\mathscr{D}_\Omega^{(1)}$. Thus, we have only to check that

$$(79)\qquad \int c(x;\xi)(\widetilde{(\nu_1-\nu)u})(\psi\varphi(x;\xi))d\xi$$

can be written by using a smooth kernel $K(x, y)\in C^\infty(N\times N)$, which depends continuously on $a, b\in\sum_C^0$ and $\varphi, \psi\in\mathscr{D}_\Omega^{(1)}$. Now, as ψ, φ are sufficiently close to the identity, one may assume that $\rho(x, \psi_1(\varphi(x;\xi)))<\varepsilon/6$. Recall that the breadth of ν_1, ν are 11ε, ε respectively. Hence $(\nu_1-\nu)(\psi_1(\varphi(x;\xi)), z)\equiv 0$ if $\rho(x, z)<\varepsilon/6$ or $\rho(x, z)>67\varepsilon/6$. Let $\nu_2(x, z)$ be a C^∞ function such that $\nu_2(x, z)\equiv 1$ on $\varepsilon/6\leqq\rho(x, z)\leqq 67\varepsilon/6$ and $\nu_2(x, z)\equiv 0$ on $\rho(x, z)<\varepsilon/12$ and $\rho(x, z)>68\varepsilon/6$. We rewrite (79) as follows:

$$\iint \tilde{c}(x;\eta, z_1)e^{-i\langle\eta|Z_1\rangle-i\phi(x;\eta)}\nu_2(x, \cdot_x Z_1)u(\cdot_x Z_1)dZ_1 d\eta\ ,$$

by the same method in §3 [8], where $\tilde{c}\in\sum_C^0$ and $\phi\in C_R^\infty(S_N^*)r$, the space of all R-valued C^∞ functions on $T_N^*-\{0\}$ of positively homogeneous of degree 1. We give a topology on $C_R^\infty(S_N^*)r$ by using the C^∞ topology on $C_R^\infty(S_N^*)$.

It is not hard to see that $\tilde{c}$ depends continuously on a, b, φ, ψ, and ϕ depends continuously on $\varphi, \psi\in\mathfrak{N}$. Set

$$K_1(x;Z_1)=\int \tilde{c}(x;\eta, Z_1)\nu_2(x, \cdot_x Z_1)e^{-i\phi(x;\eta)}e^{-i\langle\eta|Z_1\rangle}d\eta\ .$$

Set

$$K_1(x;Z_1)=\int\frac{\tilde{c}(x;\eta,Z_1)}{(1+|\eta|)^{n+2}}\nu_2(x_1,\cdot_x Z_1)e^{-i\phi(x;\eta)}(1+|\eta|)^{n+2}e^{-i\langle\eta|Z_1\rangle}đ\eta\ ,$$

and note that $\tilde{c}(x;\eta,Z_1)$ and its derivatives by Z_1 up to the order $n+2$ are continuous with respect to a, b, φ, ψ. Thus, by integration by parts, we see that $K_1(x;Z_1)$ depends continuously on a, b, φ, ψ.

§4. Inverse.

In this section, we deal with Fourier-integral operators $F(a,\varphi,\nu)$ (cf. (77)) such that φ is sufficiently close to the identity and a is sufficiently close to 1 in the Fréchet-space topology on $\sum_C^0$. The purpose of this section is to show the following:

PROPOSITION 4.1. *Notations and assumptions being as above, $F(a,\varphi,\nu)$ is invertible. (The inverse is also a Fourier-integral operator.) Moreover, if we write $F(a,\varphi,\nu)^{-1}=F(b,\varphi^{-1},\nu)+K\circ$, then $b\in\sum_C^0$, $K\in C^\infty(N\times N)$ are continuous with respet to a, φ.*

For convenience' sake, we set $F=F(a,\varphi,\nu)$, $G=F(1,\varphi^{-1},\nu)$. Then, by the composition rule, FG is a pseudo-differential operator of order 0, which can be written in the form

$$(FGu)(x)=\int_{T_x^*}c(x;\xi)\widetilde{\nu u}(x;\xi)đ\xi+(K\circ u)(x)\ . \tag{80}$$

By Lemma 3.7, one may assume that $c(x;\xi)$ and $K(x,z)$ are sufficiently close to 1 and 0 respectively. Therefore for the proof of Proposition 4.1, we have only to show that the right hand side of (80) is invertible.

For the above purpose we have to fix at first the composition rule of pseudo-differential operators, which is much simpler than that of Fourier-integral operators.

Let $A(x;\xi)$, $B(x;\xi)\in\sum_C^0$, and let

$$\begin{cases}(Fv)(x)=\displaystyle\int_{T_x^*}\int_{T_x}A(x;\xi)\frac{đz}{đX}e^{-i\langle\xi|X\rangle}\nu(x,\cdot_x X)v(\cdot_x X)đXđ\xi\ ,\\ (Gu)(y)=\displaystyle\int_{T_y^*}\int_{T_y}B(y;\eta)\frac{đz}{đY}e^{-i\langle\eta|Y\rangle}\nu(y,\cdot_y Y)u(\cdot_y Y)đYđ\eta\ , \qquad \text{(cf. (26))}\ .\end{cases} \tag{81}$$

Then, we get

$$(FGu)(x)=\iiiint Ce^{-i\phi}u(\cdot\cdot_{xX}Y)đYđ\eta đXđ\xi\ , \tag{82}$$

where

$$(83)\quad \begin{cases} \phi=\langle\xi\,|\,X\rangle+\langle\eta\,|\,Y\rangle\ , \\ C=A(x;\,\xi)\dfrac{\text{đ}z}{\text{đ}X}(x;\,X)\Big(B\dfrac{\text{đ}z}{\text{đ}Y}\Big)({}_{\cdot x}X;\,\eta,\,Y)\nu(x,\,{}_{\cdot x}X)\nu({}_{\cdot x}X,\,{}_{\cdot\cdot xX}Y)\ , \end{cases}$$

(cf. (28)) .

Set ${}_{\cdot x}Z={}_{\cdot\cdot xX}Y$. Then, $Y=S(x;\,Z,\,X)$ (cf. (31)). By (10), we set

$$Y=S_1(x;\,Z,\,X)(Z-X)\ ,$$
$$\eta'=\eta S_1(x;\,Z,\,X)\ .$$

If the breadth ε of cut off function ν is sufficiently small, then one may assume that $S_1(x;\,Z,\,X)$ is invertible. Hence, we see

$$(84)\quad \begin{cases} \phi=\langle\xi\,|\,Z\rangle-\langle\eta'-\xi\,|\,X-Z\rangle \\ C=A(x;\,\xi)B'(x;\,X,\,Z,\,\eta')\ , \end{cases}$$

where $B'=(\text{đ}z/\text{đ}X)B(\text{đ}z/\text{đ}Y)(\text{đ}\eta/\text{đ}\eta')\nu(x,\,{}_{\cdot x}X)\nu({}_{\cdot x}X,\,{}_{\cdot x}Z)(\text{đ}Y/\text{đ}Z)$. Thus, putting $\eta''=\eta'-\xi$, $X'=X-Z$, we get

$$(85)\quad \begin{cases} (FGu)(x)=\displaystyle\iint A(x;\,\xi)B''(x;\,\xi,\,Z)e^{-i\langle\xi|Z\rangle}u({}_{\cdot x}Z)\text{đ}Z\text{đ}\xi\ , \\ B''\quad=\displaystyle\iint B'(x;\,X'+Z,\,Z,\,\eta''+\xi)e^{i\langle\eta''|X'\rangle}\text{đ}X'\text{đ}\eta''\ . \end{cases}$$

This is the desired composition rule.

REMARK. By the different way of the change of variables, we get the other expression of the composition rule as follows:

$$(FGu)(x)=\iint C(x;\,\eta',\,Z)e^{-i\langle\eta'|Z\rangle}u({}_{\cdot x}Z)\text{đ}Z\text{đ}\eta'\ ,$$

where

$$C(x;\,\eta',\,Z)=\iint A(x;\,\eta'+\xi'')B'(x;\,X,\,Z,\,\eta')e^{-i\langle\xi''|X\rangle}\text{đ}X\text{đ}\xi''\ .$$

In what follows, we can apply the same method as in [5] pp. 155–157 to obtain the inverse modulo smoothing operators. Especially, we have the following:

LEMMA 4.2. *Notations being as above, if* $B(x;\,\xi)$ *is sufficiently close to* 1 *in the uniform topology on* $T_N^*-\{0\}$, *then putting* $A(x;\,\xi)=B(x;\,\xi)^{-1}$ *in* (85), *we have* $A(x;\,\xi)B''(x;\,\xi,\,Z)=1+C'(x,\,\xi,\,Z)$, $C'(x;\,\xi,\,Z)\in\tilde{\Sigma}_C^{-1}$.

Now, let $\{E, E^k, k\geqq 0\}$ be the Sobolev chain of C-valued functions on N parametrized by the non-negative integers in such a way that $E^0 = L_2(N)$ and $E=\cap E^k = C^\infty(N)$. We denote by $L(E)$ the totality of continuous linear mappings of E into itself, and by $L_{oo}(E)$ the totality of $L\in L(E)$ such that L can be extended to a bounded operator of E^k into itself for every $k\geqq 0$, and that L satisfies the inequality

$$\| Lu \|_k \leqq C \| u \|_k + D_{k-1} \| u \|_{k-1} \quad (k\geqq 1) . \tag{86}$$

The following has been known in 11.1.2 Theorem in [6], or [7] pp. 11–14:

LEMMA 4.3. *Suppose we have $L\in L_{oo}(E)$ such that C in* (86) *is less than* 1/2 *and that the operator norm of $L: E^0\to E^0$ is less than* 1. *Then the operator $1-L\in L_{oo}(E)$ is invertible as an element of $L_{oo}(E)$.*

Now, we back to our situation. Choosing $A=B(x;\xi)^{-1}$, Lemma 4.2 shows that $FG=I-L$ and L is a pseudo-differential operator of order -1. Hence, obviously $L\in L_{oo}(E)$ such that the constant C in (86) is 0.

LEMMA 4.4. *If $B(x;\xi)$ is sufficiently close to* 1, *then the operator norm of $L: E^0\to E^0$ is less than* 1.

PROOF. It is a straightforward application of Theorem 8 in [5].

By Lemmas 4.2–4, we see that $I-L$ is invertible in $L_{oo}(E)$. Namely, the sequence $\sum_{s=0}^\infty L^s$ converges as an element in $L_{oo}(E)$ (cf. [6] pp. 141–142). Thus, to complete the proof of Proposition 4.1, we have only to show that $\sum_{s=0}^\infty L^s$ is a pseudo-differential operator.

Let $\sigma: \mathbf{R}^n\to\mathbf{R}$ is a C^∞ non-negative function such that $\sigma(X)\equiv 0$ if $|X|\geqq r_0/2$, where r_0 is the injectivity radius of N, and such that $\sigma(X)\equiv 1$ if $|X|\leqq r_0/4$. For each fixed $(x;\eta)\in T_N^*-\{0\}$, we define a C^∞ function $u_{(x;\eta)}(y)$ by

$$u_{(x;\eta)}(y)=\begin{cases}\sigma({}^{x\cdot}y)e^{i\langle\eta|{}^{x\cdot}y\rangle}J\ , & \text{if } \rho(x,y)<r_0\\ 0 \qquad , & \text{if } \rho(x,y)\geqq r_0\ ,\end{cases} \tag{87}$$

by identifying $\mathbf{R}^n$ with T_x, where ${}^{x\cdot}y=\mathrm{Exp}_x^{-1}y$, and J is its Jacobian.

The following lemma is well-known (cf. [5] p. 156):

LEMMA 4.5. *Let P_a be a pseudo-differential operator with a symbol $a(x;\xi)\in\sum_C^0$. Then $a(x;\eta)-(P_a u_{(x;\eta)})(x)$ is rapidly decreasing with respect to η.*

PROOF OF PROPOSITION 4.1. Since $L(\sum_{s=0}^\infty L^s)=\sum_{s=1}^\infty L^s\in L_{oo}(E)$, $c(x;\eta)=(\sum_{s=1}^\infty L^s u_{(x;\eta)})(x)$ is a C^∞ function on $T_N^*-\{0\}$. Recall that L^s is a pseudo-differential operator of order $-s$. Therefore it is easy to see that $c(x;\xi)\in\sum_{c}^{-1}$. Let P_c be the pseudo-differential operator with a symbol c. Then, it is not hard to see that $\sum_{s=1}^{\infty}L^s-P_c$ is an operator of order $-\infty$, and hence it is a pseudo-differential operator. Thus, we get that $\sum_{s=0}^{\infty}L^s$ is a pseudo-differential operator. This completes the proof of Proposition 4.1, because continuity follows immediately from the above computations.

To complete the rules of compositions and inversions of Fourier-integral operators, we have to compute compositions $F(a,\varphi,\nu)\circ K$ and $K\circ F(a,\varphi,\nu)$, where K is a smoothing operator and $F(a,\varphi,\nu)$ is given by (77). Both of them are obviously smoothing operators, and it is easy to compute the composition $F(a,\varphi,\nu)\circ K$. Thus, in the last part of this section we shall compute $K\circ F(a,\varphi,\nu)$.

$F=F(a,\varphi,\nu)$ can be written in the form;

(88)
$$(Fu)(y)=\int_{T_y^*}\int_N a(y;\xi)e^{-i\langle\varphi_2(y;\xi)|\varphi_1(y;\xi)\cdot z\rangle}\nu(\varphi_1(y;\xi),z)u(z)\,đz\,đ\xi\ .$$

Hence $((K\circ F)u)(x)$ is given by

(89)
$$((K\circ F)u)(x)=\iiint K(x,y)a(y;\xi)\nu(\varphi_1(y;\xi),z)e^{-i\langle\varphi_2(y;\xi)|\varphi_1(y;\xi)\cdot z\rangle}u(z)\,đy\,đ\xi\,đz\ .$$

Now, let $Y={}^{\varphi_1(y;\xi)\cdot}z$ (i.e., ${}_{\cdot\varphi_1(y;\xi)}Y=z$). If we set $Z=-(d\,\mathrm{Exp}_{\varphi_1(x;\xi)})_Y Y$, then ${}_{\cdot z}Z=\varphi_1(y;\xi)$, and $Y=-(d\,\mathrm{Exp}_z)_Z Z$. Define $\zeta=(d\,\mathrm{Exp}_z)_Z^*\varphi_2(y;\xi)$, and it is clear that $\langle\varphi_2(y;\xi)|{}^{\varphi_1(y;\xi)\cdot}z\rangle=\langle\varphi_2(y;\xi)|Y\rangle=-\langle\zeta|Z\rangle$. Remark that φ is sufficiently close to the identity, and hence by the implicit function theorem, y and ξ are C^∞ functions of $(z;Z,\zeta)$. Hence (89) can be rewritten as follows:

(90)
$$((K\circ F)u)(x)=\int_N K'(x,z)u(z)\,đz\ ,$$

$$K'(x,z)=\iint K(x,y(z;Z,\zeta))a(y(z;Z,\zeta),\xi(z;Z,\zeta))\nu({}_{\cdot z}Z,z)e^{i\langle\zeta|Z\rangle}\frac{đy}{đZ}\frac{đ\xi}{đ\zeta}đZ\,đ\zeta\ .$$

$K'(x,z)$ is a smooth function on $N\times N$, because the integrand is of compact support with respect to Z. It is not hard to see that K' depends continuously on K, a, φ in the C^∞ topology.

§5. Groups of invertible Fourier-integral operators.

In this section, Fourier-integral operators of order 0 are understood as operators written in the form (19). By $G\mathscr{F}^0$ we denote the group generated by all invertible Fourier-integral operators of order 0. Let $G\mathscr{P}^0$ be the group of all invertible pseudo-differential operators of order 0. Then, obviously $G\mathscr{F}^0 \supset G\mathscr{P}^0$.

LEMMA 5.1. *$G\mathscr{P}^0$ is a normal subgroup of $G\mathscr{F}^0$.*

PROOF. Let $A \in G\mathscr{F}^0$. Although it is not clear whether A can be expressed in the form (19) or not, A is a Fourier-integral operator in the sense of [2] or [3]. Moreover, it is known that APA^{-1} is a pseudo-differential operator of order 0, if so is P. If $P \in G\mathscr{F}^0$, then obviously APA^{-1} is invertible.

Let $\mathscr{P}^{-m}$ be the space of all pseudo-differential operators of order $-m$, and let $\mathscr{P}^{-\infty} = \cap \mathscr{P}^{-m}$. For each m, $0 \leqq m \leqq \infty$, we define $G\mathscr{P}^{-m}$ as follows:

$$G\mathscr{P}^{-m} = \{1 + P \in G\mathscr{P}^0;\ P \in \mathscr{P}^{-m}\} .$$

LEMMA 5.2. *$G\mathscr{P}^{-m}$ $(m \geqq 0)$ is a normal subgroup of $G\mathscr{F}^0$. Moreover, $G\mathscr{P}^{-m}/G\mathscr{P}^{-(m+1)}$ is abelian.*

PROOF. Since $\mathscr{P}^{-m}\mathscr{P}^{-s} \subset \mathscr{P}^{-(m+s)}$, it is obvious that $G\mathscr{P}^{-m} \cdot G\mathscr{P}^{-m} \subset G\mathscr{P}^{-m}$. Let $1+P \in G\mathscr{P}^{-m}$. Then $(1+P)^{-1} = 1-(1+P)^{-1}P$ and $(1+P)^{-1}P \in \mathscr{P}^{-m}$. Hence $G\mathscr{P}^{-m}$ is a group. By the same argument as in the above lemma, we see that $G\mathscr{P}^{-m}$ is a normal subgroup of $G\mathscr{F}^0$. Note that $[\mathscr{P}^{-s}, \mathscr{P}^{-m}] \subset \mathscr{P}^{-(s+m+1)}$, and we see easily that $G\mathscr{P}^{-m}/G\mathscr{P}^{-(m+1)}$ is an abelian group.

COROLLARY 5.3. *If $m \geqq 1$, then $G\mathscr{P}^{-m}/G\mathscr{P}^{-(m+1)}$ is isomorphic to the additive group of $C^\infty(S_N^*)$. Moreover $G\mathscr{P}^0/G\mathscr{P}^{-1}$ is isomorphic to the multiplicative group of non-vanishing C^∞ functions on S_N^*.*

PROOF. Let $f, g \in C^\infty(S_N^*)$, and set $a = fr^{-m}$, $b = gr^{-s}$. Denoting by P_a, P_b pseudo-differential operators with symbols a, b respectively, the symbol of P_aP_b is given by

$$fgr^{-(m+s)} + \text{lower order terms}\ , \qquad \text{(cf. (85), and [5])}\ .$$

Hence, we obtain the desired results.

Now, let V_1, $\mathfrak{U}$, U_0 be respectively a neighborhood of 1 in $\sum_C^0$ as a Fréchet space identified with $C^\infty(\bar{D}_N^*)$, a neighborhood of the identity in

$\mathscr{D}_\Omega^{(1)}$, and a neighborhood of 0 in $C^\infty(N\times N)$. We denote by $\mathfrak{N}(V_1, \mathfrak{U}, U_0)$ the set of of all Fourier-integral operators $F(a, \varphi, \nu)+K\circ$ such that $a\in V_1$, $\varphi\in\mathfrak{U}$, $K\in U_0$. By the argument in §4, we see easily that

$$\mathfrak{N}(V_1, \mathfrak{U}, U_0)\subset G\mathscr{F}^0\,, \tag{91}$$

if V_1, $\mathfrak{U}$, U_0 are sufficiently small. Note that this proves Theorem A in the introduction.

In what follows we always assume that V_1, $\mathfrak{U}$, U_0 are sufficiently small, and hence one may assume without loss of generality that these are arcwise connected. Let $G\mathscr{F}_0^0$ be the group generated by $\mathfrak{N}(V_1, \mathfrak{U}, U_0)$. Then by the rules of compositions (§§2–3) and inversions (§4), we can conclude the following, which proves Theorem B in the introduction:

THEOREM 5.4. *Every element of $G\mathscr{F}_0^0$ can be expressed in the form* (19). *Moreover, $G\mathscr{F}_0^0$ is a topological group.*

PROOF. Let $\{V_1'\}$, $\{\mathfrak{U}'\}$, $\{U_0'\}$ be a basis of neighborhoods of 1, identity, 0 respectively in $\sum_C^0$, $\mathscr{D}_\Omega^{(1)}$, $C^\infty(N\times N)$. Remark that the argument in §4 shows also that the inversion $(F(a, \varphi, \nu)+K\circ)^{-1}$ is also continuous in a, φ, K in the C^∞ topology. Hence by Lemma 3.7, we see that $\{\mathfrak{N}(V_1', \mathfrak{U}', U_0')\}$ defines a topology on $G\mathscr{F}_0^0$ under which $G\mathscr{F}_0^0$ is a topological group. $\{\mathfrak{N}(V_1', \mathfrak{U}', U_0')\}$ is a basis of neighborhoods of the identity in $G\mathscr{F}_0^0$. To prove the first statement, we denote by S the totality of elements in $G\mathscr{F}_0^0$ which can be written in the form (19). By (91) and the composition rules in §§2–3, it is not hard to see that S is open and closed in $G\mathscr{F}_0^0$. Since $\mathfrak{N}(V_1', \mathfrak{U}', U_0')$ is assumed to be arcwise connected and hence $G\mathscr{F}_0^0$ is arcwise connected, we get $S=G\mathscr{F}_0^0$.

Let $F=F(a, \varphi, \nu)+K\circ$ be an element of $G\mathscr{F}_0^0$. Then, φ is contained in the identity component $\mathscr{D}_{\Omega,0}^{(1)}$ of $\mathscr{D}_\Omega^{(1)}$ as a topological group identified with $\mathscr{D}_\omega(S_N^*)$. We define a mapping

$$\pi\colon G\mathscr{F}_0^0 \longrightarrow \mathscr{D}_{\Omega,0}^{(1)}$$

by $\pi(F(a, \varphi, \nu)+K\circ)=\varphi^{-1}$.

THEOREM 5.5. *π is an well-defined homomorphism of $G\mathscr{F}_0^0$ onto $\mathscr{D}_{\Omega,0}^{(1)}$. The kernel $\pi^{-1}(1)$ is given by $\mathscr{P}^0\cap G\mathscr{F}_0^0$.*

PROOF. It is enough to show that π is well-defined. For that, we assume for a while that $F(a, \varphi, \nu)+K\circ=F(b, \psi, \nu)+K'\circ$. Then, $(F(b, \psi, \nu)+K'\circ)^{-1}$ can be expressed in the form $F(e, \psi^{-1}, \nu)+K''\circ$. Hence by the composition rule we get

$$I=F(*, \varphi\psi^{-1}, \nu)+K^{(3)} .$$

Remark that the wave front set of the distribution $u\mapsto(Iu)(x)$ is given by $T_x^*-\{0\}$. However, by Lemma 3.2 [8] the wave front set of $u\mapsto(F(*, \varphi\psi^{-1}, \nu)u)(x)$ is contained in $\varphi\psi^{-1}(T_x^*-\{0\})$. Thus, $T_x^*-\{0\}=\varphi\psi^{-1}(T_x^*-\{0\})$, and by Lemma 5.6 [8] we get $\varphi=\psi$. Therefore, π is well-defined. Since $\mathscr{D}_{\Omega,0}^{(1)}$ is generated by $\mathfrak{U}$, we get the surjectivity of π. Obviously, the kernel $\operatorname{Ker}\pi$ is contained in $\mathscr{P}^0$, and hence $\operatorname{Ker}\pi=\mathscr{P}^0\cap G\mathscr{F}_0^0$.

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