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On the Values of a Certain Dirichlet Series at Rational Integers

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It has known that the Riemann zeta function $\zeta(s)$ satisfies the relations

$$\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)=2\zeta(3)\ ,$$

$$2\sum_{\nu=2}^{\infty}\frac{1}{\nu^n}\left(1+\frac{1}{2}+\cdots+\frac{1}{\nu-1}\right)=n\zeta(n+1)-\{\zeta(2)\zeta(n-1)+\cdots+\zeta(n-1)\zeta(2)\}$$
$$(n=3,4,5,\cdots)$$

(see [1], [2]). In this paper we prove the following

THEOREM. *Let $f(s)$ be the function defined by the Dirichlet series as*

$$(1)\qquad f(s)=\sum_{n=2}^{\infty}n^{-s}\sum_{k<n}k^{-1}\qquad (\mathrm{Re}\ s=\sigma>1)\ .$$

Then $f(s)$ is regular in the whole s-plane except at simple poles $s=0$ and $s=1-2a$ $(a=1,2,3,\cdots)$ with residues

$$\operatorname*{Res}_{s=0}(f(s))=-\frac{1}{2}\ ,$$

$$\operatorname*{Res}_{s=1-2a}(f(s))=-\frac{B_{2a}}{2a}\qquad (a=1,2,3,\cdots)\ ,$$

where B_n are Bernoulli numbers defined by $x/(e^x-1)=\sum_{n=0}^{\infty}(B_n/n!)x^n$, and a double pole $s=1$ with residue

$$\operatorname*{Res}_{s=1}(f(s))=\gamma\qquad (\textit{Euler's constant})\ .$$

Further we have

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$$f(-2a)=\frac{1}{2}\left(1+\frac{1}{2a}\right)B_{2a} \qquad (a=1, 2, 3, \cdots).$$

PROOF OF THEOREM. The series (1) is absolutely convergent for $\sigma>1$, so that we can change the order of summation, that is,

$$f(s)=\sum_{n=1}^{\infty} n^{-1} \sum_{k=n+1}^{\infty} k^{-s} \qquad (\sigma>1). \tag{2}$$

Noticing

$$\frac{d^q}{dx^q}(x^{-s})=\frac{\Gamma(1-s)}{\Gamma(-s-q+1)}x^{-s-q} \qquad (q\geqq 0),$$

we have, from the Euler-MacLaurin sum formula,

$$\sum_{k=n+1}^{N} k^{-s}=\int_n^N x^{-s}dx+\sum_{r=1}^{2a}(-1)^r\frac{B_r}{r!}\frac{\Gamma(1-s)}{\Gamma(-s-r+2)}(N^{-s-r+1}-n^{-s-r+1})$$
$$-\frac{1}{(2a)!}\int_n^N B_{2a}(x-[x])\frac{\Gamma(1-s)}{\Gamma(-s-2a+1)}x^{-s-2a}dx \qquad (\sigma>1),$$

where a is an arbitrary positive integer and $B_n(x)$ are Bernoulli polynomials defined by $ze^{xz}/(e^z-1)=\sum_{n=0}^{\infty} B_n(x)z^n/n!$. Thus

$$\sum_{k=n+1}^{\infty} k^{-s}=\frac{n^{-s+1}}{s-1}+\sum_{r=1}^{2a}(-1)^{r-1}\frac{B_r}{r!}\frac{\Gamma(1-s)}{\Gamma(-s-r+2)}n^{-s-r+1}$$
$$-\frac{1}{(2a)!}\frac{\Gamma(1-s)}{\Gamma(-s-2a+1)}\int_n^{\infty} B_{2a}(x-[x])x^{-s-2a}dx \qquad (\sigma>1).$$

The last integral is absolutely convergent not only in $\sigma>1$, but also in $\sigma>-2a+1$, since $B_{2a}(x-[x])$ is bounded. Using the Fourier expansion of $B_{2a}(x-[x])$, namely,

$$B_{2a}(x-[x])=2(-1)^{a-1}(2a)!\sum_{k=1}^{\infty}\frac{\cos 2\pi kx}{(2\pi k)^{2a}},$$

we have

$$\sum_{k=n+1}^{\infty} k^{-s}=\frac{n^{-s+1}}{s-1}+\sum_{r=1}^{2a}(-1)^{r-1}\frac{B_r}{r!}\frac{\Gamma(1-s)}{\Gamma(-s-r+2)}n^{-s-r+1}$$
$$+2(-1)^a\frac{\Gamma(1-s)}{\Gamma(-s-2a+1)}\int_n^{\infty}\sum_{k=1}^{\infty}\frac{\cos 2\pi kx}{(2\pi k)^{2a}}x^{-s-2a}dx \qquad (\sigma>1)$$
$$=\frac{n^{-s+1}}{s-1}+\sum_{r=1}^{2a}(-1)^{r-1}\frac{B_r}{r!}\frac{\Gamma(1-s)}{\Gamma(-s-r+2)}n^{-s-r+1}$$

$$+2(-1)^a\frac{\Gamma(1-s)}{\Gamma(-s-2a+1)}\sum_{k=1}^{\infty}(2\pi k)^{-2a}\int_n^{\infty}x^{-s-2a}\cos 2\pi kxdx$$
$$(\sigma>1)\ .$$

The inversion of the order of integration and summation can be justified by the uniform convergence. Substituting $y=2\pi kx$ in the last integral, we get

$$\sum_{k=n+1}^{\infty}k^{-s}=\frac{n^{-s+1}}{s-1}+\sum_{r=1}^{2a}(-1)^{r-1}\frac{B_r}{r!}\frac{\Gamma(1-s)}{\Gamma(-s-r+2)}n^{-s-r+1}$$
$$+2(-1)^a\frac{\Gamma(1-s)}{\Gamma(-s-2a+1)}\sum_{k=1}^{\infty}(2\pi k)^{s-1}\int_{2\pi nk}^{\infty}y^{-s-2a}\cos ydy$$
$$(\sigma>1)\ .$$

Thus (2) shows

$$f(s)=\frac{\zeta(s)}{s-1}+\sum_{r=1}^{2a}(-1)^{r-1}\frac{B_r}{r!}\frac{\Gamma(1-s)}{\Gamma(-s-r+2)}\zeta(s+r)$$
$$+2(-1)^a\frac{\Gamma(1-s)}{\Gamma(-s-2a+1)}R_{2a}(s)\qquad(\sigma>1)\ , \tag{3}$$

where

$$R_{2a}(s)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}n^{-1}(2\pi k)^{s-1}\int_{2\pi nk}^{\infty}x^{-s-2a}\cos xdx\ .$$

Now, by partial integration, we have

$$\left|\int_{2\pi nk}^{R}x^{-s-2a}\cos xdx\right|\leqq R^{-\sigma-2a}+|s+2a|\left|\int_{2\pi nk}^{R}x^{-s-2a-1}\sin xdx\right|$$
$$\leqq R^{-\sigma-2a}+|s+2a|R^{-\sigma-2a-1}+|s+2a|(2\pi nk)^{-\sigma-2a-1}$$
$$+|s+2a||s+2a+1|\int_{2\pi nk}^{R}x^{-\sigma-2a-2}dx$$
$$\to|s+2a|(2\pi nk)^{-\sigma-2a-1}+\frac{|s+2a||s+2a+1|}{\sigma+2a+1}(2\pi nk)^{-\sigma-2a-1}\qquad(\sigma>-2a)$$

as $R\to\infty$, and so we obtain

$$\left|\int_{2\pi nk}^{\infty}x^{-s-2a}\cos xdx\right|<C(nk)^{-\sigma-2a-1}|s+2a|(1+|s+2a+1|)\qquad(\sigma>-2a)\ ,$$

where C is a constant independent of n, k, and s. Hence we get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left|n^{-1}(2\pi k)^{s-1}\int_{2\pi nk}^{\infty}x^{-s-2a}\cos xdx\right|$$

$$\begin{aligned}
&\leqq \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} n^{-1}(2\pi k)^{\sigma-1}C(nk)^{-\sigma-2a-1}|s+2a|(1+|s+2a+1|)\\
&\leqq C(2\pi)^{\sigma-1}\zeta(\sigma+2a+2)\zeta(2a+2)|s+2a|(1+|s+2a+1|)\\
&\leqq C(2\pi)^{A-1}\zeta(2)\zeta(2a+2)(2a+A)(2a+2+A) \qquad (\sigma>-2a,\ |s|\leqq A)\,,
\end{aligned}$$

where A is any positive constant, so that the series $R_{2a}(s)$ is absolutely and uniformly convergent in $\sigma>-2a$, $|s|\leqq A$. Thus (3) furnishes an analytic continuation of $f(s)$ into the half plane $\sigma>-2a$. The function $f(s)$ has only a pole of order 2 at $s=1$ and has simple poles at $s=0$ and at negative odd integers in $\sigma>-2a$. Since a is an arbitrary positive integer, the function $f(s)$ defined by (1) can be continued analytically to the whole s-plane. Now by (3) we have

$$\operatorname*{Res}_{s=1}(f(s))=\gamma\,,$$

$$\operatorname*{Res}_{s=0}(f(s))=\operatorname*{Res}_{s=0}\Big((-1)^0\frac{B_1}{1!}\zeta(s+1)\Big)=-\frac{1}{2}\,,$$

$$\begin{aligned}
\operatorname*{Res}_{s=1-2a}(f(s))&=\operatorname*{Res}_{s=1-2a}\Big((-1)^{2a-1}\frac{B_{2a}}{(2a)!}\frac{\Gamma(1-s)}{\Gamma(-s-2a+2)}\zeta(s+2a)\Big)\\
&=-\frac{B_{2a}}{2a}\,.
\end{aligned}$$

And, since $R_{2a}(s)$ is regular in $\sigma>-2a$ and $\lim_{s\to-2b}\Gamma(1-s)/\Gamma(-s-2a+1)=0$ $(1\leqq b<a)$, we obtain

$$\begin{aligned}
f(-2b)&=(-1)^{2b-1}\frac{B_{2b}\Gamma(2b+1)}{(2b)!\,\Gamma(2)}\zeta(0)+(-1)^{1-1}\frac{B_1}{1!}\frac{\Gamma(2b+1)}{\Gamma(2b+1)}\zeta(1-2b)\\
&=\frac{1}{2}\Big(1+\frac{1}{2b}\Big)B_{2b} \qquad (1\leqq b<a)\,.
\end{aligned}$$

REMARK. Let $g_a(s)$ be the Dirichlet series defined by

$$g_a(s)=\sum_{n=2}^{\infty} n^{-s}\sum_{k<n} k^a \qquad (\sigma>a+2)\,,$$

where a is an integer $\geqq -1$. Then we get

$$g_a(s)=\begin{cases}\dfrac{1}{a+1}\displaystyle\sum_{k=0}^{a}\binom{a+1}{k}B_k\zeta(s+k-a-1) & \text{if } a\geqq 1\,,\\ \zeta(s-1)-\zeta(s) & \text{if } a=0\,,\end{cases}$$

so that we can prove, using the functional equation for $\zeta(s)$,

$$g_a(2a+3-s)-\frac{2\cos(\pi(s-a-1)/2)\Gamma(s-a-1)}{(2\pi)^{s-a-1}}g_a(s)$$

$$=\begin{cases}\dfrac{1}{a+1}\displaystyle\sum_{k=1}^{a}\binom{a+1}{k}B_k\Big\{\zeta(a+2+k-s)-\dfrac{2\cos(\pi(s-a-1)/2)\Gamma(s-a-1)}{(2\pi)^{s-a-1}}\\ \qquad\qquad\times\zeta(s+k-a-1)\Big\} & \text{if } a\geqq 1\,,\\ \dfrac{1}{a+1}\Big\{-\zeta(3-s)+\dfrac{2\cos(\pi(s-1)/2)\Gamma(s-1)}{(2\pi)^{s-1}}\zeta(s)\Big\} & \text{if } a=0\,.\end{cases}$$

In the case $a=-1$ we have the function $f(s)=g_{-1}(s)$, for which no corresponding result seems to be known.

References

[1] W. E. BRIGGS, S. CHOWLA, A. J. KEMPNER, and W. E. MIENTKA, On some infinite series, Scripta Math., **21** (1955), 28-30.

[2] G. T. WILLIAMS, A new method of evaluating $\zeta(2n)$, Amer. Math. Monthly, **60** (1953), 19-25.

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