TOKYO J. MATH.
VOL. 7, NO. 1, 1984

On Peak Sets for the Real Part of a Function Space

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Introduction.

Let A be a function space on a compact Hausdorff space X. In this paper, we give conditions for certain families of closed subsets in X under which A is characterized. In particular, in § 2 we show that if any peak set for the real part of A is a BEP-set for A, then $A=C(X)$ (Theorem 2.2). A theorem in the case of function algebras corresponding to it has obtained by Briem [4].

Throughout this paper, X will denote a compact Hausdorff space. A is said to be a *function space* (resp. *function algebra*) on X if A is a closed subspace (resp. subalgebra) in $C(X)$ containing constant functions and separating points in X. Let A be a function space and F be a closed subset in X. We say that $A|F$ has the *norm preserving extension property* if for any $f\in A$ there is a $g\in A$ such that $g=f$ on F and $\|g\|=\|f\|_F$, where $\|g\|=\sup_{x\in X}|g(x)|$ and $\|f\|_F=\sup_{x\in F}|f(x)|$. Such a closed subset F is called an *NPEP-set* for A. For a closed subset F in X, we put $\hat{F}=\{x\in X: |f(x)|\leqq\|f\|_F\}$ for any $f\in A$.

Let A be a function space on X. $A^{\perp}$ denotes the measures μ on X such that $\int f d\mu=0$ for any $f\in A$, and E denotes the closure of $\cup_{\mu\in A^{\perp}}\operatorname{supp}\mu$. E is the smallest one in the family of closed subsets F in X which satisfy the following property: $f\in A$ whenever $f\in C(X)$ and $f(x)=0$ for any $x\in F$. We call E the *essential set* for A (see [8] for essential sets in the case of function algebras).

Let A be a function algebra on X. The following is due to Glicksberg [7]: If $A|F$ is closed in $C(F)$ for any closed subset F in X, then $A=C(X)$. In the case of function spaces A, Briem [3] has shown that $A=C(X)$ if any closed subset F in X is any NPEP-set for A.

On the other hand, when A is a function algebra on X, Briem [4] has given conditions for peak sets for the real part $\operatorname{Re} A$ of A under which A coincides with $C(X)$.

Received March 30, 1983

In this paper, we give, in § 1, a slight extension of the theorem of Briem for NPEP-sets stated above. In § 2, we give a result on peak sets for $\operatorname{Re} A$ in the case of function spaces in association with the theorem of Briem for peak sets for $\operatorname{Re} A$ in the case where A is a function algebras.

§ 1. NPEP-sets.

The following lemma is obtained by a similar way as in the proof of Briem ([3], Theorem 2).

LEMMA 1.1. *Let A be a function space and F_0 be a closed subset in X. If any closed subset F containing F_0 is an NPEP-set for A, then $\hat{F}_0 \supset \bigcup_{\mu \in A^\perp} \operatorname{supp} \mu$, and hence $\hat{F}_0 \supset E$.*

PROOF. We first have that for any $x, y \in X \backslash \hat{F}_0$, $x \neq y$, there is an $a \in A$ such that $a|(\hat{F}_0 \cup \{y\})=0$ and $a(x)=1$. For otherwise, $a_1(x)=a_2(x)$, whenever $a_1=a_2$ on $\hat{F}_0 \cup \{y\}$, $a_1, a_2 \in A$. Hence the mapping φ: $a|\hat{F}_0 \cup \{y\} \rightarrow a(x)$ is a well-defined linear functional on $A|(\hat{F}_0 \cup \{y\})$. Since $\hat{F}_0 \cup \{y\}$ is an NPEP-set for A, the norm of φ is 1. Hence $a(x)=\int_{\hat{F}_0 \cup \{y\}} a d\mu$ $(a \in A)$ for a measure μ on $\hat{F}_0 \cup \{y\}$ with $\|\mu\|=1$. We easily see that $|\mu|(\hat{F}_0)>0$. Since $x \notin \hat{F}_0$, $1=a_0(x)>\|a_0\|_{\hat{F}_0}$ for an $a_0 \in A$. We can assume that $|a_0(y)| \leqq 1$ since $\hat{F}_0 \cup \{x\}$ is an NPEP-set for A.

$$1=a_0(x)=\int_{\hat{F}_0 \cup \{y\}} a_0 d\mu \leqq \|a_0\|_{\hat{F}_0} |\mu|(\hat{F}_0)+|\mu|(\{y\})<|\mu|(\hat{F}_0)+|\mu|(\{y\})=\|\mu\|=1 .$$

From this contradiction it follows that $a|(\hat{F}_0 \cup \{y\})=0$, $a(x)=1$ for an $a \in A$.

Similarly, we can show that for any closed subset $F \supset \hat{F}_0$ and for any $x \in X \backslash F$, there is an $a \in A$ with $a|F=0$ and $a(x)=1$. From this, we have that for any closed subset $F \supset \hat{F}_0$ and any closed subset G with $F \cap G=\varnothing$, there is an $a \in A$ such that $a|F=0$, $\operatorname{Re} a|G>0$ and $a(X) \subset D=\{z \in C: |z-1/2| \leqq 1/2\}$. We put $L(F, G)=\sup \{r \in R: \operatorname{Re} a|G \geqq r$ for an $a \in A$ with $a|F=0$, $a(X) \subset D\}$. By a similar manner as in the proof of Briem ([3], Lemma 6), we have that $L(F, G)>2^{-11}$ for any closed subset $F \supset \hat{F}_0$ and any closed subset G with $F \cap G=\varnothing$. This implies that $\operatorname{supp} \mu \subset \hat{F}_0$ for any $\mu \in A^\perp$. For, if $|\mu|(X \backslash \hat{F}_0)>0$ for a $\mu \in A^\perp$, we can assume that $|\mu|(X \backslash \hat{F}_0)=1$.

Here we can find finitely many mutually disjoint closed subsets $F_1, F_2, \cdots, F_m \subset X \backslash \hat{F}_0$ and $z_i \in C$, $|z_i|=1$ $(i=1, 2, \cdots, m)$ such that the measure $\lambda=z_1|\mu|_{F_1}+\cdots+z_m|\mu|_{F_m}$ satisfies that $\|\lambda-\mu_{X \backslash \hat{F}_0}\|<2^{-12}$. Put $\nu=\mu_{\hat{F}_0}+\lambda$. Then $\|\nu-\mu\|<2^{-12}$. From the fact above, there are $b_i \in A$ such that $b_i|(\hat{F}_0 \cup F_j)=0$ $(j \neq i)$, $\operatorname{Re} b_i|F_i \geqq 2^{-11}$ and $b_i(X) \subset D$ $(i=1, 2, \cdots, m,\ j=1, 2, \cdots, m)$. If we set $a=(\bar{z}_1 b_1+\cdots+\bar{z}_m b_m)|(\hat{F}_0 \cup F)$, $F=\bigcup_{i=1}^m F_i$, then $a \in A|(\hat{F}_0 \cup F)$, $\|a\|_{\hat{F}_0 \cup F} \leqq 1$. Hence $c|\hat{F}_0 \cup F=a$, $\|c\| \leqq 1$ for a $c \in A$. An easy calculation shows that $\operatorname{Re}\int c d\nu \geqq 2^{-11}(1-2^{-12})$. On the other hand, $\operatorname{Re}\int c d\nu \leqq \left|\int c d\nu\right| = \left|\int c d\nu - \int c d\mu\right| \leqq \|\nu-\mu\| < 2^{-12}$ since $c \in A$ and $\mu \in A^{\perp}$. This contradiction proves the lemma.

From Lemma 1.1, we have

THEOREM 1.2. *Let A be a function space on X and F_0 be a closed subset in X. If F is an NPEP-set for A for any closed subset F containing F_0, then $F_0 \supset \partial_{A|E}$, where $\partial_{A|E}$ denotes the Shilov boundary for the function space $A|E$.*

PROOF. We first show that $\hat{F}_0=F_0 \cup E$. From Lemma 1.1, we have $\hat{F}_0 \supset F_0 \cup E$. If $x \notin F_0 \cup E$, there is an $a \in C(X)$ such that $a|(F_0 \cup E)=0$, $a(x)=1$, $0 \leqq a \leqq 1$ and $a|E=0$. It implies that $a \in A$ and $a(x)>\|a\|_{F_0}$. It shows that $x \notin \hat{F}_0$ and $\hat{F}_0=F_0 \cup E$. Let $\{G_\alpha : \alpha \in I\}$ be the family of closed neighborhoods of E. Then $E=\bigcap_{\alpha \in I} G_\alpha$. Let $a \in A$. Suppose that $G_{\alpha_0} \cap \{x \in F_0 : |a(x)| \geqq \|a\|_E\}=\emptyset$ for an $\alpha_0 \in I$. Then there is an $h \in C(X)$ such that $0 \leqq h \leqq 1$, $h(E)=1$ and $h(X \backslash G_{\alpha_0})=0$.

Now, since $ha=a$ on E, we have that $ha \in A$, and

$$\begin{aligned} |ha(x)| \leqq |a(x)| < \|a\|_E = \|ha\|_E \quad & \text{if} \quad x \in F_0 \cap G_{\alpha_0}, \\ ha(x)=0 \quad & \text{if} \quad x \in F_0 \backslash G_{\alpha_0}. \end{aligned}$$

It follows that $\|ha\|_E > \|ha\|_{F_0}$ and it is a contradiction since $\hat{F}_0 \supset E$. From this, for any $\alpha \in I$, $G_\alpha \cap \{x \in F_0 : |a(x)| \geqq \|a\|_E\} \neq \emptyset$. That is, $\{G_\alpha \cap \{x \in F_0 : |a(x)| \geqq \|a\|_E\}\}$ has the finite intersection property and hence $E \cap \{x \in F_0 : |a(x)| \geqq \|a\|_E\} \neq \emptyset$. Thus $\partial_{A|E} \subset E \cap F_0 \subset F_0$.

COROLLARY 1.3. *Let A be a function space and F_1, F_2 be two closed subsets in X with $F_0 \cap F_1=\emptyset$. If any closed subset containing F_0 is an NPEP-set for A and any closed subset containing F_1 is an NPEP-set for A, then $A=C(X)$.*

COROLLARY 1.4 (Briem [3]). *Let A be a function space. If any closed subset in X is an NPEP-set for A, then $A=C(X)$.*

REMARK. In the case of function algebras, the following fact corresponds to Theorem 1.2: Let A be a function algebra on X and F_0 be a closed subset in X. If $A|F$ is closed in $C(F)$ for any closed subset F

containing F_0, then $F_0 \supset \partial_{A|E}$ (cf. [9]).

§ 2. Peak sets for Re A.

We here give conditions for peak sets for Re A under which A coincides with $C(X)$ in the case where A is a function space.

When A is a function space, the following properties on a closed set F in X are equivalent (cf. [6]): (1) $\mu \in A^{\perp}$ implies $\mu_F \in A^{\perp}$. (2) A has the *bounded extension property* with respect to F, i.e., for every $f \in A|F$ and each closed set G in X with $G \cap F = \emptyset$, and for each $\varepsilon > 0$, there exists a $g \in A$ such that $g|F = f$, $\|g\| = \|f\|_F$ and $\|g\|_G < \varepsilon$. Such a subset F is called a *BEP-set* for A.

LEMMA 2.1. *Let A be a function space. If any peak set for* Re A *is a BEP-set for A, then A is self-adjoint, that is, if $f \in A$, then $\bar{f} \in A$.*

PROOF. It is sufficient to show that if a peak set F for Re A is a BEP-set for A, then F is an M-hull (cf. [1], Theorem 9.1, p. 220). In order to prove that F is an M-hull, we need to show that $F = \mathscr{F} \cap X$ for some closed face $\mathscr{F}$ in $S_A = \{L \in A^* : L(1) = 1 = \|L\|\}$ (cf. [1], Proposition 2.7, p. 158). We can construct $\mathscr{F}$ satisfying the above as follows: $\mathscr{F} = \left\{L \in A^* : L(f) = \int f d\mu \ (f \in A) \text{ for a measure } \mu \text{ such that } \mu \geqq 0,\ \|\mu\| = 1 \text{ and } \operatorname{supp} \mu \subset F\right\}$. If $\mathscr{F} \in L = (L_1 + L_2)/2$, $L_1, L_2 \in S_A$, then $L_i(f) = \int f d\mu_i$ $(f \in A)$ with some measure $\mu_i \geqq 0$ $\|\mu_i\| = 1$ $(i = 1, 2)$. Since $\mu - (\mu_1 + \mu_2)/2 \in A^{\perp}$ and F is a BEP-set for A, $\mu_F - \{(\mu_1)_F + (\mu_2)_F\}/2 \in A^{\perp}$. We here have $\mu = \mu_F$. It implies that $(\mu_1)_{X \setminus F} + (\mu_2)_{X \setminus F} \in A^{\perp}$ and hence $\int_{X \setminus F} d\mu_1 + \int_{X \setminus F} d\mu_2 = 0$. So $\mu_1(X \setminus F) = \mu_2(X \setminus F) = 0$ and $L_1, L_2 \in \mathscr{F}$. This shows that $\mathscr{F}$ is a face of S_A. To show that $F = \mathscr{F} \cap X$, let $L_x \in \mathscr{F} \cap X$ $(L_x(f) = f(x),\ f \in A)$. Then $L_x \in \mathscr{F}$, and $x \in F$ since F is a peak set for Re A, and this proves the lemma.

In association with a theorem of Briem ([4], Theorem 3), we can obtain the following in the case of function spaces.

THEOREM 2.2. *Let A be a function space. Then the following four properties are equivalent.*

(i) *Any closed subset in X containing a peak set for* Re A *is an NPEP-set for A.*

(ii) *Any peak set F_0 for* Re A *is a BEP-set for A.*

(iii) $A = C(X)$.

(iv) *Any closed subset in X is an NPEP-set for A.*

REMARK. When A is a function algebra, the property (ii) above is equivalent to (ii) in Theorem 3 of Briem ([4]).

PROOF. (i)$\rightarrow$(ii). Let F_0 be a peak set for Re A. By (i) and Lemma 1.1, $\hat{F}_0 \supset \bigcup_{\mu \in A^\perp} \operatorname{supp} \mu$. On the other hand, $\hat{F}_0 = F_0$ since F_0 is a peak set for Re A. For, if $x_0 \in \hat{F}_0 \backslash F_0$, we put $L(a|F_0) = a(x_0)$ for $a \in A$. Then L is well-defined and it is a linear functional on $A|F_0$ with $\|L\| = L(1) = 1$. Hence $a(x_0) = \int a d\nu$ $(a \in A)$ for a measure ν such that $\operatorname{supp} \nu \subset F_0$, $\nu \geqq 0$ and $\|\nu\| = 1$. Since F_0 is a peak set for Re A, there is an $f_0 \in A$ such that $\operatorname{Re} f_0 = 1$ on F_0 and $|\operatorname{Re} f_0(x)| < 1$ $(x \in X \backslash F_0)$. Hence $1 = \int \operatorname{Re} f_0 d\nu = \operatorname{Re} f_0(x_0) = |\operatorname{Re} f_0(x_0)| < 1$. From this contradiction it follows that $\hat{F}_0 = F_0$. The facts above show that for any $\mu \in A^\perp$, $\operatorname{supp} \mu \subset F_0$ and $\mu_{F_0} = \mu \in A^\perp$. It implies (ii).

(ii)$\rightarrow$(iii). By Lemma 2.1, A is self-adjoint. It implies that $A = \operatorname{Re} A \oplus i \operatorname{Re} A$. From this, $\operatorname{Re} A = A \cap C_R(X)$ and Re A is closed in $C_R(X)$. Now, a set which is both a peak set and a BEP-set for a function space B is always a sharp peak set for B, that is, for each closed subset G in X disjoint from F and for each $\varepsilon > 0$ there is an $f \in B$ such that $f = 1$ on F, $|f| < 1$ elsewhere and $|f| \leqq \varepsilon$ on G (cf. [5]).

From the facts stated above, $B = \operatorname{Re} A (= A \cap C_R(X))$ is a real function space on X and every peak set for B is a sharp peak set for B. Hence $\operatorname{Re} A = B = C_R(X)$ ([2], Theorem 5). It follows that $A = C_R(X) \oplus i C_R(X) = C(X)$.

(iii)$\rightarrow$(iv) and (iv)$\rightarrow$(i) are clear.

EXAMPLE. In connection with Theorem 2.2, we here give an example of function spaces A $(\neq C(X))$ which have the properties "F is an NPEP-set for A for any peak set F for Re A" and "F is a peak set for A whenever F is a peak set for Re A".

Let $A = \left\{ f \in C([0, 1]) : f(0) = \int_0^1 f(t) dt \right\}$. Then A is a function space on $X = [0, 1]$. We easily see that a real function g in $C([0, 1])$ belongs to Re A if and only if $g(0) = \int_0^1 g(t) dt$. From this, the following properties for a closed subset F_0 in X are equivalent:

(1) F_0 is a peak set for Re A, (2) F_0 is a peak set for A and (3) $F_0 = X$ or $F_0 \not\ni 0$.

From this, we see that any peak set F_0 for Re A is an NPEP-set for A. And if F_0 is a peak set for Re A, then F_0 is a peak set for A. But A does not coincide with $C(X)$.

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