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Another Characterization of the Two-Weight Norm Inequalities for the Maximal Operators

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Introduction

Let ν be a Borel measure on $\boldsymbol{R}^n$ and set

$$M_\alpha \nu(x) = \sup |Q|^{-\alpha} \int_Q d|\nu| \qquad (0<\alpha \leqq 1) \tag{0.1}$$

where the supremum is taken over all the cubes Q in $\boldsymbol{R}^n$ which contain x and $|Q|$ denotes the Lebesgue measure of Q. Throughout this note we deal with only cubes of the form $\prod_{j=1}^n [x_j, x_j+r)$ where $(x_1, \cdots, x_n) \in \boldsymbol{R}^n$ and $r>0$. $M_{1-\alpha/n}$, $0<\alpha<n$, is called the fractional maximal operator. When $\alpha=1$, (0.1) is the Hardy-Littlewood maximal function of ν.

Recently E. T. Sawyer [12] showed that for a nonnegative measure ω and a nonnegative function $v(x)$ on $\boldsymbol{R}^n$ there exists a positive constant C_1 independent of $f(x)$ such that

$$\left(\int_{R^n} [M_\alpha f]^q d\omega\right)^{1/q} \leqq C_1 \left(\int_{R^n} |f|^p v dx\right)^{1/p} \qquad (0<\alpha\leqq 1) \tag{0.2}$$

for all measurable functions $f(x)$ if and only if there exists a positive constant C_2 independent of cubes Q such that

$$\int_Q [M_\alpha(\chi_Q v^{1-p'})]^q d\omega \leqq C_2 \left(\int_Q v^{1-p'}\right)^{q/p} < \infty \tag{0.3}$$

for all cubes Q, where $1<p\leqq q<\infty$, $(1-p)(1-p')=1$ and χ_Q denotes the characteristic function of Q.

In the case that $p=q$, $\alpha=1$, ω is a function and $\omega=v$, as it is well known, B. Muckenhoupt [9] showed that (0.2) is valid if and only if ω satisfies A_p condition:

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(0.4)
$$\Big(|Q|^{-1}\int_Q \omega(x)dx\Big)\Big(|Q|^{-1}\int_Q \omega(x)^{1-p'}\Big)^{p-1} \leqq C_3$$

for all Q, where C_3 is independent of Q.

Consequently, in this case, (0.3) is equivalent to (0.4). R. Hunt, D. Kurtz and C. Neugebauer [7] showed elementarily this relation without using the equivalence of (0.2) and (0.3).

On the other hand, Muckenhoupt conjectured in [10, p. 319] that when $p=q$, $\alpha=1$ and $n=1$, (0.2) holds if and only if there exists C independent of I and E such that for every interval I and every subset E of I with $|E|=2^{-1}|I|$

(0.5)
$$\Big(\int_I \omega(x)dx\Big)\Big(|I|^{-1}\int_I v(x)^{1-p'}dx\Big)^p \leqq C\int_E v(x)^{1-p'}dx\ .$$

The fact is that (0.5) is sufficient, but not necessary for (0.2) in general even if we replace $\int_I \omega(x)dx$ with $\int_E \omega(x)dx$.

In this note, instead of (0.5), we shall give another necessary and sufficient condition for (0.2) to hold and we shall show the equivalence of our condition and Sawyer's condition (0.3) without using (0.2).

§1. Theorems.

At first we consider an example. Suppose $1<p<\infty$ and let $\sigma(x)$ be a nonnegative function on $\boldsymbol{R}$ such that

(1.1)
$$\sup_{R>1} |RI|^{-p}\Big(\int_{RI} \sigma dx\Big)^{p-1} < \infty \quad \text{for some interval } I$$

where RI is the interval having the same center as I but whose length is R times as large. And we set

(1.2)
$$\omega(x)=g(x)\Big\{\Big(\sup_{I\ni x,|I|\leqq 1/4} |I|^{-1}\int_I \sigma dx\Big)^p \sigma^{-1}(x) + \sup_{I\ni x,|I|>1/4} |I|^{-p}\Big(\int_I \sigma dx\Big)^{p-1}\Big\}^{-1}$$

where I denotes an interval and $g(x)$ is a nonnegative, bounded and integrable function on $\boldsymbol{R}$ such that $g(x)=1$ on $[-1, 1)$. $0\cdot\infty$ will be taken to be 0.

Then we can easily see the pair (ω, σ) satisfies (0.3) where $v=\sigma^{1-p}$, $q=p$ and $\alpha=1$. But if we set $\sigma(x)=0$ on $[-3/4, 3/4)$ and $\sigma(x)=1$ otherwise, then there cannot exist any finite positive constant C satisfying

$$\left(\int_E \omega(x)dx\right)\left(\frac{1}{2}\int_I \sigma(x)dx\right)^p \leqq C\int_E \sigma(x)dx\ , \tag{1.3}$$

where $I=[-1, 1)$ and $E=[-1/2, 1/2)$. The assumption (1.1) guarantees that $\omega\neq 0$. The construction (1.2) of ω is essentially due to Sawyer [13, p. 110]. Also refer to [2].

Muckenhoupt's conjecture suggests another characterization of the pair (ω, v) for (0.2) to hold. We state our theorems:

THEOREM 1. *Let ω and σ be nonnegative Borel measures on $\boldsymbol{R}^n$. Suppose $1\leqq q<\infty$, $0<\alpha\leqq 1$ and $0<\delta<1$. Fix a cube Q in $\boldsymbol{R}^n$. If there exists a nonnegative Borel measure μ_Q and there exists a positive constant C_Q independent of I and E such that*

$$\int_E d\omega\left(|I|^{-\alpha}\int_I d\sigma\right)^q \leqq C_Q\int_E d\mu_Q \tag{1.4}$$

for any subcube I of Q and any measurable subset E of I with measure $|E|\geqq\delta|I|$, then there exists a positive constant c_0 depending only on n, α, δ and q such that

$$\int_Q [M_\alpha(\chi_Q\sigma)(x)]^q d\omega \leqq c_0 C_Q \int_Q d\mu_Q\ . \tag{1.5}$$

From Theorem 1 we have the following:

THEOREM 2. *Let ω and σ be nonnegative Borel measures on $\boldsymbol{R}^n$ and let $\omega\neq 0$. Suppose $0<p<\infty$, $1\leqq q<\infty$ and $0<\alpha\leqq 1$. Then the following conditions* (I) *and* (II) *are equivalent:*

(I) *There exists a positive constant C_4 depending only on n, α, p, q, ω and σ such that*

$$\int_Q [M_\alpha(\chi_Q\sigma)(x)]^q d\omega \leqq C_4\left(\int_Q d\sigma\right)^{q/p} < \infty \tag{1.6}$$

for all cubes Q.

(II) *There exist positive constants C_5, C_6 and $\delta\in(0, 1)$ depending only on n, α, p, q, ω and σ, and there exist locally finite nonnegative Borel measures μ_Q for all cubes Q such that*

$$\int_E d\omega\left(|I|^{-\alpha}\int_I d\sigma\right)^q \leqq C_5\int_E d\mu_Q \tag{1.4$'$}$$

for any subcube I of Q and any measurable subset E of I with measure $|E|\geqq\delta|I|$, and

$$\int_Q d\mu_Q \leqq C_6\left(\int_Q d\sigma\right)^{q/p} . \tag{1.7}$$

If we suppose $d\sigma = v^{1-p'}dx$ where $v(x)$ is a positive function and $(1-p')(1-p)=1$, then (I) means Sawyer's condition (0.3). So we obtain the following characterization:

THEOREM 3. *Let ω be a nonnegative measure, $\omega \neq 0$, and let v be a nonnegative measurable function on $\boldsymbol{R}^n$. Set $\sigma = v^{1-p'}$ and suppose $1<p\leqq q<\infty$ and $0<\alpha\leqq 1$. Then there exists a positive constant C_1 independent of f which satisfies* (0.2) *if and only if the pair (ω, σ) satisfies the condition* (II) *in Theorem* 2.

We shall prove only Theorems 1 and 2. Theorem 3 can be also proved by the same method as Sawyer [12] and B. Jawerth [8] with slight modification.

COROLLARY (Sawyer [11]). *Suppose that ω is a nonnegative Borel measure, $\omega \neq 0$, $\sigma(x)$ is a positive function on $\boldsymbol{R}^n$, $1<p\leqq q<\infty$ and $0<\alpha\leqq 1$. If the pair (ω, σ) satisfies that there exists a constant C_7 independent of Q such that*

$$\int_Q d\omega\left(|Q|^{-\alpha}\int_Q \sigma dx\right)^q \leqq C_7\left(\int_Q \sigma dx\right)^{q/p} \tag{1.8}$$

for all cubes Q and if σ is in A_∞, that is, there exist positive constants C_8 and $\delta \in (0, 1)$ independent of Q and E such that

$$\int_E \sigma dx \geqq C_8 \int_Q \sigma dx \tag{1.9}$$

whenever E is a subset of Q with measure $|E| \geqq \delta|Q|$, then the pair (ω, σ) satisfies the condition (I) *in Theorem* 2.

Refer to [4] for A_∞ condition and see also [6] for details of our subject.

§2. Proofs of the theorems.

We first observe the easy direction (I)$\rightarrow$(II) in Theorem 2. Set

$$d\mu_Q = [M_\alpha(\chi_Q\sigma)]^q d\omega ,$$

then for any subset E of a subcube I in Q we see immediately that μ_Q satisfies (1.4)$'$ and (1.7) with $C_5=1$ and $C_6=C_4$.

(II)$\rightarrow$(I). We shall prove Theorem 1. The implication (II)$\rightarrow$(I) in Theorem 2 is an immediate consequence of Theorem 1 by (1.7). We shall use the same method as M. Christ and R. Fefferman in [3]. We begin the proof by showing a lemma which is a version of that due to Calderón and Zygmund [1].

Let $\mathscr{F}$ be a family of dyadic cubes of $\boldsymbol{R}^n$. Dyadic cubes denote the cubes of the form $\prod_{j=1}^n [k_j 2^m, (k_j+1)2^m)$ where k_j's and m are integers. We put

$$M_\alpha^d \nu(x, \mathscr{F}) = \sup |I|^{-\alpha} \int_I d|\nu|$$

where ν is a Borel measure on $\boldsymbol{R}^n$ and the supremum is taken over all cubes I which belong to $\mathscr{F}$ and contain x. If any cube I in $\mathscr{F}$ does not contain x, we put $M_\alpha^d \nu(x, \mathscr{F})=0$. $I \setminus E$ will denote the set $\{x; x \in I$ and $x \notin E\}$.

Lemma. *Suppose ν is a finite Borel measure on $\boldsymbol{R}^n$ and $\lambda>1$. Then for every integer k, satisfying $\{M_\alpha^d \nu(x, \mathscr{F})>\lambda^k\} \neq \emptyset$, there exists a subfamily $\mathscr{F}_k$ of dyadic cubes $\{I_j^k\}$ in $\mathscr{F}$ and a family of measurable subsets $\{E_j^k\}$ of $\boldsymbol{R}^n$ such that*

(i) $\{E_j^{2k}\}_{k,j}$ *and* $\{E_j^{2k+1}\}_{k,j}$ *are respectively pairwise disjoint,*

(ii) $$E_j^k \subset I_j^k \quad and \quad |I_j^k \setminus E_j^k| \leqq \lambda^{-1/\alpha} |I_j^k| ,$$

(iii) $$|I_j^k|^{-\alpha} \int_{I_j^k} d|\nu| > \lambda^k ,$$

(iv) $$M_\alpha^d \nu(x, \mathscr{F}) \leqq \lambda^{k+2} \qquad on \ E_j^k ,$$

and

(v) $$\{x ; M_\alpha^d \nu(x, \mathscr{F}) \neq 0\} \subset \bigcup_{k,j} E_j^k .$$

Proof of Lemma. Let $E^k=\{x; M_\alpha^d \nu(x, \mathscr{F})>\lambda^k\}$, then there exists a family of maximal dyadic cubes $\{I_j^k\}_j$ in $\mathscr{F}$ such that

$$\bigcup_j I_j^k = E^k \tag{2.1}$$

and

$$|I_j^k|^{-\alpha} \int_{I_j^k} d|\nu| > \lambda^k . \tag{2.2}$$

We divide $\{I_j^k\}_j$ into three classes:

$$(\mathscr{F}_k) \qquad |I_j^k|^{-\alpha}\int_{I_j^k} d|\nu| \leqq \lambda^{k+1}\,,$$

$$(\mathscr{F}_k') \qquad \lambda^{k+1} < |I_j^k|^{-\alpha}\int_{I_j^k} d|\nu| \leqq \lambda^{k+2}\,,$$

and

$$(\mathscr{F}_k'') \qquad |I_j^k|^{-\alpha}\int_{I_j^k} d|\nu| > \lambda^{k+2}\,.$$

Then $\mathscr{F}_k' \subset \mathscr{F}_{k+1}$ and, if $I_j^k \in \mathscr{F}_k''$, we see $I_j^k \subset E^{k+2}$. Therefore, we have

$$E^k \setminus E^{k+2} \subset \Big\{ \bigcup_{I_j^k \in \mathscr{F}_k} I_j^k \setminus E^{k+2} \Big\} \cup \Big\{ \bigcup_{I_j^{k+1} \in \mathscr{F}_{k+1}} I_j^{k+1} \setminus E^{k+2} \Big\}\,.$$

We set $E_j^k = I_j^k \setminus E^{k+2}$ for $I_j^k \in \mathscr{F}_k$. Then $\{M_\alpha^d \nu \neq 0\} \subset \bigcup_{k,j} E_j^k$ and $\{E_j^{2k}\}_{k,j}$ and $\{E_j^{2k+1}\}_{k,j}$ are respectively pairwise disjoint families.

Also we see that

$$I_j^k \cap E^{k+2} = \bigcup_{J_l^{k+2} \subset I_j^k} J_l^{k+2} \qquad \text{for } I_j^k \in \mathscr{F}_k$$

where the maximal dyadic cubes J_l^{k+2} satisfy

$$(2.2)' \qquad |J_l^{k+2}|^{-\alpha}\int_{J_l^{k+2}} d|\nu| > \lambda^{k+2}\,.$$

Noticing $0<\alpha\leqq 1$ and $I_j^k \in \mathscr{F}_k$, we have

$$\begin{aligned} |I_j^k \cap E^{k+2}| &= \sum_{J_l^{k+2} \subset I_j^k} |J_l^{k+2}| \\ &\leqq \Big(\lambda^{-(k+2)} \sum_{J_l^{k+2} \subset I_j^k} \int_{J_l^{k+2}} d|\nu|\Big)^{1/\alpha} \qquad \text{(by (2.2)')} \\ &\leqq \Big(\lambda^{-(k+2)} \int_{I_j^k} d|\nu|\Big)^{1/\alpha} \\ &\leqq \lambda^{-1/\alpha} |I_j^k|\,. \end{aligned}$$

So we have (i), (ii) and (iii) for E_j^k's and I_j^k's in $\mathscr{F}_k$. (iv) and (v) are immediate. This completes the proof of Lemma.

Having prepared Lemma, we can prove Theorem 1 by reducing the argument of the maximal operator M_α to that of the dyadic maximal operator M_α^d as the routine argument.

Let $\mathscr{F}(Q)$ be the family of all dyadic cubes I such that $|I|<2^{2n}|Q|$, $|I\cap Q| \geqq 2^{-2n}|I|$ and $l_j \geqq ((1+\delta)/2)^{1/(n-1)} \max\{l_1, \cdots, l_n\}$, $j=1, \cdots, n$, where l_j is the j-th side length of the rectangle $I\cap Q$. That is, $I\cap Q$ is 'almost a cube'.

Since we may assume σ is locally finite, applying Lemma we have the families of dyadic cubes $\{I_j^k\}$ and subsets $\{E_j^k\}$ of $\boldsymbol{R}^n$ which satisfy the conditions (i), (ii), (iii), (iv) and (v) for $M_\alpha^d(\chi_Q\sigma)(x, \mathscr{F}(Q))$ and $\lambda > 2^{\alpha(2n+1)}(1-\delta)^{-\alpha}$. Thus we have that

$$\int_Q [M_\alpha^d(\chi_Q\sigma)(x, \mathscr{F}(Q))]^q d\omega \leqq \int_{\cup_{k,j}E_j^{2k}\cap Q} + \int_{\cup_{k,j}E_j^{2k+1}\cap Q} \qquad \text{(by (v))}. \tag{2.3}$$

And

$$\begin{aligned} \int_{\cup_{k,j}E_j^{2k}\cap Q} [M_\alpha^d(\chi_Q\sigma)]^q d\omega &= \sum_{k,j} \int_{E_j^{2k}\cap Q} [M_\alpha^d(\chi_Q\sigma)]^q d\omega \\ &\leqq \sum_{k,j} \lambda^{(2k+2)q} \int_{E_j^{2k}\cap Q} d\omega \qquad \text{(by (iv))} \\ &\leqq \lambda^{2q} \sum_{k,j} \int_{E_j^{2k}\cap Q} d\omega \Big(|I_j^{2k}|^{-\alpha} \int_{I_j^{2k}} \chi_Q d\sigma \Big)^q \qquad \text{(by (iii))}. \end{aligned} \tag{2.4}$$

Let $\tilde{I}_j^{2k}$ be the least cube such that $I_j^{2k} \cap Q \subset \tilde{I}_j^{2k} \subset Q$. Then, because $|I_j^{2k}| \simeq |\tilde{I}_j^{2k}|$, the above expression is majorized by

$$C_{n,\alpha} \lambda^{2q} \sum_{k,j} \int_{E_j^{2k}\cap Q} d\omega \Big(|\tilde{I}_j^{2k}|^{-\alpha} \int_{\tilde{I}_j^{2k}} d\sigma \Big)^q .$$

From our assumption of $\mathscr{F}(Q)$ we see that

$$|\tilde{I}_j^{2k} \setminus (I_j^{2k} \cap Q)| \leqq \frac{1-\delta}{2} |\tilde{I}_j^{2k}| ,$$

and we see from (ii) that

$$\begin{aligned} |(I_j^{2k} \cap Q) \setminus E_j^{2k}| &< 2^{-(2n+1)}(1-\delta)\,|I_j^{2k}| \\ &\leqq \frac{1-\delta}{2} |I_j^{2k} \cap Q| \\ &\leqq \frac{1-\delta}{2} |\tilde{I}_j^{2k}| . \end{aligned}$$

Hence we have $|E_j^{2k} \cap Q| \geqq \delta\,|\tilde{I}_j^{2k}|$. Therefore, by the assumption (1.4) we obtain that the last expression of (2.4) is majorized by

$$\begin{aligned} &C_{n,\alpha} \lambda^{2q} C_Q \sum_{k,j} \int_{E_j^{2k}\cap Q} d\mu_Q \\ &\leqq C_{n,\alpha} \lambda^{2q} C_Q \int_Q d\mu_Q \qquad \text{(by (i))}. \end{aligned}$$

By the same argument we get also

$$\int_{\cup_{k,j}E_j^{2k+1}\cap Q}[M_\alpha(\chi_Q\sigma)(x, \mathscr{F}(Q))]^q d\omega \leqq C_{n,\alpha}\lambda^{2q}C_Q\int_Q d\mu_Q \,.$$

Hence we have

$$\int_Q [M_\alpha^d(\chi_Q\sigma)(x, \mathscr{F}(Q))]^q d\omega \leqq c_0' C_Q \int_Q d\mu_Q \,. \tag{2.5}$$

Next we fix x in Q. For every cube I containing x there exists a subcube J of Q such that $|J|\leqq|I|$ and $I\cap Q\subset J$. Hence

$$M_\alpha(\chi_Q\sigma)(x)\leqq \sup_{x\in I\subset Q}|I|^{-\alpha}\int_I \chi_Q d\sigma \,.$$

Fix a subcube I of Q which contains x. Let $\tilde{I}$ be the cube having the same center as I with measure $2^{4n}|Q|$. Let k and r be integers such that $2^{kn}<|I|\leqq 2^{(k+1)n}$ and $2^{rn}<|\tilde{I}|\leqq 2^{(r+1)n}$. We put

$$S_I=\{t\in Q_0\,;\ \text{there exists a dyadic cube } I_d \text{ in } \mathscr{F}(Q+t) \text{ such that } I+t\subset I_d\subset \tilde{I} \text{ and } |I_d|=2^{(k+2)n}\}$$

where $Q_0=\prod[-2^{r+1}, 2^{r+1})$, and by a geometrical observation we find at least $2^{(r-(k+8))n}$ cubes with the side length $2^k\{1-((1+\delta)/2)^{1/(n-1)}\}$ which are pairwise disjoint and are contained in S_I. This observation is due to C. Fefferman and E. Stein [5] as is well known. Also see [8, p. 383].

Let $\tau_t\sigma$ and $\tau_t\omega$ denote the translations by t of σ and ω respectively. Then we have for any integer K

$$\begin{aligned}\sup_{I\ni x, |I|>2^K}|I|^{-\alpha}\int_I \chi_Q d\sigma \\ \leqq C_{n,\alpha}2^{(-r+K+8)n}\sum_{l=1}^{N} M_\alpha^d(\chi_{Q+t_l}\tau_{t_l}\sigma)(x+t_l, \mathscr{F}(Q+t_l))\end{aligned} \tag{2.6}$$

where $N=2^{(r-K+2)n}\{1-((1+\delta)/2)^{1/(n-1)}\}^{-n}$ and t_l's are the suitable lattice points in Q_0

Since the pair $(\tau_t\sigma, \tau_t\omega)$ satisfies (1.4) with $\tau_t\mu_Q$ and the same constant C_Q for the cube $Q+t$ and since $\int_{Q+t}d\tau_t\mu_Q=\int_Q d\mu_Q$ for any t, using (2.5) and (2.6) we have

$$\int_Q\Big(\sup_{I\ni x, |I|>2^K}|I|^{-\alpha}\int_I\chi_Q d\sigma\Big)^q d\omega\leqq C(n, \alpha, \delta, q)C_Q\int_Q d\mu_Q$$

where the constant $C(n, \alpha, \delta, q)$ is independent of K. This implies (1.5) when $K\to-\infty$, and we complete the proof of Theorem 1.

Proof of Corollary. Fix a cube Q. Let E be a subset of a subcube I in Q. From (1.8) we have

$$\int_E d\omega\left(|I|^{-\alpha}\int_I d\sigma\right)^q \leqq C_7\left(\int_I d\sigma\right)^{q/p} .$$

If $|E|\geqq\delta|I|$, by (1.9) the right hand side of the above is majorized by $C_9\left(\int_E d\sigma\right)^{q/p}$. Hence we obtain, because $q/p\geqq 1$,

$$\int_E d\omega\left(|I|^{-\alpha}\int_I d\sigma\right)^q \leqq C_9\left(\int_Q d\sigma\right)^{q/p-1}\int_E d\sigma .$$

The above inequality implies that the pair (ω, σ) satisfies the condition (II) in Theorem 2 with $\mu_Q=\left(\int_Q d\sigma\right)^{q/p-1}\sigma$. Then Theorem 2 implies the conclusion of Corollary.

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