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# On Some Differential Geometric Characterizations of the Center of a Lie Group

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## 1. Introduction.

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Then J. Milnor [1] and K. Uesu [3] proved an interesting characterization of the center of $\mathfrak{g}$ which asserts that an element $X$ in $\mathfrak{g}$ belongs to the center of $\mathfrak{g}$ if and only if for any left invariant Riemannian metric on $G$, the inequality $K(X, Y)\geq 0$ holds for all $Y$ in $\mathfrak{g}$, where $K(X, Y)$ denote the sectional curvature of the plane section spanned by $X$ and $Y$.

In this article we shall prove two analogous characterizations of the center of a Lie group:

THEOREM A. *For a connected Lie subgroup $H$ of $G$, the following conditions* (A-1) *and* (A-2) *are equivalent*;

(A-1) *$H$ is contained in the center of $G$,*

(A-2) *$H$ is totally geodesic with respect to any left invariant Riemannian metric on $G$.*

THEOREM B. *If $G$ is nilpotent or compact, then for each $X\in\mathfrak{g}$ the following conditions* (B-1) *and* (B-2) *are equivalent*;

(B-1) *$X$ belongs to the center of* $\mathfrak{g}$,

(B-2) *the inequality* $\mathrm{Ric}(X)\geq 0$ *holds for any left invariant Riemannian metric on $G$, where* $\mathrm{Ric}(X)$ *denotes the Ricci curvature in the direction $X$.*

## 2. Proof of Theorem A.

In this section the following range of indices will be used;

$$A, B, C, \cdots=1, 2, 3, \cdots, n=\dim G\,,$$

$$i, j, k, \cdots=1, 2, 3, \cdots, p=\dim H\,,$$

$$\alpha, \beta, \gamma, \cdots=p+1, \cdots, n\,.$$

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A basis $\{X_A\}$ of $\mathfrak{g}$ is called *adapted* provided that $X_i \in \mathfrak{h}$. We denote the corresponding structure constants by $\xi_{ABC}$. If $G$ is equipped with a left invariant Riemannian metric and if $\{X_A\}$ is an orthonormal adapted basis, then the second fundamental form $\sigma$ is given by ([2])

$$\sigma(X_i, X_j) = \frac{1}{2} \sum (\xi_{\alpha ij} + \xi_{\alpha ji}) X_\alpha . \tag{2.1}$$

Thus we easily obtain

PROPOSITION 2.2. *The following* (1) *and* (2) *are equivalent;*

(1) *$H$ is totally geodesic with respect to any left invariant Riemannian metric on $G$,*

(2) *the structure constants satisfy equalities $\xi_{\alpha ij} + \xi_{\alpha ji} = 0$ for each adapted basis of $\mathfrak{g}$.*

This proposition shows the implication (A-1) $\Rightarrow$ (A-2).

For the converse, we make the following change of basis, for an arbitrary fixed adapted basis $\{X_A\}$:

$$\begin{aligned} \bar{X}_i &= \sum b_i^j X_j , \\ \bar{X}_\alpha &= \sum a_\alpha^j X_j + \sum c_\alpha^\beta X_\beta , \end{aligned} \tag{2.3}$$

where $(b_i^j)$ and $(c_\alpha^\beta)$ are non-singular matrices of degree $p$ and $n-p$ respectively and $(a_\alpha^j)$ is a $(p, n-p)$-matrix. Then $\{\bar{X}_A\}$ also is an adapted basis and we denote the corresponding structure constants by $\bar{\xi}_{ABC}$. By a simple calculation, we have the following relation between $\xi_{\alpha ij}$ and $\bar{\xi}_{\alpha ij}$:

$$\bar{\xi}_{\alpha ij} = \sum \{(a_\alpha^h b_i^k \xi_{hkl} + b_i^k c_\alpha^\beta \xi_{\beta kl}) \bar{b}_l^j - a_\varepsilon^p b_i^k \bar{b}_p^j c_\alpha^\beta \bar{c}_\gamma^\varepsilon \xi_{\beta k\gamma}\} , \tag{2.4}$$

here $(\bar{b}_l^j)$ (resp. $(\bar{c}_\gamma^\varepsilon)$) denote the inverse matrix of $(b_l^j)$ (resp. $(c_\gamma^\varepsilon)$). Since the equalities $\bar{\xi}_{\alpha ij} + \bar{\xi}_{\alpha ji} = 0$ hold for all $(a_\alpha^j)$, $(b_i^j)$ and $(c_\alpha^\beta)$ by assumption for $H$ and Proposition 2.2, we obtain $\xi_{iAB} = 0$. This completes the proof of Theorem A. (q.e.d.)

### 3. Proof of Theorem B.

We first give some criterion for the existence of a left invariant Riemannian metric such that $\mathrm{Ric}(X) < 0$ for a given $X \in \mathfrak{g}$. Assume that $X$ does not belong to the center of $\mathfrak{g}$, and let $\{X_i\}$ be a basis of $\mathfrak{g}$ with $X_1 = X$. For each $n$-tuple $\Theta = (\theta_1, \cdots, \theta_n)$ of positive real numbers, we set $X_i^\Theta = X_i / \sqrt{\theta_i}$. Then we have ([1])

$$\begin{aligned} \mathrm{Ric}_\Theta(X_1^\Theta) = \sum_{i \geq 2, k \geq 1} \frac{1}{4\theta_1\theta_i} \Big\{ & -3\theta_k \xi_{1ik}^2 + 2\xi_{1ik}(\theta_1 \xi_{ik1} + \theta_i \xi_{k1i}) \\ & + \frac{1}{\theta_k} (\theta_1 \xi_{ik1} - \theta_i \xi_{k1i})^2 - \frac{4\theta_1\theta_i}{\theta_k} \xi_{k11} \xi_{kii} \Big\} , \end{aligned} \tag{3.1}$$

where $\mathrm{Ric}_\Theta$ denotes the Ricci curvature of the left invariant Riemannian metric for which the basis $\{X_i^\Theta\}$ is orthonormal. If we set

$$N(X)=\{Y\in\mathfrak{g} \mid [X, Y]\neq 0\},$$

$$A(X)=\{Y\in N(X)\mid X, Y, [X, Y] \text{ are linerly independent}\},$$

$$B(X)=\{Y\in N(X) \mid [X, Y]=\alpha X+\beta Y \text{ and } \beta\neq 0\},$$

$$C(X)=\{Y\in N(X) \mid [X, Y]=\alpha X\},$$

then $N(X)\neq\varnothing$ and $N(X)=A(X)\cup B(X)\cup C(X)$ (disjoint).

In the case where $A(X)\cup B(X)\neq\varnothing$, if we take $X_2$ in $A(X)\cup B(X)$, the inequality $\mathrm{Ric}_\Theta(X_1^\Theta)<0$ holds for suitable $\Theta$ by (3.1).

In the case where $A(X)\cup B(X)=\varnothing$, we define real valued functions $\lambda_X$ and $\Lambda_X$ by

$$[X, Y]=\lambda_X(Y)X, \tag{3.2}$$

$$\Lambda_X(Y)=\lambda_X(Y)\cdot\mathrm{trace}\,\mathrm{ad}(Y), \qquad Y\in\mathfrak{g}. \tag{3.3}$$

Then $\mathfrak{g}=C(X)\cup\mathfrak{z}_X$ (disjoint), here $\mathfrak{z}_X$ denote the centralizer of $\{X\}$. Now let us take $X_2$ in $C(X)=N(X)$ and set $\theta_1=\theta$, $\theta_3=\cdots=\theta_n=\theta^2$. Then (3.1) implies

$$\lim_{\theta\to\infty}\mathrm{Ric}_\Theta(X_1^\Theta)=\frac{1}{\theta_2}\Lambda_X(X_2).$$

Therefore if $\Lambda_X(Y)<0$ holds for some $Y\in C(X)$, there exists a certain $\Theta$ such that $\mathrm{Ric}_\Theta(X_1^\Theta)<0$. Conversely assume that there exists a left invariant Riemannian metric such that $\mathrm{Ric}(X)<0$. If we take an orthonormal basis $\{X_1\,(=X), X_2, \cdots, X_n\}$ with $X_2\in C(X)$ and $X_\alpha\in\mathfrak{z}_X$, $\alpha\geq 3$, the corresponding structure constants $\xi_{ijk}$ satisfy $\xi_{12i}=\xi_{1\alpha k}=0$ for $i\geq 2$, $k\geq 1$ and $\alpha\geq 3$. Thus we obtain from (3.1)

$$\mathrm{Ric}(X)=\Lambda_X(X_2)+\frac{1}{4}\left(\sum_{\beta\geq 3}\xi_{2\beta 1}^{2}+\sum_{\alpha\geq 3}\xi_{\alpha 2 1}^{2}+\sum_{\alpha\geq 3,\beta\geq 3}\xi_{\alpha\beta 1}^{2}\right),$$

so that $\Lambda_X(X_2)<0$.

Summarizing the above argument, we obtain

LEMMA 3.4. *For each $X$ not belonging to the center of $\mathfrak{g}$, the following conditions* (1) *and* (2) *are equivalent*;

(1) *there exists a left invariant Riemannian metric on $G$ such that* $\mathrm{Ric}(X)<0$,

(2) $N(X)\neq C(X)$ *or* $N(X)=C(X)$ *and* $\Lambda_X(Y)<0$ *for some* $Y\in C(X)$.

Now if $G$ is nilpotent or compact, its Lie algebla $\mathfrak{g}$ does not admit such an element $X$ as is not contained in the center and satisfies the condition $N(X)=C(X)$. Therefore Lemma 3.4 implies Theorem B. (q.e.d.)

REMARK. In general, Theorem B does not hold for unimodular or solvable Lie groups.

## References


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