TOKYO J. MATH.
VOL. 15, NO. 1, 1992

The C^1 Uniform Pseudo-Orbit Tracing Property

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(Communicated by J. Tomiyama)

Introduction.

Let M be a closed C^∞ manifold and denote by $\mathrm{Diff}^1(M)$ the set of all diffeomorphisms of M endowed with the C^1 topology. For $\delta>0$, a sequence $\{x_i\}_{i=a}^{b-1}$ $(-\infty\leq a<b\leq\infty)$ is called a *δ-pseudo-orbit* for $f\in\mathrm{Diff}^1(M)$ if $d(f(x_i), x_{i+1})<\delta$ for $a\leq i\leq b-1$, where d is a Riemannian distance on M. Given $\varepsilon>0$, a sequence $\{x_i\}_{i=a}^{b}$ is said to be *f-ε-traced* by a point $x\in M$ if $d(f^i(x), x_i)<\varepsilon$ for $a\leq i\leq b$. We say that f has the *pseudo-orbit tracing property* (abbrev. POTP) if for $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit for f can be f-ε-traced by some point in M. For compact spaces these are independent of the compatible metrics used. We say that f satisfies C^1 *uniform pseudo-orbit tracing property* (abbrev. C^1-UPOTP) if there is a C^1 neighborhood $\mathfrak{U}(f)$ of f with the property that for $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of $g\in\mathfrak{U}(f)$ is g-ε-traced by some point. By definition it is checked that C^1-UPOTP is stronger than POTP.

Robinson [5] proved that if $f\in\mathrm{Diff}^1(M)$ satisfies Axiom A and strong transversality, then f has POTP. We show the following.

THEOREM. *If $f\in\mathrm{Diff}^1(M)$ satisfies Axiom A and strong transversality, then f satisfies C^1-UPOTP.*

Combining the above theorem and a result proved in [6] we have the following corollary.

COROLLARY. *If the dimension of M is ≤ 3, then the set of all diffeomorphisms having C^1-UPOTP is characterized as the set of all Axiom A diffeomorphisms satisfying strong transversality.*

§1. Preliminary results.

Let $\Omega(f)$ be the non-wandering set of an Axiom A diffeomorphism f. The *local stable* and *unstable manifolds* are defined by

Received February 20, 1991

$$W_\varepsilon^s(x,f)=\{y\in M : d(f^n(x),f^n(y))\le\varepsilon \text{ for all } n\ge 0\}\,,$$
$$W_\varepsilon^u(x,f)=\{y\in M : d(f^{-n}(x),f^{-n}(y))\le\varepsilon \text{ for all } n\ge 0\}$$

for $x\in\Omega(f)$. It is well known that there are $0<\lambda<1$ and $\varepsilon_0>0$ such that for every $0<\varepsilon\le\varepsilon_0$ and $x\in\Omega(f)$,

$$d(f^n(y),f^n(z))\le\lambda^n d(y,z) \qquad \text{for } y,z\in W_\varepsilon^s(x,f) \text{ and } n\ge 0\,,$$
$$d(f^{-n}(y),f^{-n}(z))\le\lambda^n d(y,z) \qquad \text{for } y,z\in W_\varepsilon^u(x,f) \text{ and } n\ge 0\,.$$

For $x\in\Omega(f)$ the *stable* and *unstable manifolds* are defined by

$$W^s(x,f)=\bigcup_{n\ge 0} f^{-n}(W_{\varepsilon_0}^s(f^n(x),f))\,,$$
$$W^u(x,f)=\bigcup_{n\ge 0} f^{n}(W_{\varepsilon_0}^u(f^{-n}(x),f))\,.$$

The $\Omega(f)$ is decomposed as a union $\Omega(f)=\Lambda_1(f)\cup\cdots\cup\Lambda_l(f)$ of basic sets $\Lambda_i(f)$. Then it is well known that $W^s(\Lambda_i(f),f)\cap W^u(\Lambda_i(f),f)=\Lambda_i(f)$ for $1\le i\le l$ and $M=\bigcup_{i=1}^l W^\sigma(\Lambda_i(f),f)$, where

$$W^\sigma(\Lambda_i(f),f)=\bigcup_{x\in\Lambda_i(f)} W^\sigma(x,f) \qquad \text{for } \sigma=s,u\,.$$

An Axiom A diffeomorphism f satisfies *strong transversality* if and only if the stable manifold $W^s(x,f)$ and the unstable manifold $W^u(x,f)$ are transversal for all $x\in M$ (i.e. $T_xW^s(x,f)+T_xW^u(x,f)=\mathrm{T}_xM$). A *cycle* for a family $\{\Lambda_i(f)\}_{i=1}^l$ is a subsequence $\Lambda_{i_1}(f),\cdots,\Lambda_{i_k}(f)$ such that $\Lambda_{i_1}(f)=\Lambda_{i_k}(f)$ and $W^u(\Lambda_{i_j}(f),f)\cap W^s(\Lambda_{i_{j+1}}(f),f)\ne\emptyset$ for $1\le j<k$. Recall that the *chain recurrent set* for f, $R(f)$, is the set of $x\in M$ such that for every $\delta>0$ there is a δ-pseudo-orbit of f from x to x. Strong transversality implies no cycles (see [7]). Thus f is Ω-stable and $R(f)$ coincides with $\Omega(f)$ (see [1] and [8]).

Let λ be as above. For $\varepsilon>0$ with $0<\varepsilon+\lambda<1$, there is a compact neighborhood V_i of $\Lambda_i(f)$ $(1\le i\le l)$ on which there exists an extended continuous splitting

$$T_{V_i}M=\tilde{E}_i^s\oplus\tilde{E}_i^u$$

of $T_{\Lambda_i(f)}M=E_i^s\oplus E_i^u$ (see [3]), and there is a C^1 neighborhood $\mathfrak{U}(f)\subset\mathrm{Diff}^1(M)$ of f such that for every $g\in\mathfrak{U}(f)$ and $x\in V_i\cap g^{-1}(V_i)$, D_xg can be written as

$$D_xg=\begin{pmatrix} A_x & B_x \\ C_x & K_x\end{pmatrix}:\ \tilde{E}_i^s(x)\oplus\tilde{E}_i^u(x)\longrightarrow\tilde{E}_i^s(g(x))\oplus\tilde{E}_i^u(g(x))$$

satisfying $\max\{\|A_x\|,\|K_x^{-1}\|\}<\lambda$ and $\max\{\|B_x\|,\|C_x\|\}<\varepsilon$. By choosing $\mathfrak{U}(f)$ small enough, we may assume that there is a compact neighborhood U_i of $\Lambda_i(f)$ $(1\le i\le l)$ such that for every $g\in\mathfrak{U}(f)$, we have $\Lambda_i(f)\subset U_i\subset V_i\cap g^{-1}(V_i)$.

Put $\delta=\min\frac{1}{4}\{1-(\lambda+\varepsilon),\lambda^{-1}-\varepsilon-1\}>0$. Then there are a C^1 neighborhood

$\mathfrak{U}_1(f)\subset\mathfrak{U}(f)$ and $\alpha_1>0$ such that for every $g\in\mathfrak{U}_1(f)$, if $0<\alpha\leq\alpha_1$ and $\|v\|<\alpha$ ($v\in T_xM$, $x\in M$), then

$$\|\exp_{g(x)}^{-1}\circ g\circ\exp_x v-D_xg(v)\|\leq\delta\|v\| .$$

For $\alpha>0$, define $\tilde{B}_i(x,\alpha)=\tilde{E}_i^s(x,\alpha)\times\tilde{E}_i^u(x,\alpha)$, where $\tilde{E}_i^\sigma(x,\alpha)=\{v\in\tilde{E}_i^\sigma(x):\|v\|\leq\alpha\}$ ($\sigma=s,u$). Then there is $0<\alpha_2<\alpha_1$ such that for every $x\in\bigcup_{i=1}^l U_i$,

$$\tilde{B}_i(x,\alpha_2)\subset\{v\in T_xM:\|v\|\leq\alpha_1\}$$

and

$$g(\exp_x\tilde{B}_i(x,\alpha_2))\subset\exp_{g(x)}\tilde{B}_i(g(x),\alpha_1)\qquad(g\in\mathfrak{U}_1(f)) .$$

PROPOSITION 1.1. *Under the above notations, for every $0<\alpha\leq\alpha_2$ there exists $\beta>0$ such that if $\{x_j\}_{j=0}^\infty\subset U_i$ $(1\leq i\leq l)$ is a β-pseudo-orbit of $g\in\mathfrak{U}_1(f)$, then there is $y\in M$ such that $d(g^j(y),x_j)<\alpha$ for all j.*

PROOF. Fix $1\leq i\leq l$. For $0<\alpha\leq\alpha_2$ and $x\in U_i$ put $B_i(x,\alpha)=\exp_x\tilde{B}_i(x,\alpha)$ and $\beta'=\min\frac{1}{8}\{1-\lambda-\varepsilon,\lambda^{-1}-\varepsilon-1\}\alpha>0$. Then there is $0<\beta\leq\beta'$ such that $d(x,y)<\beta$ $(x,y\in U_i)$ implies

$$\rho_1(\exp_x\tilde{E}_i^\sigma(x,\alpha),\exp_y\tilde{E}_i^\sigma(y,\alpha))<\beta'\qquad(\sigma=s,u) ,$$

where ρ_1 denotes the standard C^1 metric.

Let $\{x_j\}_{j=0}^\infty\subset U_i$ be a β-pseudo-orbit of $g\in\mathfrak{U}_1(f)$. Then $g(B_i(x_j,\alpha))$ stretches across the box $B_i(x_{j+1},\alpha)$ in the unstable direction and contracts on the stable direction. The choice of β implies

$$z_k\in g(g(\cdots g(g(B_i(x_0,\alpha))\cap B_i(x_1,\alpha))\cap B_i(x_2,\alpha)\cdots)\cap B_i(x_{k-1},\alpha))\cap B_i(x_k,\alpha)\neq\emptyset$$

for $k>0$. Put $y_k=g^{-k}(z_k)$. Then $y_k\in B_i(x_0,\alpha)$, $g(y_k)\in B_i(x_1,\alpha)$, $\cdots$, $g^k(y_k)\in B_i(x_k,\alpha)$. Therefore $d(g^j(y),x_j)\leq\alpha$ for all $j\geq 0$ where $y=\lim_{k\to\infty}y_k$.

Since f is Ω-stable, every $g\in\mathfrak{U}_1(f)$ has the spectral decomposition $\Omega(g)=\Lambda_1(g)\cup\cdots\cup\Lambda_l(g)$. It is known (see [3]) that there exist a C^1 neighborhood $\mathfrak{U}_2(f)\subset\mathfrak{U}_1(f)$ and a compact neighborhood $\hat{U}_i\subset U_i$ of $\Lambda_i(f)$ $(1\leq i\leq l)$ satisfying the following properties (1.1), (1.2) and (1.3).

For every $g\in\mathfrak{U}_2(f)$ each of $\hat{U}_i$ contains $\Lambda_i(g)$ and

(1.1) there is a continuous extension $T_{\hat{U}_i}M=\hat{E}_{i,g}^s\oplus\hat{E}_{i,g}^u$ of $T_{\Lambda_i(g)}M=E_{i,g}^s\oplus E_{i,g}^u$ such that

$$\begin{cases}D_xg(\hat{E}_{i,g}^s(x))=\hat{E}_{i,g}^s(g(x)) \text{ and } \|D_xg|_{\hat{E}_{i,g}^s(x)}\|<\lambda & \text{if } x\in\hat{U}_i\cap g^{-1}(\hat{U}_i) , \text{ and}\\ D_xg^{-1}(\hat{E}_{i,g}^u(x))=\hat{E}_{i,g}^u(g^{-1}(x)) \text{ and } \|D_xg^{-1}|_{\hat{E}_{i,g}^u(x)}\|<\lambda & \text{if } x\in\hat{U}_i\cap g(\hat{U}_i) .\end{cases}$$

(1.2) There exist $\eta>0$ and submanifolds $\hat{W}_\eta^\sigma(x,g)$ $(x\in\hat{U}_i)$ for $\sigma=s,u$ such that

$$\begin{cases} g(\hat{W}^s_\eta(x, g)) \subset \hat{W}^s_\eta(g(x), g) \quad \text{and} \quad d(g(x), g(y)) < \lambda d(x, y) \ (y \in \hat{W}^s_\eta(x, g)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \in \hat{U}_i \cap g^{-1}(\hat{U}_i)\,, \quad \text{and} \\ g^{-1}(\hat{W}^u_\eta(x, g)) \subset \hat{W}^u_\eta(g^{-1}(x), g) \quad \text{and} \quad d(g^{-1}(x), g^{-1}(y)) < \lambda d(x, y) \ (y \in \hat{W}^u_\eta(x, g)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } x \in \hat{U}_i \cap g(\hat{U}_i)\,. \end{cases}$$

(1.3) There exists $\gamma > 0$ such that if $d(x, y) < \gamma$ $(x, y \in \hat{U}_i)$, then $\hat{W}^s_\eta(x, g)$ and $\hat{W}^u_\eta(y, g)$ meet transversely.

Let $B(\Lambda_i(f))$ be a compact neighborhood such that $\Lambda_i(f) \subset B(\Lambda_i(f)) \subset \hat{U}_i \cap g^{-1}(\hat{U}_i) \cap g(\hat{U}_i)$ for $g \in \mathfrak{U}_2(f)$. For subspaces E and F of T_xM $(x \in B(\Lambda_i(f)))$ define

$$\tan \measuredangle (F, E) = \sup\{\|w_2\|/\|w_1\| : w_1 \in E\,,\ w_2 \in E^\perp \text{ and } w_1 + w_2 \in F - \{0\}\}\,.$$

Then we can find $\theta_{1,i} > 0$ such that for every $g \in \mathfrak{U}_2(f)$, $\tan \measuredangle (E^u_{i,g}, E^{s\perp}_{i,g}) < \theta_{1,i}$ (see [4]). Thus, by (1.1) there exists $\theta_{2,i} > 0$ such that

$$\tan \measuredangle (\hat{E}^u_{i,g}, \hat{E}^{s\perp}_{i,g}) < \theta_{2,i} \qquad (g \in \mathfrak{U}_2(f))\,.$$

LEMMA 1.2. *Let $\theta_2 = \max\{\theta_{2,i} : 1 \le i \le l\}$. For $0 < \theta < \theta_2^{-1}(2 + \theta_2)^{-1}$ there is $K(\theta) > 0$ such that for every $v \in T_xM$ $(x \in B(\Lambda_i(f)))$ and $g \in \mathfrak{U}_2(f)$, if $\tan \measuredangle (v, \hat{E}^s_{i,g}(x)) < \theta$ and $\{x, g(x), \cdots, g^N(x)\} \subset B(\Lambda_i(f))$ for some $N > 0$, then*

$$\tan \measuredangle (D_x g^{-N}(v), \hat{E}^s_{i,g}(g^{-N}(x))) \le K(\theta) \cdot \lambda^{2N}\,.$$

For the proof see [6, Claim 1].

Since $\mathfrak{U}_2(f)$ is very small, we take it satisfying $\operatorname{Ind} \Lambda_i(f) = \operatorname{Ind} \Lambda_i(g)$ $(1 \le i \le l)$ for all $g \in \mathfrak{U}_2(g)$. Here $\operatorname{Ind} \Lambda$ denotes the dimension of the stable subbundle E^s of a basic set Λ. Let $B_r(A)$ denote the closed neighborhood of a compact set A of M with radius $r > 0$.

LEMMA 1.3. *There are a C^1 neighborhood $\mathfrak{U}_3(f) \subset \mathfrak{U}_2(f)$ and constants $\theta > 0$ and $r_0 > 0$ $(B_{r_0}(\Lambda_i(f)) \subset B(\Lambda_i(f))$ for $1 \le i \le l)$ such that for $g \in \mathfrak{U}_3(f)$ and $1 \le i \ne j \le l$ $(\operatorname{Ind} \Lambda_i(g) \ge \operatorname{Ind} \Lambda_j(g))$, if $x \in \Lambda_j(g)$ and $y \in W^u(x, g) \cap B_{r_0}(\Lambda_i(f))$, then there is a linear subspace $H_y \subset T_y W^u(x, g)$ $(\dim H_y = \dim M - \operatorname{Ind} \Lambda_i(g))$ such that*

(i) *if $\operatorname{Ind} \Lambda_i(g) \ge \dim M - \operatorname{Ind} \Lambda_i(g)$, then $\tan \measuredangle (H_y, \tilde{E}^s_i(y)) \ge \theta$ and*

(ii) *if $\operatorname{Ind} \Lambda_i(g) < \dim M - \operatorname{Ind} \Lambda_i(g)$, then $\tan \measuredangle (H_y, \tilde{E}^{s\perp}_i(y)) \le \theta^{-1}$.*

PROOF. If this is false, for every $n > 0$ there are $g_n \in \operatorname{Diff}^1(M)$ $(\rho_1(g_n, f) < 1/n)$, $1 \le i_n \ne j_n \le l$ $(\operatorname{Ind} \Lambda_{i_n}(g_n) \ge \operatorname{Ind} \Lambda_{j_n}(g_n))$, $x_n \in \Lambda_{j_n}(g_n)$ and $y_n \in W^u(x_n, g_n) \cap B_{1/n}(\Lambda_{i_n}(f))$ such that for every linear subspace $H_{y_n} \subset T_{y_n} W^u(x_n, g_n)$ $(\dim H_{y_n} = \dim M - \operatorname{Ind} \Lambda_{i_n}(g_n))$,

$$\tan \measuredangle (H_{y_n}, \tilde{E}^s_{i_n}(y)) < \frac{1}{n} \qquad \text{if } \operatorname{Ind} \Lambda_{i_n}(g_n) \ge \dim M - \operatorname{Ind} \Lambda_{i_n}(g_n)$$

and

$$\tan \measuredangle (H_{y_n}, \tilde{E}^{s\perp}_{i_n}(y_n)) > n \qquad \text{if } \operatorname{Ind} \Lambda_{i_n}(g_n) < \dim M - \operatorname{Ind} \Lambda_{i_n}(g_n) .$$

We assume, without loss of generality, that $i_n = i$, $j_n = j$ for all $n > 0$. By (1.2) and (1.3) there are $z_n \in \Lambda_i(g_n)$ and $w_n = \hat{W}^u_\eta(y_n, g_n) \cap \hat{W}^s_\eta(z_n, g_n)$. Notice that y_n converges to a point of $\Lambda_i(f)$. Since $\Lambda_i(g_n) \to \Lambda_i(f)$ $(n \to \infty)$ and $y_n \in W^u(\Lambda_j(g_n), g_n)$, we can find a strictly increasing sequence $J_n > 0$ such that $g_n^{-k}(y_n) \in B(\Lambda_j(f))$ $(0 \le k \le J_n)$ and $g_n^{-J_n - 1}(y_n) \notin B(\Lambda_i(f))$. Let us put

$$r_1 = \inf_{g \in \mathfrak{U}_2(f)} \{ d(x, \Lambda_i(f)) : x \in B(\Lambda_i(f)) \text{ and } g^{-1}(x) \notin B(\Lambda_i(f)) \} .$$

Since

$$d(g_n^{-j}(y_n), g_n^{-j}(w_n)) \le \lambda^j d(y_n, w_n) \tag{1.4}$$

for $0 \le j \le J_n$, there is $N > 0$ such that for all $n \ge N$, $g_n^{-J_n}(y_n) \in B(\Lambda_i(f)) \setminus B_{r_1}(\Lambda_i(f))$ and $g_n^{-J_n}(w_n) \notin B_{r_1/2}(\Lambda_i(f))$. Thus there exists $c > 0$ such that

$$\begin{cases} g_n^{-j}(B_c(g_n^{-J_n}(y_n))) \cap B_c(g_n^{-J_n}(y_n)) = \varnothing , \\ g_n^{j}(B_c(g_n^{-J_n}(w_n))) \cap B_c(g_n^{-J_n}(w_n)) = \varnothing \end{cases} \tag{1.5}$$

for every $n > 0$ and $j > 0$.

We deal with only the case $\operatorname{Ind} \Lambda_i(g_n) \ge \dim M - \operatorname{Ind} \Lambda_i(g_n)$ for all $n > 0$ (since the other case follows in a similar way). For every $n > 0$, let $H_{y_n} \subset T_{y_n} W^u(x_n, g_n)$ be a linear subspace such that $\dim H_{y_n} = \dim M - \operatorname{Ind} \Lambda_i(g_n)$ and $H_{y_n} \not\subset \hat{E}^s_{i,g_n}(y_n)$. Take $0 < \theta' < \theta_2^{-1}(2 + \theta_2)^{-1}$. Then there is $N' > N$ such that for every $n \ge N'$, $\tan \measuredangle (H_{y_n}, \hat{E}^s_{i,g_n}(y_n)) \le \theta'$ (since $\tan \measuredangle (\hat{E}^s_{i,g_n}(y_n), \tilde{E}^s_i(y_n)) \to 0$ as $n \to \infty$). By lemma 1.2

$$\tan \measuredangle (D_{y_n} g_n^{-J_n}(H_{y_n}), \hat{E}^s_{i,g_n}(g_n^{-J_n}(y_n))) \to 0$$

as $n \to \infty$, and so

$$\tan \measuredangle (D_{y_n} g_n^{-J_n}(H_{y_n}), \delta_{g_n^{-J_n}(y_n) \circ g_n^{-J_n}(w_n)} \circ \hat{E}^s_{i,g_n}(g_n^{-J_n}(w_n))) \to 0$$

as $n \to \infty$, where $\delta_{x \circ y} : T_x M \to T_y M$ denotes the parallel transform. From (1.4) and (1.5) there are $n > N'$ and $\tilde{g}_n$ arbitrarily near to f such that $W^u(x_n, \tilde{g}_n) \cap W^s_\eta(z_n, \tilde{g}_n) \ne \varnothing$ and $W^u(x_n, \tilde{g}_n)$ does not meet transversally to $W^s_\eta(z_n, \tilde{g}_n)$, thus contradicting.

LEMMA 1.4. *There are a C^1 neighborhood $\mathfrak{U}_4(f) \subset \mathfrak{U}_3(f)$ and compact neighborhoods $\Lambda_i(f) \subset B_i \subset B_{r_0/4}(\Lambda_i(f))$ $(1 \le i \le l)$ such that for every $0 \le \varepsilon \le \varepsilon_1$, there exists $0 < \delta \le \varepsilon$ such that $g \in \mathfrak{U}_4(f)$, $1 \le i \ne j \le l$ $(\operatorname{Ind} \Lambda_i(g) \ge \operatorname{Ind} \Lambda_j(g))$, $y \in W^u(x, g) \cap B_i$ $(x \in \Lambda_j(g))$ and $d(z, y) < \delta$ $(z \in B_i)$ imply $C_y \cap \exp_z \tilde{E}^s_i(z, \varepsilon) \ne \varnothing$. Here C_y denotes the connected component of y in $W^u(x, g) \cap B_{r_0}(\Lambda_i(f))$.*

PROOF. If this is false, then there is $0 < \varepsilon \le \varepsilon_1$ such that for every $n > 0$ there are $g_n \in \operatorname{Diff}^1(M)$, $1 \le i_n \ne j_n \le l$ $(\operatorname{Ind} \Lambda_{i_n}(g_n) \ge \operatorname{Ind} \Lambda_{j_n}(g_n))$, $x_n \in \Lambda_{j_n}(g_n)$, $y_n \in W^u(x_n, g_n) \cap B_{1/n}(\Lambda_{i_n}(f))$ and $z_n \in B_{1/n}(\Lambda_{i_n}(f))$ satisfying

$$\rho_1(g_n, f) < \frac{1}{n}, \qquad d(z_n, y_n) < \frac{1}{n}$$

and

$$C_{y_n} \cap \exp_{z_n} \tilde{E}^s_{i_n}(z_n, \varepsilon) = \varnothing . \tag{1.6}$$

We assume, without loss of generality, that $i_n = i$, $j_n = j$ for all $n > 0$ and put $w_n = \hat{W}^s_\eta(y_n, g_n) \cap \hat{W}^u_\eta(z_n, g_n)$. Thus $d(y_n, w_n) \to 0$ as $n \to \infty$ (since $d(z_n, y_n) \to 0$ as $n \to \infty$). Since z_n converges to $\Lambda_i(f)$, as in the proof of lemma 1.3, we can find a strictly increasing sequence $J_n > 0$ such that $g_n^{-k}(y_n) \in B_{r_0/2}(\Lambda_i(f))$ $(0 \leq k \leq J_n)$, $g_n^{-J_n-1}(y_n) \notin B_{r_0/2}(\Lambda_i(f))$. By lemma 1.3, for every $n > 0$ there is an $\mathrm{Ind}\,\Lambda_i(g_n)$-dimensional subdisk $D(g_n^{-J_n}(y_n)) \subset W^u(g_n^{-J_n}(x_n), g_n)$ centered at $g_n^{-J_n}(y_n)$ such that $\tan \sphericalangle(T_x D(g_n^{-J_n}(y_n)), \tilde{E}^s_i(x)) \geq \theta$ or $\tan \sphericalangle(T_x D(g_n^{-J_n}(y_n)), \tilde{E}^{s\perp}_i(x)) \leq \theta^{-1}$ for $x \in D(g_n^{-J_n}(y_n))$. We can assume that $\inf_{n>0} \operatorname{diam} D(g_n^{-J_n}(y_n)) > 0$, where $\operatorname{diam} D$ $(D \subset W^u(x))$ denotes the diameter of D with respect to the metric on $W^u(x)$ induced from $\|\cdot\|$. Put $C_\eta(g_n^{-J_n}(y_n)) = D(g_n^{-J_n}(y_n)) \cap B_\eta(g_n^{-J_n}(y_n))$ and let $C_\eta(g_n^{-J_n+j}(y_n))$ be the connected component of $g_n^{-J_n+j}(y_n)$ in $g_n(C_\eta(g_n^{-J_n+j-1}(y_n))) \cap B_\eta(g_n^{-J_n+j}(y_n))$ for $1 \leq j \leq J_n$. Then, by (1.2) $C_\eta(y_n)$ converges to $\hat{W}^u_\eta(w_n, g_n)$ and so $C_{y_n} \cap \exp_{z_n} \tilde{E}^s_i(z_n, \varepsilon) \neq \varnothing$ for sufficiently large n, thus contradicting.

§ 2. The behavior of pseudo-orbits.

The purpose of this section is to investigate the behavior of pseudo-orbits of an Axiom A diffeomorphism f satisfying strong transversality. To do so we use a filtration, i.e. a sequence of a compact subsets $\{M_i\}_{i=0}^{l}$ of M and a sequence of integers $\{m_i\}_{i=1}^{l}$ such that

$$\varnothing = M_0 \subset M_1 \subset \cdots \subset M_l = M ,$$

$$f(M_i) \subset \operatorname{int} M_i , \qquad M_i \subset \bigcup_{1 \leq j \leq i} W^s(\Lambda_j(f)) ,$$

$$f^{m_i}(M_{i-1}) \subset \operatorname{int} M_i \quad \text{and}$$

$$\Lambda_i(f) \subset \operatorname{int} M_i \setminus f^{-m_i}(M_{i-1}) \subset B_i \qquad \text{for} \quad 1 \leq i \leq l .$$

Fix $\beta_0 > 0$ such that for every $1 \leq i \leq l$

$$B_{\beta_0}(f(M_i)) \subset \operatorname{int} M_i \quad \text{and}$$

$$B_{\beta_0}(f(f^{-m_i}(M_{i-1}))) \subset \operatorname{int} f^{-m_i}(M_{i-1}) ,$$

and choose $\gamma_0 > 0$ such that $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_0)$ imply

$$B_{\beta_0}(g(M_i)) \subset \operatorname{int} M_i \quad \text{and}$$

$$B_{\beta_0}(g(f^{-m_i}(M_{i-1}))) \subset \operatorname{int} f^{-m_i}(M_{i-1})$$

for every $1 \le i \le l$.

PROPOSITION 2.1. *Under the above notations, there are constants* $\gamma_3 > 0$, $L > 0$ *and* $\beta_3 > 0$ *such that for* $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_3)$, $2 \le j \le l-1$ *and a* β*-pseudo-orbit* $\{x_i\}$ *of* g $(0 < \beta \le \beta_3)$, *if there exists* $i_1 \in \boldsymbol{Z}$ *such that* $x_i \in B_{j+1}$ $(i \le i_1)$ *and* $x_{i_1+1} \notin B_{j+1}$, *then* $x_{i_1+L} \in \mathrm{int}\, M_j$ *and*

(i) $x_{i_1+2L} \in \mathrm{int}\, M_{j-1}$ *when* $x_{i_1+L} \in f^{-m_j}(M_{j-1})$,

(ii) $x_{i_1+3L} \in \mathrm{int}\, M_{j-1}$ *when* $x_{i_1+L} \notin f^{-m_j}(M_{j-1})$ *and* $\mathrm{Ind}\, \Lambda_j(f) < \mathrm{Ind}\, \Lambda_{j+1}(f)$, *and furthermore*

(iii) *one of the following* (a) *and* (b) *holds when* $x_{i_1+L} \notin f^{-m_j}(M_{j-1})$ *and* $\mathrm{Ind}\, \Lambda_j(f) \ge \mathrm{Ind}\, \Lambda_{j+1}(f)$;

(a) $x_{i_1+L+n} \in B_j$ *for all* $n \ge 0$,

(b) *there exists* $n > 0$ *such that* $x_{i_1+L+k} \in B_j$ $(0 < k < n)$, *and* $x_{i_1+2L+n} \in \mathrm{int}\, M_{j-1}$.

The proof is divided into three lemmas.

LEMMA 2.2. *There are* $0 < \gamma_1 < \gamma_0$, $L_1 > 0$ *and* $0 < \beta_1 \le \beta_0$ *such that for* $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_1)$ *and a* β*-pseudo-orbit* $\{x_i\}$ *of* g $(0 < \beta \le \beta_1)$, *if there exist* $i_1 \in \boldsymbol{Z}$ *and* $1 \le j \le l-1$ *such that* $x_i \in B_{j+1}$ $(i \le i_1)$ *and* $x_{i_1} \notin B_{j+1}$, *then* $x_{i_1+L_1} \in \mathrm{int}\, M_j$.

PROOF. First we prove that there are $0 < \gamma_1 < \gamma_0$, $L_1 > 0$ and $0 < \beta_1 \le \beta_0$ such that for $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_1)$ and a β-pseudo-orbit $\{x_i\}$ of g $(0 < \beta \le \beta_1)$, if there exist $1 \le j \le l-1$ and $i_1 \in \boldsymbol{Z}$ such that $x_i \in B_{j+1}$ $(i \le i_1)$, and $x_{i_1+1} \notin B_{j+1}$, then there exists $0 < l_1 \le L_1$ satisfying $x_{i_1+l_1} \in \mathrm{int}\, M_j$.

Suppose that this is false. For every $n > 0$ there are $g_n \in \mathrm{Diff}^1(M)$ $(d(f, g_n) < 1/n)$, $1 \le j_n \le l$, a $(1/n)$-pseudo-orbit $\{x_i^n\}$ of g_n and an integer i_i^n such that $x_i^n \in B_{j_n+1}$ $(i \le i_1^n)$, $x_{i_i^n+1}^n \notin B_{j_n+1}$ but $x_{i_1^n+i}^n \notin \mathrm{int}\, M_{j_n}$ for all $0 < i \le n$. We may assume that $j_n = j$ for all $n > 0$ and

$$x = \lim_{n\to\infty} x_{i_n+1}^n \notin \Lambda_{j+1}(f)\,. \tag{2.1}$$

Since $x_{-1} = \lim_{n\to\infty} x_{i_n}^n \in B_{j+1}$, we have

$$\begin{aligned} d(f(x_{-1}), x) \le d(f(x_{-1}), f(x_{i_n}^n)) &+ d(f(x_{i_n}^n, g_n(x_{i_n}^n)) \\ &+ d(g_n(x_{i_n}^n), x_{i_n+1}) + d(x_{i_n+1}, x) \to 0 \qquad (n \to \infty) \end{aligned}$$

and so $f(x_{-1}) = x$. Since $x_{-2} = \lim_{n\to\infty} x_{i_n-1}^n \in B_{j+1}$, we have $f(x_{-2}) = x_{-1}$, and so $f^2(x_{-2}) = x$. Inductively we have $f^{-i}(x) \in B_{j+1}$ for all $i \ge 0$ and so

$$x \in W^u(\Lambda_{j+1}(f))\,. \tag{2.2}$$

Put $x_i = \lim_{n\to\infty} x_{i_n+1+i}^n \notin \mathrm{int}\, M_j$ for $i \ge 1$. As above we have for all $i > 0$

$$f^i(x) = x_i \notin \mathrm{int}\, M_j\,. \tag{2.3}$$

Since $x \in W^s(\Lambda_{j+1}(f))$ and $f^{-i}(x)$ converges to a point of $\Lambda_{j+1}(f) \subset \mathrm{int}\, M_{j+1}$ as $i \to \infty$, we have $x \in M_{j+1}$. Thus $x \in \bigcup_{1 \le k \le j+1} W^s(\Lambda_k(f))$ (since $M_{j+1} \subset \bigcup_{1 \le k \le j+1} W^s(\Lambda_k(f))$). However $x \notin \bigcup_{1 \le k \le j} W^s(\Lambda_k(f))$. Indeed, if there is $1 \le k \le j$ such that $x \in W^s(\Lambda_k(f))$, then $f^i(x)$ converges to a point of $\Lambda_k(f) \subset \operatorname{int} M_k \subset \operatorname{int} M_j$ as $i \to \infty$, which is contrary to (2.3). Thus $x \in W^s(\Lambda_{j+1}(f)) \cap W^u(\Lambda_{j+1}(f)) = \Lambda_{j+1}(f)$ by (2.2). This is a contradiction since $x \notin \Lambda_{j+1}(f)$.

Thus there is $0 < l_1 \le L_1$ such that $x_{i_1+l_1} \in \operatorname{int} M_j$, and so, by the choice of $\beta_0 > 0$, we have $x_{i_1+l_1+n} \in \operatorname{int} M_j$ for all $n \ge 0$.

LEMMA 2.3. *There are $0 < \gamma_2 \le \gamma_1$, $0 < \beta_2 \le \beta_1$ and $L_2 > 0$ such that for $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_2)$, $1 \le j \le l-1$ and a β-pseudo-orbit $\{x_i\}$ of g $(0 < \beta \le \beta_2)$,*

(i) *if there exists $k \in \mathbf{Z}$ such that $x_k \in \operatorname{int} M_j$ and $x_i \notin f^{-m_j}(M_{j-1})$ $(i \ge k)$, then $x_i \in B_j$ for $i \ge k$, and*

(ii) *if there exist $k \in \mathbf{Z}$ and $i_2 \ge k$ such that $x_k \in \operatorname{int} M_j$ and $x_{i_2} \in f^{-m_j}(M_{j-1})$, then $x_{i_2+L_2} \in \operatorname{int} M_{j-1}$.*

PROOF. Take and fix $1 \le j \le l$. For $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_1)$ and a β-pseudo-orbit $\{x_i\}$ of g $(0 < \beta \le \beta_1)$, if there is $k \in \mathbf{Z}$ such that $x_k \in \operatorname{int} M_j$ and $x_i \notin f^{-m_j}(M_{j-1})$ $(i \ge k)$, then we have $x_i \in B_j$ since

$$\Lambda_j(f) \subset \operatorname{int} M_j \setminus f^{-m_j}(M_{j-1}) \subset B_j .$$

This proves (i).

To prove (ii), we check that there are $0 < \gamma_2 \le \gamma_1$, $0 < \beta_2 \le \beta_1$ and $L_2 > 0$ such that for $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_2)$, $1 \le j \le l-1$ and a β-pseudo-orbit $\{x_i\}$ of g $(0 < \beta \le \beta_2)$, if there are $k \in \mathbf{Z}$ and $i_2 \ge k$ with $x_k \in \operatorname{int} M_j$ and $x_{i_2} \in f^{-m_j}(M_{j-1})$, then there exists $0 < l_2 \le L_2$ such that $x_{i_2+l_2} \in \operatorname{int} M_{j-1}$.

If this is false, for $n > 0$ there are $g_n \in \mathrm{Diff}^1(M)$ $(d(f, g_n) < 1/n)$, $1 \le j_n \le l-1$, a $(1/n)$-pseudo-orbit $\{x_i^n\}$ of g_n and integers $k_n \in \mathbf{Z}$ and $i_2^n \ge k_n$ such that $x_{k_n}^n \in \operatorname{int} M_{j_n}$, $x_{i_2^n}^n \in f^{-m_{j_n}}(M_{j_n-1})$ and

$$x_{i_2^n+i}^n \notin \operatorname{int} M_{j_n-1} \qquad \text{for} \quad 0 < i \le n . \tag{2.4}$$

We may assume that $j = j_n$ for $n > 0$. The choice of β_0 implies $x_{i_2^n+i}^n \in f^{-m_j}(M_{j-1})$ for $i \ge 0$, and so $x = \lim_{n \to \infty} x_{i_2^n}^n \in f^{-m_j}(M_{j-1})$. Thus

$$f^{m_j+1}(x) \in f(M_{j-1}) \subset \operatorname{int} M_{j-1} . \tag{2.5}$$

Since $d(f, g_n) < 1/n$ and $d(g_n(x_{i_2^n+i}^n), x_{i_2^n+i+1}) < 1/n$ for $i \ge 0$, we have $f^i(x) \notin \operatorname{int} M_{j-1}$ for $i > 0$ by (2.4). This contradicts (2.5).

If there exists $0 < l_2 \le L_2$ with $x_{i_2+l_2} \in \operatorname{int} M_j$, then $x_{i_2+l_2+i} \in \operatorname{int} M_j$ for $i \ge 0$, and so $x_{i_2+L_2} \in \operatorname{int} M_j$.

LEMMA 2.4. *Let $2 \le j \le l-1$ and L_1 be as in lemma 2.2. Suppose that $\operatorname{Ind} \Lambda_j(f) < \operatorname{Ind} \Lambda_{j+1}(f)$. Then there are $0 < \gamma_j \le \gamma_2$, $0 < \beta_j \le \beta_2$ and $L_j > 0$ such that for $g \in \mathrm{Diff}^1(M)$ $(d(f, g) < \gamma_j)$ and a β-pseudo-orbit $\{x_i\}$ of g $(0 < \beta \le \beta_j)$, if there exists $i_3 \in \mathbf{Z}$*

satisfying $x_i \in B_{j+1}(i \leq i_3)$ *and* $x_{i_3+1} \notin B_{j+1}$, *then* $x_{i_3+L_1+L_j} \in f^{-m_j}(M_{j-1})$.

PROOF. Under the assumption of this lemma, we see that there are $0<\gamma_j\leq\gamma_2$, $0<\beta_j\leq\beta_2$ and $L_j>0$ such that for $g\in \mathrm{Diff}^1(M)$ $(d(f,g)<\gamma_j)$ and a β-pseudo-orbit $\{x_i\}$ of g $(0<\beta\leq\beta_j)$, if there exists $i_3\in \boldsymbol{Z}$ satisfying $x_i\in B_{j+1}$ $(i\leq i_3)$ and $x_{i_3+1}\notin B_{j+1}$, then there exists $0<l_j\leq L_j$ such that $x_{i_3+L_1+l_j}\in f^{-m_j}(M_{j-1})$.

If this is false, for $n>0$ there are $g_n\in \mathrm{Diff}^1(M)$ $(d(f,g_n)<1/n)$, a $(1/n)$-pseudo-orbit $\{x_i^n\}$ of g_n and $i_3^n\in \boldsymbol{Z}$ such that $x_i^n\in B_{j+1}$ $(i\leq i_3)$, $x_{i_3^n+1}^n\notin B_{j+1}$ and $x_{i_3^n+L_1+i}^n\notin f^{-m_j}(M_{j-1})$ for $0<i\leq n$.

By lemma 2.2, $x_{i_3^n+L_1}^n\in \mathrm{int}\, M_j$ and so $x_{i_3^n+L_1+i}^n\in \mathrm{int} M_j$ for $i\geq 0$. Thus for every $0<i\leq n$,

$$x_{i_3^n+L_1+i}^n\in \mathrm{int}\, M_j\setminus f^{-m_j}(M_{j-1})\subset B_j. \tag{2.6}$$

Put $x=\lim_{n\to\infty} x_{i_3^n}^n$. Then $f^{-i}(x)\in B_{j+1}$ for every $i\geq 0$. On the other hand, by (2.6) we have $f^{L_1+i}(x)\in B_j$ for $i\geq 0$ and so $x\in W^s(\Lambda_j(f))$. Thus $x\in W^s(\Lambda_j(f))\cap W^u(\Lambda_{j+1}(f))\neq\emptyset$ and $\mathrm{Ind}\,\Lambda_j(f)\geq \mathrm{Ind}\,\Lambda_{j+1}(f)$. This is a contradiction.

By the choice of β_0 it is clear that $x_{i_3+L_1+l_j}\in f^{-m_j}(M_{j-1})$ implies $x_{i_3+L_1+L_j}\in f^{-m_j}(M_{j-1})$.

By the same manner stated in the proof of lemma 2.4 we can prove that for $2\leq i\neq j\leq l-1$ $(\mathrm{Ind}\,\Lambda_i(f)<\mathrm{Ind}\,\Lambda_j(f))$, there are $0<\gamma_{ij}\leq\gamma_2$, $0<\beta_{ij}\leq\beta_2$ and $L_{ij}>0$ such that for $g\in\mathrm{Diff}^1(M)$ $(d(f,g)<\gamma_{ij})$ and a β-pseudo-orbit $\{x_i\}$ of g $(0<\beta\leq\beta_{ij})$, if there are $i_3, i_4\in\boldsymbol{Z}$ $(i_3<i_4)$ such that $x_i\in B_{j+1}$ $(i\leq i_3)$ and $x_{i_4}\in \mathrm{int}\, M_i$, then $x_{i_4+L_{ij}+L_1}\in f^{-m_i}(M_{i-1})$. Now put

$$\gamma_3=\min\{\gamma_{ij}: 2\leq i\neq j\leq l-1 \text{ with } \mathrm{Ind}\,\Lambda_i(f)<\mathrm{Ind}\,\Lambda_j(f)\},$$

$$\beta_3=\min\{\beta_{ij}: 2\leq i\neq j\leq l-1 \text{ with } \mathrm{Ind}\,\Lambda_i(f)<\mathrm{Ind}\,\Lambda_j(f)\},$$

$$L_3=\max\{L_{ij}: 2\leq i\neq j\leq l-1 \text{ with } \mathrm{Ind}\,\Lambda_i(f)<\mathrm{Ind}\,\Lambda_j(f)\} \quad \text{and}$$

$$L=\max\{L_1, L_2, L_3\}.$$

Then Proposition 2.1 is concluded.

3. Proof of the theorem.

Let γ_3, L and β_3 be as in proposition 2.1 and put $L'=3lL$. Take and fix $\mathfrak{U}(f)\subset\mathfrak{U}_4(f)\cap\{g\in\mathrm{Diff}^1(M): d(f,g)<\gamma_3\}$. Then there are $0<\beta_4\leq\beta_3$ and $K>0$ such that $g\in\mathfrak{U}(f)$, $d(x,y)<\beta_4$ $(x,y\in M)$ implies $d(g(x),g(y))\leq Kd(x,y)$. Fix $0<\varepsilon\leq\beta_4$ and take $0<\varepsilon_1\leq\varepsilon$ satisfying

$$(1+K+K^2+\cdots+K^{L'})\cdot 2\varepsilon_1<\varepsilon.$$

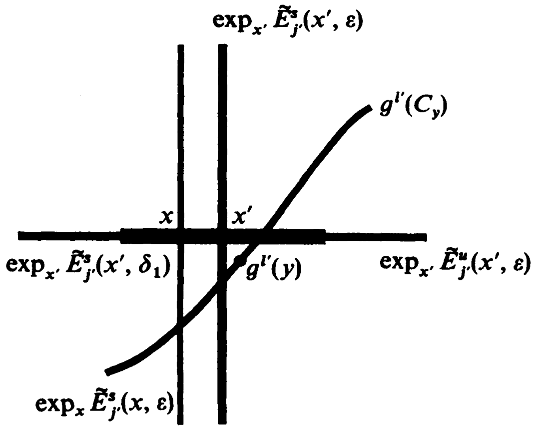
For every $g\in\mathfrak{U}(f)$, $x\in\Lambda_j(g)$ $(1<j\leq l)$ and $y\in W^u(x,g)\cap B_j$, we denote by C_y the connected component of y in $B_{\varepsilon_1}(y)\cap W^u(x,g)$. Let d_u be a metric on $W^u(x,g)$ induced

from$\|\cdot\|$ and put $\delta'=\min_{1<j\leq l}\delta'_j$, where

$$\delta'_j=\inf_{\substack{g\in\mathfrak{U}(f)\\ x\in\Lambda_j(g)\\ y\in W^u(x,g)\cap B_j}}\ \min_{0\leq i\leq L'} d_u(g^i(\partial C_y), g^i(y))>0\,.$$

Let $\theta>0$ and $0<\delta(\varepsilon)\leq\varepsilon$ be numbers given in lemmas 1.3 and 1.4 respectively. It is easy to see that for $\delta_1=\sin(\theta/2)\cdot\min\{\delta', \delta(\varepsilon)\}$, if there are $x'\in B_{j'}$ $(1\leq j'<j\leq l)$ and $0<l'\leq L'$ such that $d(g^{l'}(y), x')<\delta_1$, then for every $x\in\exp_{x'}\tilde{E}^s_{j'}(x', \delta_1)$,

$$\exp_x\tilde{E}^s_{j'}(x, \varepsilon)\cap g^{l'}(C_y)\neq\varnothing\,.$$



Pick $0<\delta_2<\delta_1$ such that

$$(1+K+K^2+\cdots+K^{L'})\delta_2<\delta_1$$

and let $0<\delta_3=\delta_3(\delta_2/2)<\delta_2/2$ be a number as in proposition 1.1.

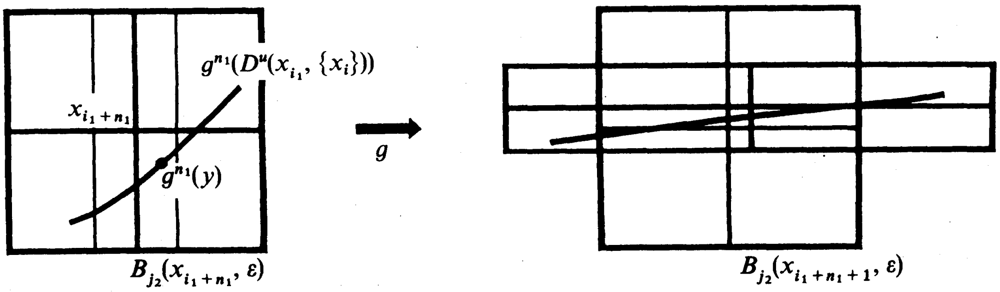
For every $g\in\mathfrak{U}(f)$ and a δ-pseudo-orbit $\{x_i\}$ of g $(0<\delta\leq\delta_3)$, we may assume that there are $1\leq j_1\leq l$ and $i_1\in\boldsymbol{Z}$ such that $x_i\in B_{j_1}(i\leq i_1)$ and $x_{i_1+1}\notin B_{j_1}$. Thus, by proposition 2.1 there are $1\leq j_1\neq j_2\leq l$ (Ind $\Lambda_{j_2}(f)\geq$ Ind $\Lambda_{j_1}(f)$) and $0\leq n_1\leq L'$ such that $x_{i_1+n_1-1}\notin B_{j_2}$ and $x_{i_1+n_1}\in B_{j_2}$. Moreover, there exists $0\leq\bar{n}_1=\bar{n}_1(\{x_i\})\leq\infty$ such that $x_{i_1+n_1+i}\in B_{j_2}$ for $0\leq i\leq\bar{n}_1$. By proposition 1.1

$$D^u(x_{i_1}, \{x_i\})=\{x\in M: d(g^{-i}(x), x_{i_1-i})\leq\varepsilon \text{ for all } i\geq 0\}$$

contains a $(\dim M-\operatorname{Ind}\Lambda_{j_1}(g))$-dimensional disk. Put

$$y=D^u(x_{i_1}, \{x_i\})\cap\exp_{x_{i_1}}\tilde{E}^s_{j_1}(x_{i_1}, \varepsilon_1)\,.$$

Then $d(y, x_{i_1})<\delta_2$ and $d(g^{n_1}(y), x_{i_1+n_1})<\delta_1$.



For every

$$z \in g(g^{n_1}(D^u(x_{i_1}, \{x_i\}))) \cap B_{j_2}(x_{i_1+n_1}, \varepsilon),$$

we have $z \in B_{j_2}(x_{i_1+n_1+1}, \varepsilon)$, $g^{-1}(z) \in B_{j_2}(x_{i_1+n_1}, \varepsilon)$ and $d(g^{-i-1}(z), x_{i_1+n_1-i}) < \varepsilon$ for all $i \geq 0$. By the same reason for every

$$z \in g(g(g^{n_1}(D^u(x_{i_1}, \{x_i\})) \cap B_{j_2}(x_{i_1+n_1+1}, \varepsilon)) \cap B_{j_2}(x_{i_1+n_1+2}, \varepsilon),$$

we have $z \in B_{j_2}(x_{i_1+n_1+2}, \varepsilon)$, $g^{-1}(z) \in B_{j_2}(x_{i_1+n_1+1}, \varepsilon)$, $g^{-2}(z) \in B_{j_2}(x_{i_1+n_1}, \varepsilon)$ and $d(g^{-2-i}(z), x_{i_1+n_1-i}) < \varepsilon$ for all $i \geq 0$. Repeating this way $\{x_i\}$ is g-ε-traced for $i \leq i_1 + n_1 + \bar{n}_1$. For the case $0 \leq \bar{n}_1 < \infty$, there are $0 < n_2 \leq L'$ and $1 \leq j_3 \neq j_2 \leq l$ (Ind $\Lambda_{j_3}(f) \geq$ Ind $\Lambda_{j_1}(f)$) such that $x_{i_1+n_1+\bar{n}_1+n_2-1} \notin B_{j_3}$ and $x_{i_1+n_1+\bar{n}_1+n_2} \in B_{j_3}$. Furthermore, by proposition 2.1 (iii) there is $0 < \bar{n}_2 = \bar{n}_2(\{x_i\}) \leq \infty$ such that $x_{i_1+n_1+\bar{n}_1+n_2+i} \in B_{j_3}$ $(0 \leq i \leq \bar{n}_2)$. Since δ is small enough, we can repeat the above arguments to get a g-ε-tracing point of $\{x_i\}$ for all $i \leq i_1 + n_1 + \bar{n}_1 + n_2 + \bar{n}_2$. Thus, by induction $\{x_i\}$ is g-ε-traced.

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