TOKYO J. MATH.
VOL. 20, NO. 1, 1997

Extrinsic Hyperspheres of Naturally Reductive Homogeneous Spaces

Koji TOJO

Chiba University

(Communicated by T. Nagano)

0. Introduction.

In [2], Chen investigated extrinsic spheres of (locally) symmetric spaces and obtained the following.

THEOREM. *Let N be an n-dimensional submanifold in a locally symmetric space $\tilde{M}$. Then N is an extrinsic sphere in $\tilde{M}$ if and only if N is an extrinsic hypersphere in a $(n+1)$-dimensional totally geodesic submanifold of constant sectional curvature.*

On the other hand the author has proved in [7] the following. Let G be a compact simple Lie group and K a closed subgroup of G. If the normal homogeneous space G/K contains a totally geodesic hypersurface N, then G/K is a space with constant sectional curvature. Then, in this paper, we treat a similar problem in case that G/K is a naturally reductive homogeneous space and N is an extrinsic hypersphere.

The paper is organized as follows. In Section 2 we write the Levi-Civita connections of homogeneous spaces in terms of the Lie algebra. In Section 3, using a result of Section 2, we shall describe circles of homogeneous spaces in terms of Lie algebras. Section 4 is devoted to prove the following theorem.

MAIN THEOREM. *If a naturally reductive homogeneous space G/K admits an extrinsic hypersphere, then G/K is a space with constant sectional curvature.*

1. Preliminaries.

In this section we recall some basic facts with respect to the Levi-Civita connection on Riemannian manifolds.

Let (M, g) be an n-dimensional Riemannian manifold and ∇ the Levi-Civita connection of (M, g). Let $\{e_1, \cdots, e_n\}$ be a local orthonormal frame field and $\{\omega^1, \cdots, \omega^n\}$

Received May 15, 1995
Revised September 19, 1995

their dual 1-forms. Associated with $\{e_1, \cdots, e_n\}$, there uniquely exist local 1-forms $\{\omega_i^j\}$ $(i, j=1, \cdots, n)$, which are called the connection forms, such that

$$\omega_i^j + \omega_j^i = 0 \tag{1.1}$$

$$d\omega^i + \sum_{j=1}^{n} \omega_j^i \wedge \omega^j = 0\,. \tag{1.2}$$

Then the following holds (see [5]).

$$\nabla_{e_i} e_j = \sum_{k=1}^{n} \omega_j^k(e_i) e_k\,. \tag{1.3}$$

Next, let G be a Lie group and K a closed subgroup of G such that $\mathrm{Ad}(K)$ is compact. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then there exist an $\mathrm{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ and an $\mathrm{Ad}(K)$-invariant scalar product $\langle\ ,\ \rangle$ on $\mathfrak{p}$. Then

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \tag{1.4}$$

$$\langle [u, x], y\rangle + \langle [u, y], x\rangle = 0 \qquad (u \in \mathfrak{k},\ x, y \in \mathfrak{p})\,. \tag{1.5}$$

Moreover, under the canonical identification of $\mathfrak{p}$ with the tangent space $T_o G/K$ $(o=\{K\})$ of homogeneous space G/K, the scalar product $\langle\ ,\ \rangle$ can be extended to a G-invariant metric on G/K. Let Λ be the connection function of $(G/K, \langle\ ,\ \rangle)$ (cf. [6]). Then for $x, y \in \mathfrak{p}$,

$$\Lambda(x)(y) = \tfrac{1}{2}[x, y]_{\mathfrak{p}} + U(x, y) \tag{1.6}$$

where

$$\langle U(x, y), z\rangle = \tfrac{1}{2}\{\langle [z, x]_{\mathfrak{p}}, y\rangle + \langle [z, y]_{\mathfrak{p}}, x\rangle\} \qquad (z \in \mathfrak{p})\,. \tag{1.7}$$

Furthermore, the curvature tensor R is given by

$$R(x, y)z = [[x, y]_{\mathfrak{k}}, z] + \Lambda([x, y]_{\mathfrak{p}})(z) - [\Lambda(x), \Lambda(y)](z)\,. \tag{1.8}$$

In particular, let $(G/K, \langle\ ,\ \rangle)$ be a naturally reductive homogeneous space. Then it satisfies an additional condition:

$$U(x, y) = 0 \qquad (x, y \in \mathfrak{p})\,. \tag{1.9}$$

Then $c(t) = \tau(\exp tx)(o)$ $(x \in \mathfrak{p})$ is a geodesic of $(G/K, \langle\ ,\ \rangle)$ where $\tau(g)$ $(g \in G)$ denotes the left transformation of G/K. Moreover the curvature tensor R is given by

$$R(x, y)z = [[x, y]_{\mathfrak{k}}, z] + \tfrac{1}{2}[[x, y]_{\mathfrak{p}}, z]_{\mathfrak{p}} - \tfrac{1}{4}[x, [y, z]_{\mathfrak{p}}]_{\mathfrak{p}} + \tfrac{1}{4}[y, [x, z]_{\mathfrak{p}}]_{\mathfrak{p}}\,. \tag{1.10}$$

2. The Levi-Civita connections on homogeneous spaces.

In this section we shall write the Levi-Civita connections of Riemannian homogeneous spaces in terms of the Lie algebras.

Let $(G/K, \langle\ ,\ \rangle)$ be a homogeneous space with a G-invariant metric $\langle\ ,\ \rangle$ as stated in Section 1. Let $\pi: G \to G/K$ be the canonical projection and W an open subset in $\mathfrak{p}$ such that $0 \in W$ and the mapping

$$\pi \circ \exp: W \to \pi(\exp W)$$

is diffeomorphic. Let $\{e_\alpha\}_{\alpha \in A}$ be a basis of $\mathfrak{k}$ and $\{e_i\}_{i \in I}$ an orthonormal basis of $(\mathfrak{p}, \langle\ ,\ \rangle)$. In this section we use the following convention on the range of indices, unless otherwise stated:

$$i, j, k, \cdots \in I\,, \quad \alpha, \beta, \gamma, \cdots \in A\,, \quad p, q, r, \cdots \in I \cup A\,.$$

Let $\{X_\alpha\}$ and $\{X_i\}$ be the left invariant vector fields on G such that $(X_\alpha)_e = e_\alpha$ and $(X_i)_e = e_i$ (e is the identity of G). Furthermore we define an orthonormal frame field $\{E_i\}$ on $\pi(\exp W)$ and the mapping $\mu: \pi(\exp W) \to \exp W$ as follows:

$$(E_i)_{\pi(\exp x)} = \tau(\exp x)_*(e_i)$$

$$\mu(\pi(\exp x)) = \exp x \qquad (x \in W)\,.$$

Then since $\pi_*(X_i) = E_i$, $\pi_*(X_\alpha) = 0$ and $\pi_* \mu_* = id$, we can put

$$\mu_*(E_i) = X_i + \sum_\alpha \eta_{\alpha i} X_\alpha\,. \tag{2.1}$$

Let $\{\omega^\alpha\}$, $\{\omega^i\}$ and $\{\theta^i\}$ be the dual 1-forms of $\{X_\alpha\}$, $\{X_i\}$ and $\{E_i\}$, respectively. Then it is easy to see

$$\mu^*(\omega^i) = \theta^i\,. \tag{2.2}$$

Set $[X_p, X_q] = \sum_r c^r_{pq} X_r$. Then the following is known as the equation of Maurer-Cartan (cf. [5]):

$$d\omega^p = -\frac{1}{2} \sum_{q,r} c^p_{qr} \omega^q \wedge \omega^r\,. \tag{2.3}$$

Now for the sake of completeness we shall show the following known fact.

Lemma 2.1. *Let $\{\theta^i_j\}$ be the connection forms of $(G/K, \langle\ ,\ \rangle)$ associated with $\{E_i\}$. Then*

$$\theta^i_j = -\mu^* \left\{ \sum_\alpha c^i_{j\alpha} \omega^\alpha + \frac{1}{2} \sum_k (c^i_{jk} - c^j_{ik} - c^k_{ij}) \omega^k \right\}.$$

Proof. It follows from (1.4) and (1.5) that

$$c^\beta_{j\alpha} = 0\,, \qquad c^i_{j\alpha} + c^j_{i\alpha} = 0\,. \tag{2.4}$$

Moreover since $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, we get

$$c^i_{\alpha\beta} = 0\,. \tag{2.5}$$

From equations (2.2), (2.3), (2.4) and (2.5) it follows that

$$d\theta^i=\mu^* d\omega^i=-\sum_j \mu^*\left\{\sum_\alpha c^i_{j\alpha}\omega^j\wedge\omega^\alpha+\frac{1}{2}\sum_k (c^i_{jk}-c^j_{ik}-c^k_{ij})\omega^j\wedge\omega^k\right\}$$

$$=\sum_j \mu^*\left\{\sum_\alpha c^i_{j\alpha}\omega^\alpha+\frac{1}{2}\sum_k (c^i_{jk}-c^j_{ik}-c^k_{ij})\omega^k\right\}\wedge\theta^j$$

(note that $\sum_{j,k}(c^k_{ij}+c^j_{ik})\omega^j\wedge\omega^k=0$). Put $\theta^i_j=-\mu^*(\sum_\alpha c^i_{j\alpha}\omega^\alpha+\frac{1}{2}\sum_k (c^i_{jk}-c^j_{ik}-c^k_{ij})\omega^k\}$. Then it is easy to check $\theta^i_j+\theta^j_i=0$. Consequently, by (1.1) and (1.2), the connection forms coincide with $\{\theta^i_j\}$. $\square$

By (1.3), (2.1) and the above lemma, we have the following.

PROPOSITION 2.2.

$$\nabla_{E_i}E_j=\sum_k\left\{\sum_\alpha c^k_{\alpha j}\eta_{\alpha i}+\frac{1}{2}(c^k_{ij}-c^j_{ik}-c^i_{jk})\right\}E_k .$$

Next we shall rewrite Proposition 2.2 in terms of the bracket operation [,] of $\mathfrak{g}$.

For $x\in W$, we define $z^i_x(t)\in W$ and $h^i_x(t)\in K$ ($t\in\mathbf{R}$ with $|t|$ small enough) to satisfy the following:

$$\exp x\cdot\exp te_i=\exp z^i_x(t)\cdot h^i_x(t) \tag{2.6}$$

with $z^i_x(0)=x$ and $h^i_x(0)=e$. Then

$$\mu_*(E_i)_{\pi(\exp x)}=\frac{d}{dt}\bigg|_0 \mu(\pi(\exp x\cdot\exp te_i))$$

$$=\frac{d}{dt}\bigg|_0 \mu(\pi(\exp z^i_x(t)))=(\exp_*)_x\left(\frac{d}{dt}\bigg|_0 z^i_x(t)\right).$$

Here, the differential map $\exp_*$ of exp has the following form (see [4]):

LEMMA 2.3. *Let $x, y\in\mathfrak{g}$. Then*

$$(\exp_*)_x(y)=(L_{\exp x})_*\circ\Phi_x(y) ,$$

where $\Phi_x(y)=\sum_{n=0}^\infty((-1)^n/(n+1)!)(\mathrm{ad}\,x)^n(y)$.

Thus we have

$$\mu_*(E_i)_{\pi(\exp x)}=(L_{\exp x})_*\circ\Phi_x\left(\frac{d}{dt}\bigg|_0 z^i_x(t)\right). \tag{2.7}$$

On the other hand, (2.6) and Lemma 2.3 give

$$(L_{\exp x})_*\circ\Phi_x\left(\frac{d}{dt}\bigg|_0 z^i_x(t)\right)=(L_{\exp x})_*(e_i)-(L_{\exp x})_*\left(\frac{d}{dt}\bigg|_0 h^i_x(t)\right). \tag{2.8}$$

Considering (2.1), (2.7) and (2.8), we obtain

$$\frac{d}{dt}\Big|_0 h_x^i(t) = -\sum_\alpha \eta_{\alpha i}(\exp x) e_\alpha . \tag{2.9}$$

Therefore, by (2.9) and Proposition 2.2, we have

$$(\nabla_{E_i} E_j)_{\pi(\exp x)} = \tau(\exp x)_* \left\{ \Lambda(e_i)(e_j) - \left[\frac{d}{dt}\Big|_0 h_x^i(t),\ e_j \right] \right\} . \tag{2.10}$$

REMARK. For $x \in \mathfrak{p}$ ($|x|$: small), the mapping $p_\mathfrak{p} \circ \Phi_x \colon \mathfrak{p} \to \mathfrak{p}$ is an isomorphism ($p_\mathfrak{p} \colon \mathfrak{g} \to \mathfrak{p}$ denotes the canonical projection.). So we can assume that for each $x \in W$ the mapping $p_\mathfrak{p} \circ \Phi_x$ is an isomorphism. Therefore we can regard the equation (2.8) as a characterization of $(d/dt)|_0 z_x^i(t)$ ($\in \mathfrak{p}$) and $(d/dt)|_0 h_x^i(t)$ ($\in \mathfrak{k}$).

For $X \in \mathfrak{p}$, we denote by X_* the vector field on $\pi(\exp W)$ defined by

$$(X_*)_{\pi(\exp x)} = \tau(\exp x)_*(X) .$$

Then the following theorem is easily derived from the above arguments.

THEOREM 2.4. *Let $x \in W$ and $X, Y \in \mathfrak{p}$. Then*

$$(\nabla_{X_*} Y_*)_{\pi(\exp x)} = \tau(\exp x)_* \{\Lambda(X)(Y) - [h_x(X), Y]\} .$$

Here $h_x(X) = -p_\mathfrak{k} \circ \Phi_x \circ (p_\mathfrak{p} \circ \Phi_x)^{-1}(X)$ ($p_\mathfrak{k} \colon \mathfrak{g} \to \mathfrak{k}$ denotes the canonical projection).

COROLLARY 2.5. *Let $x \in W$ and $X, Y \in \mathfrak{p}$. Then*

$$(\nabla_{\tau(\exp x)_* \circ p_\mathfrak{p} \circ \Phi_x(X)} Y_*) = \tau(\exp x)_* \{\Lambda(p_\mathfrak{p} \circ \Phi_x(X))(Y) + [p_\mathfrak{k} \circ \Phi_x(X), Y]\} .$$

PROOF.

$$\begin{aligned}
(\nabla_{\tau(\exp x)_* \circ p_\mathfrak{p} \circ \Phi_x(X)} Y_*) &= \sum_i \langle p_\mathfrak{p} \circ \Phi_x(X), e_i \rangle (\nabla_{E_i} Y_*)_{\pi(\exp x)} \\
&= \sum_i \langle p_\mathfrak{p} \circ \Phi_x(X), e_i \rangle \tau(\exp x)_* \{\Lambda(e_i)(Y) - [h_x(e_i), Y]\} \\
&= \tau(\exp x)_* \left\{ \Lambda(p_\mathfrak{p} \circ \Phi_x(X))(Y) - \left[\sum_i \langle p_\mathfrak{p} \circ \Phi_x(X), e_i \rangle h_x(e_i),\ Y \right] \right\} .
\end{aligned}$$

From Theorem 2.4 we get

$$\sum_i \langle p_\mathfrak{p} \circ \Phi_x(X), e_i \rangle h_x(e_i) = h_x(p_\mathfrak{p} \circ \Phi_x(X)) = -p_\mathfrak{k} \circ \Phi_x(X) .$$

This completes the proof of the corollary. □

3. Circles of homogeneous spaces.

Let (M, g) be a Riemannian manifold and ∇ the Levi-Civita connection of (M, g). A curve $c(t)$ of (M, g) parameterized by arc length is called a *circle* if there exist a unit vector field $\tilde{\xi}(t)$ along $c(t)$ and nonzero constant λ such that

$$\nabla_{c'(t)}c'(t) = \lambda\tilde{\xi}(t), \qquad \nabla_{c'(t)}\tilde{\xi}(t) = -\lambda c'(t). \tag{3.1}$$

In this section we shall give an asymptotic expansion of a circle of a Riemannian homogeneous space.

Let $(G/K, \langle\ ,\ \rangle)$ be a homogeneous space with a G-invariant Riemannian metric $\langle\ ,\ \rangle$ as stated in Section 1. Let $c(t)$ be a circle of $(G/K, \langle\ ,\ \rangle)$ such that $c(0)=o$ and $c'(0)=x$ $(x \in \mathfrak{p})$. Using the same notation as in Section 2 we can put

$$c(t) = \pi(\exp X(t)), \quad (X(t) \in W) \qquad \text{for} \quad t \text{ with } |t| \text{ small enough}.$$

Then it follows from Lemma 2.3 that

$$c'(t) = \tau(\exp X(t))_* \circ p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t)). \tag{3.2}$$

Thus we get

$$X(0) = 0, \qquad X'(0) = x. \tag{3.3}$$

Moreover, by (3.1) and Corollary 2.5 we have

$$\begin{aligned}\lambda\tilde{\xi}(t) = \tau(\exp X(t))_*\Big\{&\Lambda(p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t)))(p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t))) \\ &+ \frac{d}{dt}p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t)) + [p_{\mathfrak{k}} \circ \Phi_{X(t)}(X'(t)), p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t))]\Big\}.\end{aligned} \tag{3.4}$$

Set

$$\begin{aligned}F(t) = {}&\frac{d}{dt}(p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t))) + [p_{\mathfrak{k}} \circ \Phi_{X(t)}(X'(t)), p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t))] \\ &+ \Lambda(p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t)))(p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t))).\end{aligned} \tag{3.5}$$

Then (3.4) gives

$$F(0) = \lambda\xi \qquad (\xi = \tilde{\xi}_o). \tag{3.6}$$

Furthermore (3.1) is rewritten as follows.

LEMMA 3.1. *Let $c(t) = \pi(\exp X(t))$ $(X(t) \in W$, $|t|$: small enough) be a circle of $(G/K, \langle\ ,\ \rangle)$. Then $X(t)$ satisfies the following.*

$$-\lambda^2 p_{\mathfrak{p}} \circ \Phi_{X(t)}(X'(t)) = [p_{\mathfrak{k}} \circ \Phi_{X(t)}(X'(t)), F(t)]$$

$$+\frac{d}{dt}F(t)+\Lambda(p_{\mathfrak{p}}\circ\Phi_{X(t)}(X'(t)))(F(t)).$$

From (3.4), (3.5) and Lemma 3.1 it is easy to see

$$\tilde{\xi}(t)=\tau(\exp X(t))_*\{\xi-t(\lambda x+\Lambda(x)(\xi))+o(t)\}. \tag{3.7}$$

Here $o(t^n)$ denotes an infinitesimal of order higher than t^n. Also it follows from (3.2), (3.3), (3.5) and (3.6) that

$$X(t)=tx+\frac{t^2}{2!}(\lambda\xi-\Lambda(x)(x))+o(t^2)$$

$$\text{and}\quad c'(t)=\tau(\exp X(t))_*\{x+t(\lambda\xi-\Lambda(x)(x))+o(t)\}.$$

4. Proof of Main Theorem.

Let N be an n-dimensional submanifold of an m-dimensional Riemannian manifold (M, g). Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections on N and M, respectively. Then the second fundamental form σ is given by

$$\sigma(X, Y)=\tilde{\nabla}_X Y-\nabla_X Y, \tag{4.1}$$

where X and Y are vector fields tangent to N.

N is said to be *umbilical* if $\sigma(X, Y)=g(X, Y)H$, where $H=(1/n)\mathrm{trace}(\sigma)$ is the mean curvature vector of N. Moreover N is said to be an *extrinsic sphere* if N is umbilical and its mean curvature vector is nonzero and parallel with respect to the normal connection $\nabla^\perp$. In particular, if N is an extrinsic hypersphere, then

$$\sigma(X, Y)=\lambda g(X, Y)\tilde{\xi}, \tag{4.2}$$

for some nonzero constant λ, where $\tilde{\xi}$ is a unit vector field normal to N. Then for each $p\in N$ the equation of Codazzi gives

$$R(T_pN, T_pN)T_pN\subset T_pN \tag{4.3}$$

where R denotes the curvature tensor of (M, g).

Suppose that N is an extrinsic hypersphere of M. Let $c(t)$ ($|t|$: small enough) be a geodesic of N parameterized by arc length. Then (4.1) and (4.2) give

$$\tilde{\nabla}_{c'(t)}c'(t)=\lambda\tilde{\xi}, \qquad \tilde{\nabla}_{c'(t)}\tilde{\xi}=-\lambda c'(t).$$

Thus $c(t)$ is a circle of (M, g).

Now, let $(G/K, \langle\ ,\ \rangle)$ be a naturally reductive homogeneous space as in Section 1 and suppose that $(G/K, \langle\ ,\ \rangle)$ admits an extrinsic hypersphere N through $o=\{K\}$. Put $V=T_oN$ (then V is a hyperplane of $\mathfrak{p}$). Let $c(t)=\pi(\exp X(t))$ ($X(t)\in W$, $|t|$: small enough) be a geodesic of N with $c(0)=o$ and $c'(0)=x$ ($x\in V$, $|x|=1$). Then by (1.6), (1.9) and

(3.7) we can write

$$\tilde{\xi}(c(t))=\tau(\exp X(t))_*\{\xi-t(\lambda x+A(x)(\xi))+o(t)\}\,. \tag{4.4}$$

REMARK 4.1. If for $\xi\in\mathfrak{p}$ there exists an extrinsic hypersphere N such that it is tangent to $\xi^\perp$ at o, then for any $x(\perp\xi)$ there exists an extrinsic hypersphere such that its normal vector $\hat{\xi}$ at o is not normal to x and ξ. In fact, by (4.4), there is $t\in\mathbf{R}$ such that the normal vector

$$\tau(\exp X(t))_*^{-1}\{\tilde{\xi}_{\pi(\exp X(t))}\}$$

at o of the extrinsic hypersphere $\tau(\exp X(t))_*^{-1}(N)$ is not normal to x and ξ.

Let S be the unit sphere of $\mathfrak{p}$ and λ a nonzero constant. Let E_λ be the set of all $\xi\in S$ such that there exists an extrinsic hypersphere tangent to $\xi^\perp$ whose principal curvature equals λ.

LEMMA 4.2. *E_λ contains an open subset of S.*

PROOF. Let ξ be an element of E_λ and N an extrinsic hypersphere associated to ξ. As before $\tilde{\xi}$ denotes the unit normal vector field of N such that $\tilde{\xi}(o)=\xi$. Let $c_x(t)=\pi(\exp X(t,x))$ $(X(t,x)\in W)$ be a geodesic of N such that $c_x(0)=o$ and $c'_x(t)=x$ $(x\in T_oN,\ |x|=1)$. Let ε be a positive number such that $\pi(\exp X(t,x))$ is contained in N for any t $(|t|<\varepsilon)$ and for any $x\in(S\cap T_oN)$. Then we define a smooth mapping $f\colon U_\varepsilon\to S$ $(U_\varepsilon=\{x\in T_oN : |x|<\varepsilon\})$ as follows:

$$f(tx)=\tau(\exp X(t,x))_*^{-1}\{\tilde{\xi}(\pi(\exp X(t,x)))\}\,.$$

Then as in Remark 4.1 we can see that $f(tx)\in\mathfrak{p}$ is a unit vector which is normal to the extrinsic hypersphere $\tau(\exp X(t,x))^{-1}(N)$ at o. Thus we get $f(U_\varepsilon)\subset E_\lambda$. It follows from (4.4) that the differential mapping of f at $0\in U_\varepsilon$ is given by (under the identification of T_0U_ε with $T_\xi S$)

$$(f_*)_0(x)=-\lambda x-A(x)(\xi)\,.$$

By (1.9) the mapping $(f_*)_0$ is an isomorphism. Thus the inverse function theorem implies that there is an open subset of S contained in E_λ. $\square$

For $\xi\in S$, set

$$h(\xi)=\sum_{i,j,k=1}^{n}(\langle R(e_i,e_j)e_k,\xi\rangle)^2\,,$$

where $\{e_1,\cdots,e_n\}$ denotes an orthonormal basis of $\xi^\perp$. Then h is a well-defined real analytic function on S. From (4.3) we can easily check that $h(E_\lambda)=\{0\}$. Therefore, by Lemma 4.2, h is identically zero on S. Thus any hyperplane of $\mathfrak{p}$ is curvature invariant and hence $(G/K,\langle\ ,\ \rangle)$ has constant sectional curvature.

ACKNOWLEDGEMENTS. The author would like to express special thanks to the referee for his useful advice.

References

[1] M. BERGER, Les variétés Riemanniennes homogènes normales simplement connexes à courbure strictement positive, Ann. Scuola Norm. Sup. Pisa **15** (1961), 179–246.
[2] B. Y. CHEN, Extrinsic spheres in Riemannian manifolds, Houston J. Math. **5** (1979), 319–324.
[3] B. Y. CHEN and L. VERSTRAELEN, Hypersurfaces of symmetric spaces, Bull. Inst. Math. Acad. Sinica **8** (1980), 201–236.
[4] S. HELGASON, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press (1978).
[5] S. KOBAYASHI and K. NOMIZU, *Foundations of Differential Geometry II*, Interscience Publishers (1969).
[6] K. NOMIZU, Invariant affine connections on homogeneous spaces, Amer. J. Math. **76** (1954), 33–65.
[7] K. TOJO, Normal homogeneous spaces admitting totally geodesic hypersurfaces, to appear in J. Math. Soc. Japan.

Present Address:
GENERAL EDUCATION (NATURAL SCIENCE), CHIBA INSTITUTE OF TECHNOLOGY,
SHIBAZONO, NARASHINO-SHI, CHIBA, 275 JAPAN.