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On Some Tubes over J-holomorphic Curves in S^6

Dedicated to Professor Koichi Ogiue on his 60th birthday

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Introduction

Let (M, g) be a Riemannian manifold. We denote by $G^p(T_mM)$ the Grassmann manifold of all oriented p-dimensional linear subspaces of the tangent space T_mM of M at $m \in M$ and by $G^p(TM)$ the Grassmann bundle $\bigcup_{m\in M} G^p(T_mM)$. Let V be a subbundle of $G^p(TM)$. A p-dimensional submanifold N of M is called a *V-submanifold* if $T_mN \in V$ holds for any $m \in N$. If a Lie group G acts on M, the action is naturally extended to the action of G on $G^p(TM)$. It seems to be an interesting problem to study V-submanifold for an orbit V of an action of G on $G^p(TM)$.

Let J be the standard almost complex structure of the 6-dimensional sphere S^6 and $\langle , \rangle$ the standard Riemannian metric. It is well-known that the group of automorphisms of $(S^6, J, \langle , \rangle)$ is isomorphic to the compact exceptional simple Lie group G_2. The complex volume form ω of the tangent space T_mS^6 at $m \in S^6$ is extended to a G_2-invariant (complex) 3-form on S^6. For a complex number κ ($|\kappa| \le 1$), we put

$$V_\kappa = \{\xi \in G^3(TS^6) : \omega(\xi) = \kappa\}\,.$$

A 2-dimensional submanifold $\varphi : M^2 \to S^6$ is said to be a J-holomorphic curve if $J(d\varphi(T_mM)) = d\varphi(T_mM)$ holds for all $m \in M$. Bryant [1] showed that for any Riemann surface M there exists a superminimal J-holomorphic curve $\varphi : M \to S^6$ which has no geodesic point. In this note we study whether a tube over a J-holomorphic curve (in the direction of first or second) normal space is a V_κ-submanifold or not. In the case of tubes in the direction of second normal space, we shall prove the following

THEOREM 1. *Let $\varphi : M^2 \to S^6$ be a J-holomorphic curve without geodesic point. If a tube $\tilde\varphi_{2,\gamma}$ over φ of radius γ is a V_κ-submanifold, then one of the following holds*

(i) $\gamma = \pi/2$ *and* $\kappa = 1$,

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(ii) *φ is a superminimal J-holomorphc curve and*

$$\kappa = \cos\gamma \, \frac{9\cos^2\gamma - 8}{4 - 3\cos^2\gamma} .$$

A submanifold N of S^6 is said to be a *totally real* submanifold if $J(T_m N) \perp T_m N$ holds for any $m \in N$. A tube over a J-holomorphic curve $\varphi : M^2 \to S^6$ is a totally real submanifold if φ is superminimal and the radius is equal to $\arccos(\sqrt{5}/3)$ (Ejiri [3]) and if the radius is equal to $\pi/2$ (Dillen and Vranken [2]).

For a 3-dimensional subspace ξ of $T_m S^6$ the condition $J(\xi) \perp \xi$ is equivalent to $|\omega(\xi)| = 1$. The second author showed that $\omega(T_m N) = \pm 1$ $(m \in N)$ holds for a 3-dimensional totally real submanifolds [5], namely a totally real submanifold is nothing but a $V_{\pm 1}$-submanifold. Recently the second author [7] proved that if a compact V_κ submanifold exists then κ is a real number.

Ejiri [3] also showed that there exists a tube $\tilde{\varphi}_{1,\gamma}$ over a J-holomorphic curve $\varphi : S^2_{1/6} \to S^6$ which is a totally real submanifold. But the value, he obtained as the radius of the tube, is incorrect. In this paper, we also study tubes in the direction of the first normal bundle over J-holomorphic curves.

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1. Cayley algebra

Take an orthonormal basis $\{E_0 = 1, E_1, \cdots, E_7\}$ of the Cayley algebra $\mathfrak{C}$ such that

$$\begin{aligned} &E_1E_2 = E_3 , \quad E_1E_4 = E_5 , \quad E_1E_7 = E_6 , \quad E_2E_5 = E_7 , \\ &E_2E_4 = E_6 , \quad E_3E_4 = E_7 , \quad E_3E_6 = E_5 . \end{aligned}$$

We put

$$\mathfrak{C}_0 = \{u \in \mathfrak{C} \ : \ u + \bar{u} = 0\}$$

where "$^-$" denotes the conjugation in $\mathfrak{C}$. The unit sphere $S^6 \subset \mathfrak{C}_0$ centered at the origin has an almost complex structure J defined by

$$J_m(X) = m \cdot X , \quad m \in S^6, \ X \in T_m S^6 .$$

We denote by G_2 the group of all automorphisms of $\mathfrak{C}$. We identify $\mathfrak{C}_0 = \sum_{i=1}^{7} \mathbf{R} E_i$ with the set of all 7-dimensional column vectors in a natural manner and consider G_2 as a subgroup of $SO(7)$.

LEMMA 2. *For a pair of mutually orthogonal unit vectors a_1, a_2 in $\mathfrak{C}_0$ put $a_3 = a_1 \cdot a_2$. Take a unit vector $a_4 \in \mathfrak{C}_0$, which is perpendicular to a_1, a_2 and a_3. If we put $a_5 = a_1 \cdot a_4$,*

$a_6 = a_2 \cdot a_4$ and $a_7 = a_3 \cdot a_4$ then the matrix

$$g = [a_1, a_2, a_3, a_4, a_5, a_6, a_7] \in SO(7)$$

is an element of G_2.

We denote by $\omega_1, \cdots, \omega_7$ the orthonormal coframe dual to $E_1, \cdots, E_7$. The complex volume form

$$\omega = (\omega_2 + \sqrt{-1}\omega_3) \wedge (\omega_4 + \sqrt{-1}\omega_5) \wedge (\omega_7 + \sqrt{-1}\omega_6)\,.$$

of the tangent space $T_{E_1}S^6$ is extended to a G_2-invariant 3-form on S^6, which we also denote by ω.

PROPOSITION 3. *Let ξ, ξ' be elements of $G^3(TS^6)$. There exists an element $g \in G_2$ such that $g(\xi) = \xi'$ if and only if $\omega(\xi) = \omega(\xi')$.*

PROOF. Without loss of generality, we may assume that ξ_1 and ξ_2 are subspaces of $T_{E_1}S^6$.

If we take a suitable oriented base v_1, v_2, v_3 of ξ, the restriction of the Kähler form $\Omega(X, Y) = \langle J(X), Y\rangle$ on ξ is of the form

$$[\langle J(v_i), v_j\rangle] = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a \geq 0\,.$$

There exists an element $h \in SU(3)$ such that

$$h(v_1) = E_2\,, \quad h(v_2) = aE_3 + bE_4 + cE_5\,, \quad h(v_3) = E_7\,.$$

Note that $\omega(\xi) = b + \sqrt{-1}c$. We take an oriented orthonormal base v_1', v_2', v_3' of ξ' and $h' \in SU(3)$ with

$$h'(v_1') = E_2\,, \quad h'(v_2') = a'E_3 + b'E_4 + c'E_5\,, \quad h'(v_3') = E_7\,.$$

Since the only element of $SU(3)$ which stabilizes E_2 and E_7 is the unit element, there exists an element $g \in SU(3)$ with $g(\xi) = \xi'$ if and only if $(a, b, c) = (a', b', c')$. q.e.d.

2. J-holomorphic curve in S^6

Let $\varphi : M^2 \to S^6$ be a J-holomorphic curve in S^6. We denote by σ the second fundamental form of a J-holomorphic curve φ. Take an orthonormal frame $e_1, e_2 = J(e_1)$ of T_mM. The length of $\sigma(e_1, e_1)_m$ and $\sigma(e_1, e_2)_m$ are equal to each other if the point m is not a geodesic point. Assume that $\varphi : M \to S^6$ is a J-holomorphic curve without geodesic point and put

$$e_3 = \sigma(e_1, e_1)/\lambda\,, \quad e_4 = \sigma(e_1, e_2)/\lambda\,, \quad e_5 = e_1e_3\,, \quad e_6 = J(e_5)$$

where $\lambda = |\sigma(e_1, e_1)|$. Since we have $\varphi(m) \times e_3 = e_4$ and $e_2 \times e_3 = -e_6$, the matrix

$$[\varphi(m), e_1, e_2, e_3, e_4, e_5, -e_6]$$

is an element of G_2 by Lemma 2.

We denote by UN_1 the bundle of all unit vectors in the first normal space. For a constant γ, we define the tube $\tilde{\varphi}_{1,\gamma}$ of radius γ in the direction of the 1-st normal bundle as follows;

$$\tilde{\varphi}_{1,\gamma} : UN_1 \to S^6 \ ; (m, X) \mapsto (\cos\gamma)\varphi(m) + \sin\gamma X\,.$$

The tube $\tilde{\varphi}_{2,\gamma}$ of radius γ in the direction of the 2-nd normal bundle is defined in a similar fashion.

Define a (local) lift $g : M \to G_2$ of φ by

$$g(m) = [e_0, e_1, e_2, e_3, e_4, e_5, -e_6]$$

where we put $e_0 = \varphi(m)$. We denote by D the covariant derivative of $\mathbf{R}^7$ and by ω_{BA} the connection form,

$$\omega_{BA}(X) = \langle D_X e_A, e_B\rangle\ , \quad 0 \le A, B \le 6\,.$$

Put $h^B_{iA} = \left\langle D_{e_i} e_A, e_B\right\rangle = \omega_{BA}(e_i)$. The pull-back of the Maurer-Cartan form of G_2 by g is of the form

$$\begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} & 0 & 0 & 0 \\ \omega_{10} & 0 & \omega_{12} & \omega_{13} & \omega_{14} & \omega_{15} & -\omega_{16} \\ \omega_{20} & \omega_{21} & 0 & \omega_{23} & \omega_{24} & \omega_{25} & -\omega_{26} \\ \omega_{30} & \omega_{31} & \omega_{32} & 0 & \omega_{34} & \omega_{35} & -\omega_{36} \\ 0 & \omega_{41} & \omega_{42} & \omega_{43} & 0 & \omega_{45} & -\omega_{46} \\ 0 & \omega_{51} & \omega_{52} & \omega_{53} & \omega_{54} & 0 & -\omega_{56} \\ 0 & -\omega_{61} & -\omega_{62} & -\omega_{63} & -\omega_{64} & -\omega_{65} & 0 \end{pmatrix} \tag{1}$$

Since (1) is a $\mathfrak{g}_2$-valued 1-form, we have the following

PROPOSITION 4. *The connection form* $\{\omega_{BA}\}$ *satisfy the following*

$$\begin{cases} \omega_{41} = -\omega_{32} \\ \omega_{42} = \ \ \omega_{31} \\ \omega_{62} = \ \ \omega_{51} \\ \omega_{61} = -\omega_{52} \\ \omega_{63} = -\omega_{54} + \omega_1 \\ \omega_{64} = -\omega_{53} - \omega_2 \\ \omega_{65} = -\omega_{43} - \omega_{21}\,. \end{cases}$$

In other words,

$$\begin{cases} h^4_{12} = h^4_{21} = h^3_{11} = -h^3_{22} (= \lambda)\,, \\ h^3_{12} = h^3_{21} = -h^4_{11} = h^4_{22} = 0\,, \\ h^5_{13} = h^6_{14} = -h^5_{24} = h^6_{23}\,, \\ h^6_{13} + h^5_{14} = h^5_{23} - h^6_{24} = 1\,, \\ h^6_{13} + h^6_{24} = h^5_{23} - h^5_{14} = 0\,. \end{cases}$$

holds. Especially, φ is a minimal immersion.

The third fundamental form σ_3 of φ is defined as follows

$$\sigma_3(u, v, w) = \langle D_u\sigma(V, W), e_5\rangle\, e_5 + \langle D_u\sigma(V, W), e_6\rangle\, e_6\,, \quad u,\ v,\ w \in T_m M\,,$$

where V, W are locally defined vector field on M with $V_m = v$ and $W_m = w$. A 2-dimensional submanifold M of S^6 is said to be a *superminimal* surface if and only if $\{\sigma_3(X, X, X) : X \in T_m M, |X| = 1\}$ is a circle in the second normal space. The condition that a surface $\varphi : M^2 \to S^6$ is a superminimal surface is equivalent to

$$h^5_{13} + h^6_{23} = h^5_{23} - h^6_{13} = 0\,.$$

Thus we have the following

COROLLARY 5. *A J-holomorphic curve $\varphi : M \to S^6$ without geodesic point is a superminimal surface if and only if*

$$h^5_{13} = h^6_{14} = h^6_{23} = -h^5_{24} = 0\,,$$
$$h^6_{13} = h^5_{14} = h^5_{23} = -h^6_{24} = 1/2\,.$$

REMARK 6. If we put $\hat{e}_1 = e_2$ and $\hat{e}_2 = e_1$ and put $\hat{e}_3 = \sigma(\hat{e}_1, \hat{e}_1)/\lambda$, $\hat{h}^5_{13} = \langle D_{\hat{e}_1}\hat{e}_3, \hat{e}_5\rangle$ etc., then we have

$$\hat{h}^5_{13} = h^5_{13}\,, \quad \hat{h}^5_{14} = 1 - h^5_{14}\,.$$

From proposition 4, if we put $h^5_{13} = \nu$ and $h^5_{14} - 1/2 = \mu$, we have

$$\omega_{53} = \nu\omega_1 + (\mu + 1/2)\omega_2$$
$$\omega_{63} = -(\mu - 1/2)\omega_1 + \nu\omega_2$$
$$\omega_{54} = (\mu + 1/2)\omega_1 - \nu\omega_2$$
$$\omega_{64} = \nu\omega_1 + (\mu - 1/2)\omega_2$$

From the Maurer-Cartan equation of G_2, we have the following

PROPOSITION 7 (Integrability condition).

$$d\omega_1 = \omega_{21} \wedge \omega_2 \tag{3.1}$$

$$d\omega_2 = \omega_{12} \wedge \omega_1 \tag{3.2}$$

$$d\omega_{21} = (2\lambda^2 - 1)\omega_1 \wedge \omega_2 \tag{3.3}$$

$$d\omega_{43} = -(2\lambda^2 - 2\mu^2 + 2\mu - 1 - 2\nu^2)\omega_1 \wedge \omega_2 \tag{3.4}$$

$$0 = J^* d \log \lambda + 2\omega_{21} - \omega_{43} \tag{3.5}$$

$$0 = J^* d \log \mu + 2\omega_{21} + 2\omega_{43} \tag{3.6}$$

where J^* is defined for 1 *form by* $J^*\alpha(X) = -\alpha(J(X))$, $X \in TM$.

COROLLARY 8. ([4])

$$\Delta \log \lambda = 5/2 - 6\lambda^2 + 2|\mathbf{III}|^2$$

$$\Delta \log |\mathbf{III}| = 1 - 4|\mathbf{III}|^2$$

where $|\mathbf{III}|$ *is the square of the length of the third fundamental form* $|\mathbf{III}|^2 = (\mu - 1/2)^2 + \nu^2$.

3. Proof of theorem 1

Let $\varphi : M \to S^6$ be a J-holomorphic curve without geodesic point. From lemma 2 an orthogonal transformation g on $\mathfrak{C}_0$ which is defined by

$$g(\varphi(x)) = E_1\,, \quad g(e_1) = E_2\,, \quad g(e_2) = E_3\,, \quad g(e_3) = E_4\,,$$
$$g(e_4) = E_5\,, \quad g(e_5) = E_6\,, \quad g(e_6) = -E_7$$

is an element of G_2. We put $\alpha = \cos\gamma$, $\beta = \sin\gamma$ and

$$A = h^5_{13}\cos t + h^6_{13}\sin t\,, \quad B = (1 - h^6_{13})\cos t + h^5_{13}\sin t\,.$$

The tangent space of the tube

$$\tilde{\varphi}_{2,\gamma} : UN_2 \to S^6$$

at $g^{-1}(\alpha E_1 + \beta(\cos t E_6 - \sin t E_7))$ is spanned by

$$U_1 = g^{-1}({}^t[\,0,\ 0,\ 0,\ 0,\ 0,\ \sin t,\ \cos t\,])\,,$$
$$U_2 = g^{-1}({}^t[\,0,\ \alpha,\ 0,\ -\beta A,\ -\beta B,\ 0,\ 0\,])\,,$$
$$U_3 = g^{-1}({}^t[\,0,\ 0,\ \alpha,\ -\beta B,\ \ \beta A,\ 0,\ 0\,])\,.$$

The orthogonal matrix

$$h = \begin{pmatrix} \alpha & 0 & 0 & 0 & 0 & \beta\cos t & -\beta\sin t \\ 0 & 0 & 0 & 0 & 0 & \sin t & \cos t \\ -\beta & 0 & 0 & 0 & 0 & \alpha\cos t & -\alpha\sin t \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & \beta\cos t & -\beta\sin t & 0 & 0 \\ 0 & 0 & 0 & \sin t & \cos t & 0 & 0 \\ 0 & 0 & -\beta & \alpha\cos t & -\alpha\sin t & 0 & 0 \end{pmatrix}$$

is an element of G_2 with $h(\alpha E_1 + \beta(\cos t E_6 - \sin t E_7) = E_1$. Substiting $h(g(U_1))$, $h(g(U_2))$, $h(g(U_3))$ into $\omega|_{E_1}$, we have

$$\begin{aligned} &\omega(U_1,\ U_2,\ U_3) \\ &\quad = \beta\{(1/2 + (h^5_{13})^2 - h^6_{13} + (h^6_{13})^2)\beta^2 - 2\alpha^2\} \\ &\qquad + \beta(1-3\alpha^2)\{(1/2 - h^6_{13})\cos(2t) + h^5_{13}\sin(2t)\} \\ &\qquad + 2\sqrt{-1}\,\alpha\beta\{h^5_{13}\cos(2t) + (h^6_{13} - 1/2)\sin(2t)\}\,. \end{aligned}$$

If we put $G = \det([\langle U_i, U_j\rangle])$, we have

$$\omega(d\tilde{\varphi}_{1,\gamma}(T(UN_2)) = \omega(U_1, U_2, U_3)/\sqrt{G}\,.$$

The imaginary part of $\omega(d\tilde{\varphi}_{2,\gamma}(T(UN_2))$ is equal to

$$2\alpha\beta \frac{h^5_{13}\cos(2t) + (h^6_{13} - 1/2)\sin(2t)}{(h^5_{13})^2 + (h^6_{13})^2 + h^6_{13} - 1/2 - (h^6_{13} - 1/2)\cos(2t) + h^5_{13}\sin(2t)}\,,$$

which is a constant if and only if one of the following holds.

- $\alpha = \cos\gamma = 0$,
- $h^6_{13} = 1/2$, $h^5_{13} = 0$.

If $\alpha = 0$ then $\omega(d\tilde{\varphi}_{2,\gamma}(T(UN_2))) = 1$, namely the tube $\tilde{\varphi}_{2,\pi/2}(UN_2))$ is a V_1 (in other word, totally real) submanifold.

If $h^6_{13} = 1/2$ and $h^5_{13} = 0$ then $\omega(d\tilde{\varphi}_{2,\gamma}(T(UN_2)))$ is equal to

$$\cos\gamma\,\frac{9\cos^2\gamma - 8}{4 - 3\cos^2\gamma}\,.$$

q.e.d.

4. Tubes $\varphi_{1,\gamma}$

First we give two typical examples of J-holomorphic curves of S^6.

EXAMPLE 1. The subgroup

$$T = \{\exp(aG_{23} + bG_{45} + cG_{76}) : a, b, c \in \mathbf{R}, a + b + c = 0\}$$

is a maximal torus of G_2, where G_{ij} $(1 \le i \ne j \le 7)$ are skew-symmetric transformations on $\mathfrak{C}_0$ defined by

$$G_{ij}(E_k) = \delta_{jk}E_i - \delta_{ik}E_j\,, \quad 1 \le k \le 7\,.$$

The mapping

$$\Phi_0 : T \to S^6; (a, b, c) \mapsto \exp(aG_{23} + bG_{45} + cG_{76})(p_0)$$

where we put $p_0 = (E_2 + E_4 + E_6)/\sqrt{3}$, is a J-holomorphic curve. Take

$$e_1 = \sqrt{3/2}(G_{23} - G_{76}),\ e_2 = \sqrt{1/2}(G_{23} - 2G_{45} + G_{76})$$

as a basis of the Lie algebra of T. We have

$$D_{e_1}e_1 = -p_0 + \sqrt{1/2}\,e_3\,, \quad D_{e_1}e_2 = \sqrt{1/2}\,e_4\,, \quad D_{e_2}e_2 = -p_0 - \sqrt{1/2}\,e_3\,,$$

where we put

$$e_3 = -\sqrt{1/6}(E_2 - 2E_4 + E_6)\,, \quad e_4 = -\sqrt{1/2}(E_2 - E_6)\,.$$

We extend e_3, e_4 to a left invariant vector fields on T, and denote them also by e_3, e_4. Put

$$e_5 = \sqrt{1/3}(E_3 + E_5 - E_7)\,, \quad e_6 = E_1\,,$$

and extend e_5, e_6 to a left invariant vector fields on T, and denote them also by e_5, e_6. We have

$$\begin{cases} h^4_{12} = h^4_{21} = h^3_{11} = -h^3_{22} = \sqrt{1/2}\,, \\ h^3_{12} = h^3_{21} = h^4_{22} = -h^4_{11} = 0\,, \\ h^5_{13} = h^6_{14} = h^6_{23} = -h^5_{24} = 0\,, \\ h^5_{14} = h^5_{23} = 1\,, \quad h^6_{13} = h^6_{24} = 0\,. \end{cases}$$

EXAMPLE 2. There exists a 3-dimensional simple Lie subgroup U_4 of $G_2(\subset SO(\mathbf{R}^7))$ whose action on $\mathbf{R}^7$ is irreducible (cf. [5]). The orbit of the subgroup U_4 through e_1 is the Veronese embedding $\Phi_1 : S^2_{1/6} \to S^6$. The length of second fundamental form of Φ_0 is $5/3$.

THEOREM 9. *Let $\varphi : M^2 \to S^6$ be a J-holomorphic curve without geodesic point. If a tube $\tilde{\varphi}_{1,\gamma}$ over φ of radius γ is a V_κ-submanifold, then one of the following holds*

(i) *φ is (locally) congruent (up to G_2) to the J-holomorphic curve $\Phi_0 : T^2 \to S^5(\subset S^6)$ and $\gamma = \pi/2$. In this case $\kappa^2 = 1/3$.*

(ii) *φ is (locally) congruent (up to G_2) to $\Phi_1 : S^2_{1/6} \to S^6$ and $0 < \gamma \leq \pi/2$. In this case*

$$\kappa = \sin\gamma \frac{6\cos^2\gamma - 2\sin^2\gamma}{\sqrt{9\cos^4\gamma - 3\cos^2\gamma\sin^2\gamma + 4\sin^4\gamma}},$$

(iii) *φ is a superminimal J-holomorphc curve $M \to S^6$ and $\gamma = \pi/2$. In this case $\kappa = 1$.*

PROOF OF THEOREM 9. The tangent space of the tube $\tilde{\varphi}_{1,\gamma} : UN_1 \to S^6$ at $g^{-1}(\alpha E_1 + \beta(\cos t E_3 + \sin t E_4))$ is spanned by

$$U_1 = g^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin t \\ -\cos t \\ 0 \\ 0 \end{pmatrix},$$

$$U_2 = g^{-1} \cdot \begin{pmatrix} 0 \\ \alpha - \beta h^3_{11}\cos t \\ -\beta h^3_{11}\sin t \\ 0 \\ 0 \\ \beta(h^5_{13}\cos t + h^5_{14}\sin t) \\ -\beta((1 - h^5_{14})\cos t + h^5_{13}\sin t) \end{pmatrix},$$

$$U_3 = g^{-1} \cdot \begin{pmatrix} 0 \\ -\beta h^3_{11}\cos t \\ \alpha + \beta h^3_{11}\sin t \\ 0 \\ 0 \\ \beta(h^5_{14}\cos t - h^5_{13}\sin t) \\ -\beta(h^5_{13}\cos t + (h^5_{14} - 1)\sin t) \end{pmatrix},$$

We put

$$h = \begin{pmatrix} \alpha & 0 & 0 & \beta\cos t & \beta\sin t & 0 & 0 \\ 0 & 0 & 0 & -\sin t & \cos t & 0 & 0 \\ \beta & 0 & 0 & -\alpha\cos t & -\alpha\sin t & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & 0 & -\beta\cos t & -\beta\sin t \\ 0 & 0 & 0 & 0 & 0 & \sin t & -\cos t \\ 0 & 0 & \beta & 0 & 0 & \alpha\cos t & \alpha\sin t \end{pmatrix}$$

and substitute $h(g(U_1))$, $h(g(U_2))$ and $h(g(U_3))$ into $\omega|_{E_1}$. By a long (and tedious) calculation we obtain

$$\begin{aligned} \omega(U_1, U_2, U_3) = \beta\{2\alpha^2 + \beta^2((h^5_{13})^2 + (h^5_{14})^2 - h^5_{14} - (h^3_{11})^2\} \\ + 2\alpha\beta^2 h^3_{11} h^5_{13}\sin(3t) - \alpha\beta^2 h^3_{11}(2h^5_{14} - 1)\cos(3t) \\ - 2\sqrt{-1}\beta^2 h^3_{11} h^5_{13}\cos(3t)\,. \end{aligned}$$

If we put $G = \det([\langle U_i, U_j\rangle])$, we have

$$\omega(d\tilde{\varphi}_{1,\gamma}(T(UN_1))) = \omega(U_1, U_2, U_3)/\sqrt{G}\,.$$

Fix a point m and assume that $\omega(d\tilde{\varphi}_{1,\gamma}(T(UN_1)))$ is a constant with respect to t. Since G does not contain any term $\cos(kt)$ and $\sin(kt)$ for $k \geq 5$, the coefficient of $\sqrt{-1}\sin(6t)$ in the expansion of $\mathrm{Im}(\omega(U_1, U_2, U_3))^2$ is equal to 0, namely we have

$$2\alpha\beta^4(h^3_{11})^2(h^5_{13})^2 = 0\,.$$

From the condition that the coefficient of $\sqrt{-1}\sin(6t)$ in the expansion of $\mathrm{Im}(\omega(U_1, U_2, U_3))^2$ is equal to 0, we have the following two cases:

- $h^5_{13} = 0$, $h^5_{14} = 1/2$,
- $h^5_{13} = 0$, $\alpha = 0$.

CASE 1. Assume that $h^5_{13} = 0$, $h^5_{14} = 1/2$ hold, namely φ is a superminimal surface. Since ω and G are

$$\begin{aligned} \omega(U_1, U_2, U_3) &= \beta(2\alpha^2 - \beta^2(\lambda^2 + 1/4))\,, \\ G &= (\alpha^2 + \beta^2(\lambda^2 + 1/4))^2 - (2\lambda\alpha\beta)^2\,. \end{aligned}$$

the function $\kappa(m) = \omega(d\tilde{\varphi}_{1,\gamma}(T(UN_1)))$ is a constant on M if and only if $\lambda = h^3_{11}$ is a constant.

From the equation of Gauss, M is a space of constant curvature. By results of Sekigawa [6] and Hashimoto [4] φ is the Veronese surface $S^2_{1/6} \to S^6$ (where $\lambda = \sqrt{5/12}$). In this case, we have

$$\kappa = \frac{\beta(6\alpha^2 - 2\beta^2)}{\sqrt{(3\alpha^2 + 2\beta^2)^2 - 15(\alpha\beta)^2}} .$$

CASE 2. Assume that $\alpha = 0$ and $h^5_{13} = 0$. We have

$$\kappa^2 = \frac{(\lambda^2 - \mu^2 + 1/4)^2}{(\lambda^2 + (\mu^2 + 1/2)^2)(\lambda^2 + (\mu^2 - 1/2)^2)}$$

where we put $\lambda = h^3_{11}$ and $\mu = h^5_{14} - 1/2$. From

$$d\kappa^2 = \frac{8\lambda\mu(\lambda^2 - \mu^2 + 1/4)\{\mu(\lambda^2 + \mu^2 - 1/4)d\lambda - \lambda(\lambda^2 + \mu^2 + 1/4)d\mu\}}{(\lambda^2 + (\mu^2 + 1/2)^2)(\lambda^2 + (\mu^2 - 1/2)^2)} \tag{4}$$

the function κ^2 on M is a constant if and only if one of the following holds

$$0 = \mu \tag{5.1}$$

$$0 = \lambda^2 - \mu^2 + 1/4 \tag{5.2}$$

$$0 = \mu(\lambda^2 + \mu^2 - 1/4)d\lambda - \lambda(\lambda^2 + \mu^2 + 1/4)d\mu . \tag{5.3}$$

Put $M_1 = \{m \in M \ : \ \mu(m) \neq 0\}$.

In the interior of $M \setminus M_1$ it falls into Case 1. So we assume that M_1 is non-empty. Since (5.2) or (5.3) holds on M_1,

$$U = \frac{\lambda^2 - \mu^2 + 1/4}{\lambda\mu}$$

is a constant (the numerator of the differential dU is equal to the left hand side of (5.3)).

From (3.5) and (3.6), we have

$$\omega_{21} = -(1/6)J^* d \log|\lambda^2\mu| .$$

Since $\omega_{21} = J^* d \log\rho$, we conclude that $\rho^6\lambda^2\mu$ is a constant.

Put $M_2 = \{m \in M_1 \ : \ d\mu_m \neq 0\}$, $M_3 = \{m \in M_2 \ : \ d\lambda_m \neq 0\}$ and assume that M_3 is not an empty set.

Since $U = (\lambda^2 - \mu^2 + 1/4)/\lambda\mu$ and $V = \rho^6\lambda^2\mu$ are constants, λ and μ are functions of ρ on M_3. Namely there exist functions $\hat{\lambda}$, $\hat{\mu}$ of variable ρ such that

$$\lambda(x) = \hat{\lambda}(\rho(x)) , \quad \mu(x) = \hat{\mu}(\rho(x)) , \quad m \in M_3 .$$

From $\rho^6\lambda^2\mu = \text{const.}$, we have

$$2\rho^2\mu d\lambda + \rho\,\lambda\,d\mu = -6\lambda\,\mu\,d\rho . \tag{6}$$

From (5.3) and (6), we have

$$\hat{\lambda}' = -6\hat{\lambda}(\hat{\lambda}^2 + \hat{\mu}^2 + 1/4)/\rho(3\hat{\lambda}^2 + 3\hat{\mu}^2 + 1/4) \tag{7.1}$$

$$\hat{\mu}' = -6\hat{\mu}(\hat{\lambda}^2 + \hat{\mu}^2 - 1/4)/\rho(3\hat{\lambda}^2 + 3\hat{\mu}^2 + 1/4) \tag{7.2}$$

If f is a function of ρ, i.e., there exists a function F of 1-variable such that $f(x) = F(\rho(x))$, $m \in M$, we have

$$(FF'' - F'^2 + (1/\rho)FF')||\mathrm{grad}\rho||^2 = F^2\Delta \log f - FF'\rho(2\lambda^2 - 1)\,.$$

Apply the above to λ and μ, we have

$$(\hat{\lambda}\hat{\lambda}'' - \hat{\lambda}'^2 + (1/\rho)\hat{\lambda}\hat{\lambda}')||\mathrm{grad}\rho||^2 = \hat{\lambda}^2(5/2 - 6\hat{\lambda}^2 + 2\hat{\mu}^2) - \hat{\lambda}\hat{\lambda}'\rho(2\hat{\lambda}^2 - 1) \tag{8.1}$$

$$(\hat{\mu}\hat{\mu}'' - \hat{\mu}'^2 + (1/\rho)\hat{\mu}\hat{\mu}')||\mathrm{grad}\rho||^2 = \hat{\mu}^2(1 - 4\hat{\mu}^2) - \hat{\mu}\hat{\mu}'\rho(2\hat{\lambda}^2 - 1) \tag{8.2}$$

From (7.1), (7.2) we have

$$\hat{\lambda}\hat{\lambda}'' - \hat{\lambda}'^2 + (1/\rho)\hat{\lambda}\hat{\lambda}' = \frac{-6\hat{\lambda}^2}{\rho(3\hat{\lambda}^2 + 3\mu^2 + 1/4)^2}(\hat{\lambda}\hat{\lambda}' + MM')\,,$$

$$\hat{\mu}\hat{\mu}'' - \hat{\mu}'^2 + (1/\rho)\hat{\mu}\hat{\mu}' = \frac{-12\hat{\mu}^2}{\rho(3\hat{\lambda}^2 + 3\mu^2 + 1/4)^2}(\hat{\lambda}\hat{\lambda}' + \hat{\mu}\hat{\mu}')\,.$$

Thus we obtain

$$2\hat{\mu}^2 \times (\text{l.h.s. of (8.1)}) - \hat{\lambda}^2 \times (\text{l.h.s. of (8.2)}) = 0$$

On the other hand, we have

$$\begin{aligned} 0 &= 2\hat{\mu}^2 \times (\text{r.h.s. of (8.1)}) - \hat{\lambda}^2 \times (\text{r.h.s. of (8.2)}) \\ &= \frac{\hat{\lambda}^2\hat{\mu}^2}{2(3\hat{\lambda}^2 + 3\hat{\mu}^2 + 1/4)}(48\hat{\lambda}^2\hat{\mu}^2 - 12\hat{\lambda}^2 + 48\hat{\mu}^4 - 8\hat{\mu}^2 - 1) \\ &= \frac{\hat{\lambda}^2\hat{\mu}^2}{2(3\hat{\lambda}^2 + 3\hat{\mu}^2 + 1/4)}(4\hat{\mu}^2 - 1)(12\hat{\mu}^2 + 12\hat{\lambda}^2 + 1) \end{aligned}$$

Thus M is a constant, which contradicts to our assumption $d\mu \neq 0$. Namely M_3 is an empty set and λ is a constant on M_2. But using (5.3), we conclude that M_2 is also an empty set. Namely λ and μ are constants on M_1. Using Corollaly 8, we conclude that $\mu = 1$ on M_1. After all if M_1 is not an empty set then $\mu = 1$ on M and λ is a constant on M. Thus we have $K = 1 - 2\lambda^2 = 0$ and φ is congruent (up to G_2) to Φ_0 (cf., [4]). q.e.d.

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