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An Upper Bound for the Garcia-Stichtenoth Numbers of Towers

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Abstract. We give several examples of towers $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ of function fields of one variable over a finite field $\mathbf{F}_q$, for which the Garcia-Stichtenoth number

$$\lambda(\mathcal{F}) = \lim_{m\to\infty} \frac{\text{number of } \mathbf{F}_q\text{-rational places of } F_m}{\text{genus of } F_m}$$

is zero. Moreover, we study an upper bound for the limit $\lambda(\mathcal{F})$.

1. Introduction

Let $\mathbf{F}_q$ be the finite field of cardinality q, where q is a power of a prime number. Let $F/\mathbf{F}_q$ be an algebraic function field of one variable with the field of constant $\mathbf{F}_q$. We shall refer to $F/\mathbf{F}_q$ as a function field. We denote by

$$g(F) = g(F/\mathbf{F}_q) \quad (\text{resp. } N(F) = N(F/\mathbf{F}_q))$$

the genus (resp. the number of $\mathbf{F}_q$-rational places, namely, places of degree one) of $F/\mathbf{F}_q$.

Garcia and Stichtenoth [2] introduced towers of function fields in order to construct sequences of codes with excellent error-correcting properties. A *tower* of function fields over $\mathbf{F}_q$ is a sequence

$$\mathcal{F} = (F_0, F_1, F_2, \cdots)$$

of function fields $F_m/\mathbf{F}_q$ having the following properties:

(i) $F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots$;
(ii) for each $m \geq 0$, F_{m+1}/F_m is a separable extension of degree $[F_{m+1} : F_m] > 1$;
(iii) for some $s \geq 0$, $F_s/\mathbf{F}_q$ is non-rational and non-elliptic.

A tower $\mathcal{F}$ over $\mathbf{F}_q$ is called *tame* if for all $m \geq 0$ and all places P of F_m, the ramification index of P in F_m/F_0 is relatively prime to the characteristic of $\mathbf{F}_q$. We say that a tower $\mathcal{F}$ is of *degree* l if it satisfies the condition:

(iv) for each m, F_{m+1}/F_m is an extension of degree l.

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Let E/F be a finite separable extension of function fields over $\mathbf{F}_q$. The Hurwitz Genus Formula (see, [6] Theorem III.4.12) states that

$$2g(E) - 2 = [E : F] \cdot (2g(F) - 2) + \deg \mathrm{Diff}(E/F)\,,$$

where $\mathrm{Diff}(E/F)$ denotes the different divisor of E/F. By using Condition (iii) and the Hurwitz Genus Formula, Garcia and Stichtenoth [2] showed that the limit

$$\lambda(\mathcal{F}) = \lambda(\mathcal{F}/\mathbf{F}_q) = \lim_{m\to\infty} N(F_m)/g(F_m) \quad (\geq 0)$$

exists. We call it the *Garcia-Stichtenoth number*. They also gave a criterion for

$$N(F_m)/g(F_m) \to 0 \quad \text{as} \ \ m \to \infty\,.$$

A tower $\mathcal{F}$ of function fields is said to be *asymptotically bad* (resp. *asymptotically good*) if

$$\lambda(\mathcal{F}) = 0 \quad (\text{resp.}\ \ \lambda(\mathcal{F}) > 0)\,.$$

These terminologies are motivated by applications to coding theory.

Garcia and Stichtenoth showed the following fact:

FACT ([3]). Let $f(X, Y) \in \mathbf{F}_q[X, Y]$ be a separable polynomial both in X and Y over $\mathbf{F}_q$ and $a = \deg_X f$ and $b = \deg_Y f$. Let $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ be a tower given recursively by $F_m = \mathbf{F}_q(x_0, \cdots, x_m)$ with

$$f(x_{i+1}, x_i) = 0 \quad \text{for}\ \ i = 0, 1, \cdots, m-1\,.$$

Assume that $[F_m : \mathbf{F}_q(x_0)] = a^m$ and $[F_m : \mathbf{F}_q(x_m)] = b^m$ for every $m \geq 0$. If $\deg_X f \neq \deg_Y f$, then the tower $\mathcal{F}$ is asymptotically bad.

In general, it is not easy to find asymptotically good sequences of codes. Kondo et alia ([7]) gave examples of sequences of one-point codes over the finite fields with good properties from towers of function fields:

EXAMPLE 1 ([7]). Let $r > 0$ be an odd integer. We define function fields $K_m/\mathbf{F}_{q^{2r}}$ recursively by $K_m = \mathbf{F}_{q^{2r}}(x_0, x_1, \cdots, x_m)$ with

$$x_{i+1}{}^{q^r+1} = x_i{}^q + x_i \quad \text{for}\ \ i = 0, 1, \cdots, m-1\,.$$

Then the sequence $\mathcal{K} = (K_0, K_1, K_2, \cdots)$ is a tower of degree $q^r + 1$ having the following properties:

- $g(K_m) = \dfrac{q^r}{2} \sum_{s=0}^{m-1} (q^{s+1} - 1)(q^r + 1)^{m-1-s}$ for $m \geq 1$;
- $N(K_m) = q^{2r+1+m} + 1$ for $m \geq 0$;
- the tower $\mathcal{K}$ is asymptotically bad: $\lambda(\mathcal{K}) = 0$;

Drinfeld and Vlăduţ proved the following asymptotic result: Setting

$$N_q(g) := \max\{N(F) \mid F/\mathbf{F}_q \text{ is a function field of genus } g\}$$

and

$$A(q) := \limsup_{g\to\infty} N_q(g)/g\,,$$

they gave the following result on the asymptotic behavior of the number of $\mathbf{F}_q$-rational places (the so-called Drinfeld-Vlăduţ bound):

$$A(q) \leq \sqrt{q} - 1\,.$$

If $q = l^2$ is a square, the inequality is in fact an equality: Ihara (independently, Tsfasman, Vlăduţ and Zink) proved that $A(l^2) = l - 1$. Their proof requires deep results from algebraic geometry and modular curves. It is obvious by the definition that $\lambda(\mathcal{F}) \leq A(q)$. A tower $\mathcal{F}$ is said to be *optimal* if $\lambda(\mathcal{F}) = A(q)$.

Our aim is to give another upper bound for the Garcia-Stichtenoth number, and to compute the numbers for several towers of function fields. Moreover, we show that the converse of Fact [3] of Garcia and Stichtenoth is false.

Section 2 gives some facts from the theory of function fields. In particular, we investigate function fields of Kummer type.

We define, in Section 3, an important subset of the set $\mathbf{N}$ of natural numbers and a real-valued function on the product of $\mathbf{N}$ and $\mathbf{N}$, and give the main result: an upper bound for the Garcia-Stichtenoth number.

In the final section, we treat several examples of towers of function fields, which are counterexamples to the converse of Fact [3], and a few examples of optimal towers.

2. Genera of function fields

Throughout this paper, we use the notation and terminology of the textbook [6] of Stichtenoth. In this section, we determine the genera of some Kummer extensions of function fields. We apply these results to computing the Garcia-Stichtenoth numbers of towers of function fields in Section 4.

Let n be an integer with $n > 1$ and $n \mid q - 1$, and let F be a finite algebraic extension of a rational function field with the field of constant $\mathbf{F}_q$. Let x be a transcendental element of F over $\mathbf{F}_q$. Suppose that $f(x), h(x) \in F$ are relatively prime polynomials over $\mathbf{F}_q$ satisfying

$$f(x)/h(x) \neq w^d \quad \text{for any } \; w \in F \quad \text{and } \; d \mid n, d > 1\,.$$

We define the function field $E = F(y)$ over $\mathbf{F}_q$ given by $y^n = f(x)/h(x)$. The field E is called a *Kummer extension* of F. Let $\mathbf{P}_F$ be the set of places of F. For a place $P' \in \mathbf{P}_E$ of $E/\mathbf{F}_q$ lying over $P \in \mathbf{P}_F$, we denote by $e(P'|P)$ (resp. by $d(P'|P)$) the ramification index (resp. the different exponent) of P' over P.

We recall two well-known facts.

FACT ([6], Dedekind's Different Theorem). With the notation above, the different exponent of P' over P is given by

$$d(P'|P) = \begin{cases} 0 & \text{if } P'|P \text{ is unramified}, \\ e(P'|P) - 1 & \text{if } P'|P \text{ is tamely ramified}. \end{cases}$$

FACT ([6], Kummer Extensions). Suppose that there exists a place $Q \in \mathbf{P}_F$ such that

$$\gcd(v_Q(f(x)/h(x)), n) = 1 .$$

Then E/F is an cyclic extension of degree n with constant field $\mathbf{F}_q$, and the genus of E is given by

$$g(E) = 1 - n + \frac{1}{2} \sum_{P \in \mathbf{P}_F} (n - \gcd(v_P(f(x)/h(x)), n)) \deg P . \tag{1}$$

We will use the next lemma in the proof of Proposition 2 and in the last section.

LEMMA 1. *Assume that*

$$\gcd(\deg f(x), \deg h(x)) = 1 \quad \text{and} \quad n = \max\{\deg f(x), \deg h(x)\} . \tag{2}$$

If $Q \in \mathbf{P}_F$ is a place with

$$v_Q(x) = -m < 0 \quad \text{and} \quad \gcd(m, n) = 1 , \tag{3}$$

then Q is totally ramified in E/F, the place $Q' \in \mathbf{P}_E$ lying over Q has

$$v_{Q'}(y) = -m \cdot (\deg f(x) - \deg h(x)) ,$$

and $v_Q(f(x)/h(x))$ and n are coprime;

$$\gcd(v_Q(f(x)/h(x)), n) = 1 .$$

PROOF. We may assume that $n = \deg f(x)$. Since

$$v_Q(f(x)/h(x)) = -m \cdot (n - \deg h(x)) , \tag{4}$$

we have $n \cdot v_{Q'}(y) = -e(Q'|Q) \cdot m \cdot (n - \deg h(x))$. It follows that $n \mid e(Q'|Q)$ from the hypotheses (2) and (3). On the other hand, we have

$$e(Q'|Q) \leq [E : F] = n ,$$

by [[6] Corollary III.1.12 (b)]. The numbers $v_Q(f(x)/h(x))$ and n are coprime by the hypotheses (2), (3) and Eq. (4). This completes the proof of Lemma 1. □

In general, it is hard to determine the genera of function fields. By the rational function field theory (see, [6] Chapter I) and the Genus Formula (1), we can prove the following proposition, which is used to calculate the Garcia-Stichtenoth numbers in the last section.

PROPOSITION 2. *Let $E = \mathbf{F}_q(x, y)$ be a Kummer extension of $F = \mathbf{F}_q(x)$ given by*

$$y^n = f(x)/h(x)\,.$$

Suppose that $f(x), h(x) \in \mathbf{F}_q[x]$ are relatively prime separable polynomials with

$$\gcd(\deg f(x), \deg h(x)) = 1 \quad \textit{and} \quad n = \max\{\deg f(x), \deg h(x)\}\,.$$

Then the genus of E is given by

$$g(E) = \frac{n-1}{2} \cdot (\deg f(x) + \deg h(x) - 1) \geq 1\,.$$

PROOF. For the infinite place $Q \in \mathbf{P}_F$, we have $v_Q(x) = -1$, and hence we get

$$\gcd(v_Q(f(x)/h(x)), n) = 1$$

by Lemma 1. Let

$$f(x) = a \cdot \prod_{i=1}^{s} p_i(x) \quad \left(\text{resp. } h(x) = b \cdot \prod_{j=1}^{t} q_j(x)\right)$$

denote the factorization of $f(x)$ (resp. $h(x)$) into a product of irreducible polynomials in $\mathbf{F}_q[x]$, where $a, b \in \mathbf{F}_q$. For i and j, choose the place $P_{p_i}, P_{q_j} \in \mathbf{P}_F$ such that a prime element for P_{p_i} is the polynomial $p_i(x)$ and that for P_{q_j} is $q_j(x)$. Then we have

$$v_{P_{p_i}}(f(x)/h(x)) = v_{P_{p_i}}(p_i(x)) = 1\,, \quad \gcd(v_{P_{p_i}}(f(x)/h(x)), n) = 1\,,$$

$$v_{P_{q_j}}(f(x)/h(x)) = -1 \quad \text{and} \quad \gcd(v_{P_{q_j}}(f(x)/h(x)), n) = 1\,.$$

If $P \in \mathbf{P}_F$ is a place different from P_{p_i}, P_{q_j} and Q, then

$$v_P(f(x)/h(x)) = 0 \quad \text{and} \quad \gcd(v_P(f(x)/h(x)), n) = n\,.$$

Since

$$\sum_{i=1}^{s} \deg P_{p_i} = \deg f(x) \quad \text{and} \quad \sum_{j=1}^{t} \deg P_{q_j} = \deg h(x)\,,$$

the Genus Formula (1) yields

$$\begin{aligned} g(E) &= 1 - n + \frac{n-1}{2} \cdot \left(\sum_{i=1}^{s} \deg P_{p_i} + \sum_{j=1}^{t} \deg P_{q_j} + \deg Q\right) \\ &= 1 - n + \frac{n-1}{2} \cdot (\deg f(x) + \deg h(x) + 1) \\ &= \frac{n-1}{2} \cdot (\deg f(x) + \deg h(x) - 1)\,. \end{aligned}$$

This completes the proof of Proposition 2. □

This proposition implies that the function field $\mathbf{F}_q(x, y)/\mathbf{F}_q$ is not rational.

3. Upper bound of towers

This section contains the heart of this paper.

DEFINITION. (1) A tower $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ over $\mathbf{F}_q$ is said to be *tame* if the extension F_m/F_0 is tame for each $m \geq 1$.

(2) The F_0-*ramification locus* of $\mathcal{F}/\mathbf{F}_q$ is defined to be

$$V_{F_0}(\mathcal{F}) = \{P \in \mathbf{P}_{F_0} \mid P \text{ is ramified in } F_s/F_0, \text{ for some } s \geq 1\},$$

where $\mathbf{P}_{F_0}$ denotes the set of places of $F_0/\mathbf{F}_q$.

(3) The set of $\mathbf{F}_q$-rational places of F_0 that split completely in each extension F_m/F_0 is denoted by $T_{F_0}(\mathcal{F})$, namely,

$$T_{F_0}(\mathcal{F}) = \{P \in \mathbf{P}_{F_0} \mid P \text{ is } \mathbf{F}_q\text{-rational and splits completely in each } F_m/F_0\}.$$

Garcia, Stichtenoth and Thomas proved the following fact.

FACT ([5]). Let $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ be a tame tower over $\mathbf{F}_q$ having properties

$$\sharp V_{F_0}(\mathcal{F}) < \infty \quad \text{and} \quad T_{F_0}(\mathcal{F}) \neq \emptyset.$$

Then the limit $\lambda(\mathcal{F})$ satisfies the inequality

$$\lambda(\mathcal{F}) \geq \frac{2 \cdot \sharp T_{F_0}(\mathcal{F})}{2g(F_0) + \displaystyle\sum_{P \in V_{F_0}(\mathcal{F})} \deg P - 2} \quad (> 0).$$

We now give an upper bound for the limit $\lambda(\mathcal{F})$.

DEFINITION. Let $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ be a tower of function fields over $\mathbf{F}_q$, and let m be a positive integer. We say that $\mathcal{F}$ has the property $\lambda(m, l)$ if it satisfies the inequalities

$$\frac{\deg \mathrm{Diff}(F_{m+i}/F_{m+i-1})}{[F_{m+i} : F_{m+i-1}] - 1} \geq [F_{m+i-1} : F_0] + [F_1 : F_0]$$

for all $i = 1, 2, \cdots, l$. We define the set

$$\Sigma_m(\mathcal{F}) = \Sigma_m(\mathcal{F}/\mathbf{F}_q) = \{l \geq 1 \mid \mathcal{F}/\mathbf{F}_q \text{ has the property } \lambda(m, l)\}.$$

A problem arises whether there exists a tower $\mathcal{F}$ over $\mathbf{F}_q$ such that the set $\Sigma_m(\mathcal{F})$ is nonempty. We will take up this problem and show affirmative examples in the last section. The following Theorem and Corollary 3 make sense only when the set $\Sigma_m(\mathcal{F})$ is nonempty.

THEOREM. *Suppose that $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ is a tower of function fields over $\mathbf{F}_q$ with $\Sigma_m(\mathcal{F}) \neq \emptyset$.*

(1) *The inequality*

$$\sum_{i=1}^{l}[F_{m+l} : F_{m+i}] \cdot \deg \mathrm{Diff}(F_{m+i}/F_{m+i-1}) \geq [F_{m+l} : F_0] \cdot R(m,l)$$

holds for every $l \in \Sigma_m(\mathcal{F})$, *where*

$$R(m,l) := l - \sum_{i=1}^{l} \frac{1}{[F_{m+i} : F_{m+i-1}]} + \frac{[F_{m+l} : F_m] - 1}{[F_{m+l} : F_1]}\,.$$

(2) *The inequality*

$$2g(F_{m+l}) - 2 \geq [F_{m+l} : F_m] \cdot (2g(F_m) - 2) + [F_{m+l} : F_0] \cdot R(m,l)$$

holds for every $l \in \Sigma_m(\mathcal{F})$.

(3) *We obtain*

$$g(F_n) \geq [F_n : F_m] \cdot (g(F_m) - 1) + \frac{[F_n : F_0]}{2} \cdot R(m,l) + 1$$

for every $n \geq m + l$. *In particular*,

$$\frac{2 \cdot N(F_n)}{[F_n : F_m] \cdot (2g(F_m) - 2) + [F_n : F_0] \cdot R(m,l) + 2} \geq \frac{N(F_n)}{g(F_n)}\,.$$

PROOF. (1) This is an immediate consequence of the definitions of $\Sigma_m(\mathcal{F})$ and $R(m,l)$.

(2) The Hurwitz Genus Formula yields

$$2g(F_{m+l}) - 2 = [F_{m+l} : F_m] \cdot (2g(F_m) - 2) + \sum_{i=1}^{l}[F_{m+l} : F_{m+i}] \cdot \deg \mathrm{Diff}(F_{m+i}/F_{m+i-1})\,.$$

Applying (1), we complete the proof of (2).

(3) The Hurwitz Genus Formula yields

$$\begin{aligned} 2g(F_n) - 2 &= [F_n : F_{m+l}] \cdot (2g(F_{m+l}) - 2) + \deg \mathrm{Diff}(F_n/F_{m+l}) \\ &\geq [F_n : F_{m+l}] \cdot (2g(F_{m+l}) - 2)\,. \end{aligned}$$

Then the desired result follows from (2). □

REMARK. (1) If $l_1 < l_2$, then $R(m,l_1) < R(m,l_2)$.

(2) Let m_1 and m_2 be positive integers with $m_1 < m_2$. If $\mathcal{F}/\mathbf{F}_q$ is of degree l, then $R(m_1,l) > R(m_2,l)$.

For a tower $\mathcal{F} = (F_0, F_1, F_2, \cdots)$ and any $m \geq 1$, we have

$$N(F_m) \leq [F_m : F_{m-1}] \cdot N(F_{m-1})\,.$$

Hence we obtain

$$0 \le N(F_m)/[F_m : F_0] \le N(F_{m-1})/[F_{m-1} : F_0] .$$

This shows that the limit $\lim_{n\to\infty} N(F_n)/[F_n : F_0]$ exists.

COROLLARY 3. *With the above notation, we have*

$$\frac{2t}{2g(F_0) + R(m,l) + d_m - 2} \ge \lambda(\mathcal{F}) ,$$

where t and d_m are defined by

$$t := \lim_{n\to\infty} \frac{N(F_n)}{[F_n : F_0]} \quad \textit{and} \quad d_m := \frac{\deg \mathrm{Diff}(F_m/F_0)}{[F_m : F_0]} ,$$

respectively.

REMARK. If $m_1 \le m_2$, then

$$d_{m_2} = d_{m_1} + \frac{\deg \mathrm{Diff}(F_{m_2}/F_{m_1})}{[F_{m_2} : F_0]} .$$

4. Examples

Let $p > 2$ be a prime number, and let

$$ax^2 + bx + c \quad \text{and} \quad \alpha x + \beta$$

be separable polynomials in $\mathbf{F}_p[x]$ with $\alpha x + \beta \nmid ax^2 + bx + c$ and $\alpha \cdot a \ne 0$. We consider the sequence $\mathcal{F}$ of function fields over $\mathbf{F}_{p^2}$ defined recursively by a quadratic equation

$$y^2 = \frac{ax^2 + bx + c}{\alpha x + \beta} .$$

For the sequence $\mathcal{F}$, we get

PROPOSITION 4. *The sequence $\mathcal{F}$ over $\mathbf{F}_{p^2}$ is a tame tower of degree* 2.

PROOF. It is sufficient to prove that

(i) $\mathbf{F}_{p^2}$ is the field of constant of F_m, for all $m \ge 0$;

(ii) F_{m+1}/F_m is a cyclic and tame extension of degree 2, for all $m \ge 0$.

We show the claims (i) and (ii) by induction on m: For the rational function field $F_0 = \mathbf{F}_{p^2}(x_0)$, the assumption (3) in Lemma 1 holds (i.e., $v_{Q_0}(x_0) = -1$). Therefore the place Q_0 is totally ramified in F_1/F_0 (denote the place of F_1 over Q_0 by $Q_1 \in \mathbf{P}_{F_1}$). Then $v_{Q_1}(x_1) = -1$. Hence the field of constant of F_1 is $\mathbf{F}_{p^2}$. It follows from the Kummer Extension Theory that F_1/F_0 is a cyclic and tame extension of degree 2. We now suppose that the claims (i) and (ii) are true for $m-1$, that is, there exists a place $Q_{m-1} \in \mathbf{P}_{F_{m-1}}$ such that $v_{Q_{m-1}}(x_{m-1}) = -1$.

By Lemma 1, we know that $\mathbf{F}_{p^2}$ is the field of constant of F_m. It is easily seen, from the Kummer Extension Theory, that F_m/F_{m-1} is a cyclic and tame extension of degree 2. □

EXAMPLE 2. Consider the tame tower $\mathcal{F}$ over $\mathbf{F}_9$ (resp. over $\mathbf{F}_{25}$) of degree 2 defined recursively by the equation

$$y^2 = \frac{x(x+1)}{x-1} .$$

The tower $\mathcal{F}$ over $\mathbf{F}_9$ (resp. over $\mathbf{F}_{25}$) has the following properties:

(1) $g(F_0) = 0$ and $g(F_1) = 1$ by Proposition 2;

(2) $N(F_m) = 8 \cdot (m+1)$, for all $m \geq 1$ (resp. for all $m \geq 2$);

(3) The rational places P_i of F_1 are totally ramified in F_2/F_1:

- $1/x_0, 1/x_1 \in P_1$;
- $x_0, x_1 \in P_2$;
- $x_0 + 1, x_1 \in P_3$;
- $x_0 - 1, 1/x_1 \in P_4$.

Thus, by Dedekind's Different Theorem, we have

$$\deg \mathrm{Diff}(F_2/F_1) \geq 4 , \quad \text{and} \quad 1 \in \Sigma_1(\mathcal{F}) .$$

Corollary 3 implies the tower $\mathcal{F}$ over $\mathbf{F}_9$ (resp. over $\mathbf{F}_{25}$) is asymptotically bad.

REMARK. (1) It can be shown that

$$g(F_m/\mathbf{F}_9) = 2^{m+2} - 4m - 3 \quad \text{for all } m \geq 1 .$$

By the definition, the Garcia-Stichtenoth number of the tower $\mathcal{F}$ is

$$\lambda(\mathcal{F}/\mathbf{F}_9) = \lim_{m\to\infty} \frac{N(F_m/\mathbf{F}_9)}{g(F_m/\mathbf{F}_9)} = \lim_{m\to\infty} \frac{8 \cdot (m+1)}{2^{m+2} - 4m - 3} = 0 .$$

(2) It follows from Fact [5] of Garcia, Stichtenoth and Thomas, that the tower $\mathcal{F}$ over $\mathbf{F}_{49}$ as in Example 2 is optimal, that is, $\lambda(\mathcal{F}/\mathbf{F}_{49}) = A(49) = 6$.

EXAMPLE 3. The tame tower $\mathcal{F}$ over $\mathbf{F}_9$ (resp. over $\mathbf{F}_{49}$) of degree 2 defined recursively by the equation

$$y^2 = \frac{(x+1)(x-1)}{x}$$

has the following properties:

(1) The genera of F_0 and F_1 are $g(F_0) = 0$ and $g(F_1) = 1$;

(2) for all $m \geq 0$, the number of rational places of $F_m/\mathbf{F}_9$ is

$$N(F_m/\mathbf{F}_9) = \begin{cases} 7 \cdot 2^{m/2+1} - 4 & \text{if } m \equiv 0 \pmod 2 , \\ 5 \cdot 2^{(m+1)/2+1} - 4 & \text{if } m \equiv 1 \pmod 2 \end{cases}$$

(resp. $N(F_m/\mathbf{F}_{49}) = 88$ for $m \geq 3$);

(3) the rational places P_i of F_1 are totally ramified in F_2/F_1:

- $1/x_0, 1/x_1 \in P_1$;
- $x_0, 1/x_1 \in P_2$;
- $x_0 + 1, x_1 \in P_3$;
- $x_0 - 1, x_1 \in P_4$.

Therefore, we have $\deg \mathrm{Diff}(F_2/F_1) \geq 4$ and $1 \in \Sigma_1(\mathcal{F})$.

As a result, the tower $\mathcal{F}$ over $\mathbf{F}_9$ (resp. over $\mathbf{F}_{49}$) is asymptotically bad by Corollary 3.

REMARK. (1) It can be shown that, for all $m \geq 0$

$$g(F_m/\mathbf{F}_9) = \begin{cases} 2^{m+2} - 7 \cdot 2^{m/2} + 3 & \text{if } m \equiv 0 \pmod 2, \\ 2^{m+2} - 5 \cdot 2^{(m+1)/2} + 3 & \text{if } m \equiv 1 \pmod 2. \end{cases}$$

(2) By Fact [5], the tower $\mathcal{F}$ over $\mathbf{F}_{25}$ as in Example 3 is optimal, namely, $\lambda(\mathcal{F}/\mathbf{F}_{25}) = A(25) = 4$.

REMARK. Let $q = 2, 4, 8$. Consider the tower $\mathcal{K} = (K_0, K_1, K_2, \cdots)$ of generalized Klein Quartic function fields over $\mathbf{F}_q$ given by $K_m := \mathbf{F}_q(x_0, \cdots, x_m)$, with

$$x_i \cdot x_{i+1}^3 + x_{i+1} + x_i^3 = 0, \quad \text{for } i = 0, 1, \cdots, m-1.$$

By Corollary 3, the tower $\mathcal{K}/\mathbf{F}_q$ is asymptotically bad (see, [8]).

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