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A Cauchy-Euler Type Factorization of Operators

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Abstract. A Cauchy-Euler type factorization property which is closely related with the Hyers-Ulam stability problem is introduced in the algebra of all linear self maps of a commutative algebra without order. Several examples of linear self maps with such a property are given in this note.

1. Introduction

In 1940, Ulam (cf. [5, 6]) posed the following problem: Let f be an approximate linear map. Does there exist an exact linear map near to f? In the following year, Hyers [2] gave an answer to the problem for additive maps between two Banach spaces. The stability problem of this kind is called "Hyers-Ulam stability problem", and many mathematicians have considered this problem for several functional equations. Alsina and Ger [1] are the first authors who considered Hyers-Ulam stability for certain differential equations. After that, Miura, Miyajima and Takahasi [4] proved the following stability result of this kind:

Let $P(z) = \sum_{k=0}^{n} \lambda_k z^k$ be a polynomial and $D = \frac{d}{dt}$. Then the differential equation $P(D)f = 0$ on $\mathbf{R}$ has the Hyers-Ulam stability if and only if the equation $P(z) = 0$ has no purely imaginary solution.

Recently, Kim and Chung [3] proved the following stability result:

Any Cauchy-Euler differential equation $\sum_{k=0}^{n} \lambda_k t^k f^{(k)}(t) = 0$ has the Hyers-Ulam stability on an arbitrary bounded interval in $\mathbf{R}^+$.

Both of the proofs essentially depend on the fact that the above differential operators can be factorized suitably. The former can be factorized as follows:

$$P(D) = \lambda_n (D - \alpha_1 I) \cdots (D - \alpha_n I)$$

by the Fundamental Theorem of Algebra. The latter can be factorized as follows:

$$\sum_{k=0}^{n} \lambda_k t^k D^k = \lambda_n (tD - \alpha_1 I) \cdots (tD - \alpha_n I)\,.$$

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Here $\alpha_1, \ldots, \alpha_n$ are the roots of the corresponding characteristic equation. Now the following question naturally arises:Which operators can be factorized suitably?

2. A problem

Let A be a complex commutative algebra without order and $L(A)$ the algebra of all linear self maps of A. In this case, we pose the following

PROBLEM. Given $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbf{C}$, $a \in A$, $T \in L(A)$, can one find $\lambda, \alpha_1, \ldots, \alpha_n \in \mathbf{C}$ such that $\sum_{k=0}^{n} \lambda_k a^k T^k = \lambda(aT - \alpha_1 I) \cdots (aT - \alpha_n I)$?

We say that the pair (a, T) has the Cauchy-Euler type factorization property if $\sum_{k=0}^{n} \lambda_k a^k T^k$ is factorized in the above sense for each $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbf{C}$ ($n = 2, 3, \ldots$). We wish to give a characterization for the pair (a, T) to have the Cauchy-Euler type factorization property.

3. The main result

The following result gives a sufficient condition for the pair (a, T) to have the Cauchy-Euler type factorization property.

THEOREM 3.1. *Let n be a positive integer. Let $\beta_1, \beta_2, \ldots, \beta_n \in \mathbf{C}$, $a \in A$ and $T \in L(A)$ be such that*

$$(*) \qquad a^{k+1} T x = aT(a^k x) + (\beta_1 - \beta_{k+1}) a^k x$$

for all $x \in A$ and $1 \le k \le n - 1$. Then

$$\sum_{k=0}^{n} \lambda_k a^k T^k = \lambda_n (\widetilde{T} - \alpha_1 I) \cdots (\widetilde{T} - \alpha_n I)$$

holds for each $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbf{C}$, where $\widetilde{T} = aT + \beta_1 I$ and $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ are the roots of the corresponding characteristic equation $\lambda_0 + \sum_{k=1}^{n} \lambda_k (t - \beta_1) \cdots (t - \beta_k) = 0$.

PROOF. The resulting equation for $n = 1$ is trivial. Suppose that $1 \le k \le n - 1$ and $a^k T^k = (\widetilde{T} - \beta_1 I) \cdots (\widetilde{T} - \beta_k I)$. Then

$$\begin{aligned}
(\widetilde{T} - \beta_1 I) \cdots (\widetilde{T} - \beta_{k+1} I) x &= (\widetilde{T} - \beta_{k+1} I)(\widetilde{T} - \beta_1 I) \cdots (\widetilde{T} - \beta_k I) x \\
&= (\widetilde{T} - \beta_{k+1} I)(a^k T^k x) \\
&= \widetilde{T}(a^k T^k x) - \beta_{k+1} a^k T^k x \\
&= aT(a^k T^k x) + \beta_1 a^k T^k x - \beta_{k+1} a^k T^k x \\
&= aT(a^k T^k x) + (\beta_1 - \beta_{k+1}) a^k T^k x \\
&= a^{k+1} T T^k x \text{ (by } (*))
\end{aligned}$$

$$=a^{k+1}T^{k+1}x$$

for all $x \in A$. Therefore we have $a^kT^k = (\widetilde{T} - \beta_1 I)\cdots(\widetilde{T} - \beta_k I)$ for all $1 \le k \le n$ and then

$$\begin{aligned}\sum_{k=0}^{n} \lambda_k a^k T^k =&\lambda_0 I + \sum_{k=1}^{n} \lambda_k a^k T^k \\ =&\lambda_0 I + \sum_{k=1}^{n} \lambda_k (\widetilde{T} - \beta_1 I)\cdots(\widetilde{T} - \beta_k I) \\ =&\lambda_n (\widetilde{T} - \alpha_1 I)\cdots(\widetilde{T} - \alpha_n I)\end{aligned}$$

which completes the proof. Q. E. D.

4. Examples

(1) Let M be a multiplier on A and $a \in A$. Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ be the roots of the characteristic equation $\sum_{k=0}^{n} \lambda_k t^k = 0$. Then we have

$$\sum_{k=0}^{n} \lambda_k a^k M^k = \lambda_n (aM - \alpha_1 I)\cdots(aM - \alpha_n I)\,.$$

Therefore (a, M) has the Cauchy-Euler type factorization property.

PROOF. Take $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in Theorem 3.1. In this case, it is clear by the definition of multipliers that $(*)$ holds and then the desired result is obtained from Theorem 3.1. Q. E. D.

(2) Suppose that A has an identity element e. Let D be a derivation on A with $D(a) = e$ for some $a \in A$. Let $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ be the roots of the characteristic equation
$\lambda_0 + \sum_{k=1}^{n} \lambda_k t(t-1)\cdots(t-k+1) = 0$. Then

$$\sum_{k=0}^{n} \lambda_k a^k D^k = \lambda_n (aD - \alpha_1 I)\cdots(aD - \alpha_n I).$$

Therefore (a, D) has the Cauchy-Euler type factorization property.

PROOF. Take $\beta_k = k - 1$ $(k = 1, 2, \ldots, n)$ in Theorem 3.1. Since

$$\begin{aligned}aD(a^kx) + (\beta_1 - \beta_{k+1})a^kx =&aD(a^kx) - ka^kx \\ =&a(ka^{k-1}x + a^kDx) - ka^kx \\ =&a^{k+1}Dx\,,\end{aligned}$$

it follows that $(*)$ holds and then the desired result is obtained from Theorem 3.1. Q. E. D.

(3) Let M and D be a multiplier and a derivation on A respectively. Suppose that A has an identity e and that $D(a) = e$ for some $a \in A$. Then $(a, \lambda D + M)$ has the Cauchy-Euler type factorization property for each $\lambda \in \mathbf{C}$.

PROOF. Put $T = \lambda D + M$ and take $\beta_k = \lambda(k-1)$ $(k = 1, 2, \ldots, n)$ in Theorem 3.1. Since

$$\begin{aligned} aT(a^k x) + (\beta_1 - \beta_{k+1})a^k x =& a(\lambda D + M)(a^k x) - \lambda k a^k x \\ =& a(\lambda k a^{k-1} x + \lambda a^k Dx) + a^{k+1} Mx - \lambda k a^k x \\ =& \lambda a^{k+1} Dx + a^{k+1} Mx \\ =& a^{k+1} Tx\,, \end{aligned}$$

it follows that $(*)$ holds and then the desired result is obtained from Theorem 3.1. Q. E. D.

(4) Let H and D be a homomorphism and a derivation on A respectively. Suppose that A has an identity e and that $H(a) = a$ and $D(a) = e$ for some $a \in A$. Then $(a, \lambda D + \mu H)$ has the Cauchy-Euler type factorization property for each $\lambda, \mu \in \mathbf{C}$.

PROOF. Put $T = \lambda D + \mu H$ and take $\beta_k = \lambda(k-1)$ $(k = 1, 2, \ldots, n)$ in Theorem 3.1. Since

$$\begin{aligned} aT(a^k x) + (\beta_1 - \beta_{k+1})a^k x =& a(\lambda D + \mu H)(a^k x) - \lambda k a^k x \\ =& a(\lambda k a^{k-1} x + \lambda a^k Dx) + \mu a H(a^k) Hx - \lambda k a^k x \\ =& \lambda a^{k+1} Dx + \mu a^{k+1} Hx \\ =& a^{k+1} Tx\,, \end{aligned}$$

it follows that $(*)$ holds and then the desired result is obtained from Theorem 3.1. Q. E. D.

(5) Let H and M be a homomorphism and a multiplier on A respectively. Suppose that $H(a) = a$ for some $a \in A$. Then $(a, bH + M)$ has the Cauchy-Euler type factorization property for each $b \in A \cup \mathbf{C}$.

PROOF. Put $T = bH + M$ and take $\beta_1 = \beta_2 = \cdots = \beta_n = 0$ in Theorem 3.1. Since

$$\begin{aligned} aT(a^k x) + (\beta_1 - \beta_{k+1})a^k x =& a(bH + M)(a^k x) \\ =& baH(a^k x) + aM(a^k x) \\ =& baH(a^k)H(x) + a^{k+1} Mx \\ =& a^{k+1} bH(x) + a^{k+1} Mx \\ =& a^{k+1} T(x)\,, \end{aligned}$$

it follows that $(*)$ holds and then the desired result is obtained from Theorem 3.1. Q. E. D.

Here we give an example of (5). Put $(Hf)(t) = f(1-t)$ for each $t \in \mathbf{R}$ and $f \in C^\infty(\mathbf{R})$. Then H is a homomorphism of $C^\infty(\mathbf{R})$ into itself. Put $p(t) = (t - \frac{1}{2})^2$ for each $t \in \mathbf{R}$. Then p is a fixed point of H. Given $g, h \in C^\infty(\mathbf{R})$, define $(T_{g,h} f)(t) = g(t) f(1-t) + h(t) f(t)$ for each $f \in C^\infty(\mathbf{R})$. In this case, $T_{g,h}$ becomes a linear self map of $C^\infty(\mathbf{R})$ and we have the following factorization:

$$\sum_{k=0}^{n} \lambda_k \left(t - \frac{1}{2}\right)^{2k} T_{g,h}^k = \lambda_n \left(\left(t - \frac{1}{2}\right)^2 T_{g,h} - \alpha_1 I\right) \cdots \left(\left(t - \frac{1}{2}\right)^2 T_{g,h} - \alpha_n I\right)$$

for each $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbf{C}$, where $\alpha_1, \ldots, \alpha_n \in \mathbf{C}$ are the roots of the corresponding characteristic equation $\sum_{k=0}^{n} \lambda_k t^k = 0$.

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