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Totally Geodesic Submanifolds in Compact Symmetric Spaces of Rank Two

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Abstract. In 1978 B. Y. Chen and T. Nagano obtained the local classification of the maximal totally geodesic submanifolds in compact connected irreducible symmetric spaces of rank two. In this paper, we investigate their global classification.

1. Introduction

Totally geodesic submanifolds in symmetric spaces are also symmetric spaces and they are the so-called subspaces in the category of symmetric spaces. The classification of the maximal totally geodesic submanifolds in compact symmetric spaces of rank one was obtained by J. A. Wolf in [12]. In [4] B. Y. Chen and T. Nagano obtained the local classification of the maximal totally geodesic submanifolds in compact symmetric spaces of rank two, but table of them is defective. The main idea of their method is to make use of "polars" and "meridians" of compact symmetric spaces. In the present paper we make the global classification of the maximal totally geodesic submanifolds in compact symmetric spaces of rank two by inheriting Chen-Nagano's method. There are some partial results for the case of higher rank. Borel and Siebenthal [1] classified maximal Lie subgroups of maximal rank in compact simple Lie groups and by using this result Ikawa and Tasaki [7] classified the maximal totally geodesic submanifolds of maximal rank in compact symmetric spaces $M = G/K$ with rank M = rank G.

In Section 2, we review the basic concept of symmetric spaces and introduce certain totally geodesic submanifolds in compact symmetric spaces which were defined by Chen-Nagano ([4]), this is to say "polars" and "meridians". In Section 3, we refer to known results of maximal subspaces in compact symmetric spaces. In Section 4, we explain our method which is an extension of Chen-Nagano's method to a global one. In Section 5, by using the method, we give the list of all maximal subspaces in compact symmetric spaces of rank two.

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2. Preliminaries

2.1. Basic concept of symmetric spaces. Let M and N be compact connected irreducible Riemannian symmetric spaces and let $f : M \to N$ be a totally geodesic isometric immersion. Then, M is a totally geodesic submanifold of N locally. It is clear that rank $M \leq$ rank N, where the rank of a compact symmetric space is the dimension of its maximal torus. Let G_M and G_N be the groups of isometries of M and N respectively, then $f : M \to N$ induces the Lie algebra monomorphism $\mathfrak{g}_M \to \mathfrak{g}_N$, where $\mathfrak{g}_M$ and $\mathfrak{g}_N$ are the Lie algebras of G_M and G_N respectively.

We prepare some terminologies and notations. Let U be a compact connected semisimple Lie group and let σ be an involutive automorphism of U. We put

$$U_\sigma = \{u \in U \mid \sigma(u) = u\}$$

and denote the identity component of U_σ by U_σ^0.

If a closed Lie subgroup L of U satisfies $U_\sigma^0 \subset L \subset U_\sigma$, then (U, L) is called a *symmetric pair of compact type*. We can take a σ-invariant and bi-invariant Riemannian metric on U, which naturally induces a U-invariant Riemannian metric on the homogeneous space $N = U/L$, then N is a Riemannian symmetric space. Conversely, any compact semisimple Riemannian symmetric space is constructed in this manner.

We denote by $\mathfrak{u}$ and $\mathfrak{l}$ the Lie algebras of U and L, respectively. The involutive automorphism σ of U induces an involutive automorphism of $\mathfrak{u}$, which we also denote by σ. A σ-invariant and bi-invariant Riemannian metric on U induces a σ-invariant and $\mathrm{Ad}(U)$-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{u}$. The relation $U_\sigma^0 \subset L \subset U_\sigma$ implies

$$\mathfrak{l} = \{X \in \mathfrak{u} \mid \sigma(X) = X\}.$$

If we put

$$\mathfrak{p} = \{X \in \mathfrak{u} \mid \sigma(X) = -X\},$$

we obtain the following orthogonal direct sum decomposition of $\mathfrak{u}$ since σ is isometric and involutive:

$$\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$$

which we call the *canonical decomposition* of $\mathfrak{u}$ with respect to $(\mathfrak{u}, \sigma)$ or *canonical decomposition* of N.

We define a restricted root system of a Riemannian symmetric space.

Let $N = U/L$ be an irreducible compact Riemannian symmetric space and let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of N. Let A be a maximal flat totally geodesic submanifold through the origin o in N, so called a *maximal torus*, and we denote its dimension by $r(N)$. We call $r(N)$ the *rank* of N. Under the identification between the tangent space T_oN of N at o and $\mathfrak{p}$, T_oA is identified with a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$. Now, we identify $\mathfrak{a}$ and $\mathfrak{a}^*$ with respect to the inner product $\langle\ ,\ \rangle$ on $\mathfrak{p}$ induced by the Riemannian metric of N.

DEFINITION 2.1. Let α be a linear form on $\mathfrak{a}$, and let

$$\mathfrak{u}(\alpha) := \{X \in \mathfrak{u} \mid (\operatorname{ad} H)^2 X = -\alpha(H)^2 X \quad \text{for all} \quad H \in \mathfrak{a}\}.$$

A non-zero linear form α is said to be a *restricted root* of N with respected to $\mathfrak{a}$ if $\mathfrak{u}(\alpha) \neq 0$. The set $R(N)$ of restricted roots is called the *restricted root system* of N with respect to $\mathfrak{a}$.

Let N be a Riemannian symmetric space. If we denote by U the identity component of the isometry group of N and by L the isotropy subgroup at some point o in N, then N is a homogeneous space U/L. Let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of the Lie algebra $\mathfrak{u}$ of U and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$.

DEFINITION 2.2 ([6]). We define the subset $\mathfrak{a}_L$ of $\mathfrak{a}$ as follows:

$$\mathfrak{a}_L = \{H \in \mathfrak{a} \mid \exp H \in L\}.$$

$\mathfrak{a}_L$ is called the *unit lattice* in $\mathfrak{a}$.

THEOREM 2.1 ([6]). *Let $N = U/L$ be a compact simply connected irreducible Riemannian symmetric space with rank $N = r$. Let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of the Lie algebra $\mathfrak{u}$ of U and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then the unit lattice $\mathfrak{a}_L$ in $\mathfrak{a}$ is spanned by $\Sigma(N) = \{\alpha_1, \ldots, \alpha_r\}$, where $\Sigma(N)$ is a fundamental root system of N. This is to say*,

$$\mathfrak{a}_L = \{\alpha_1, \ldots, \alpha_r\}_{\mathbf{Z}}.$$

LEMMA 2.1. *Let $M = G/K$ and $N = U/L$ be compact simply connected irreducible Riemannian symmetric spaces with $R(M) = R(N)$. If M is a totally geodesic submanifold of N, then $\mathfrak{a}_K = \mathfrak{a}_L$.*

PROOF. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ and $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decompositions of M and N respectively and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. By the assumption, we can assume that $\mathfrak{m}$ is a subspace of $\mathfrak{p}$ which contains $\mathfrak{a}$. Thus $\mathfrak{a}_K = \mathfrak{a}_L$. □

EXAMPLE 2.1. Let $M = SU(n+1)/SO(n+1)$ and $N = SU(n+1) \times SU(n+1)/\triangle$ where $\triangle$ is the diagonal subgroup of $SU(n+1) \times SU(n+1)$. Then M is a totally geodesic submanifold of N with $R(M) = R(N)$. In this case

$$\mathfrak{a}_K = \left\{ \sum_{i=1}^{n+1} x_i \varepsilon_i \,\middle|\, \sum_{i=1}^{n+1} x_i = 0, x_i \in \mathbf{Z}, (1 \leq i \leq n+1) \right\},$$

$$\mathfrak{a}_L = \left\{ 2\sum_{i=1}^{n+1} x_i \varepsilon_i \,\middle|\, \sum_{i=1}^{n+1} x_i = 0, x_i \in \mathbf{Z}, (1 \leq i \leq n+1) \right\},$$

where $\{\varepsilon_i | 1 \leq i \leq n+1\}$ denotes the standard orthonormal basis of $\mathbf{R}^{n+1}$.

2.2. Certain totally geodesic submanifolds in compact symmetric spaces. We introduce a polar and the meridian in a compact symmetric space which were defined by Chen-Nagano.

DEFINITION 2.3 ([4]). Let o be a point of a symmetric space N. Then we call a connected component of the fixed point set of s_o, the symmetry at o, in N a *polar* of o. We denote it by N^+ or $N^+(p)$ if N^+ contains a point p. We also call a connected component of the fixed point set of $s_p \circ s_o$ in N through p the *meridian* of $N^+(p)$ in N and denote it by $N^-(p)$ or simply by N^-. When a polar consists of a single point, we call it a *pole*.

REMARK 2.1. Polars and meridians are totally geodesic submanifolds in N; they are thus symmetric spaces. And they were determined for every compact connected irreducible Riemannian symmetric space ([4], [9] and [10]). One of the most important properties of these totally geodesic submanifolds is that N is determined by any pair of $(N^+(p), N^-(p))$ completely ([10]). We note that N^- has the same rank as N.

DEFINITION 2.4 ([5]). Let N be a compact connected Riemannian symmetric space and o be a point in N. And we suppose that there is a pole p of o in N. Then we call the set consisting of the midpoints of the geodesic segments from o to p the *centrosome* and denote it by $C(o, p)$ or simply by C. Here, each connected component of the centrosome is a totally geodesic submanifold of N.

DEFINITION 2.5. Let M be a totally geodesic submanifold of N and let p be a point in M. We denote by $T_p^\perp M$ the orthogonal complement of T_pM in T_pN. If there is a totally geodesic submanifold $M^\perp$ of N through p whose tangent space at p coincides with $T_p^\perp M$, then $M^\perp$ is called the *orthogonal complement* to M in N at p.

REMARK 2.2. A polar $N^+(p)$ and the meridian $N^-(p)$ are the orthogonal complements to each other in N at p.

DEFINITION 2.6 ([8]). Let N be a Riemannian manifold and let M be a submanifold in N. M is a *reflective submanifold* if M is a connected component of the fixed-point set of some involutive isometry of N.

REMARK 2.3. Any reflective submanifold is a totally geodesic submanifold. In addition, if N is Riemannian symmetric space, then a reflective submanifold M in N is a Riemannian symmetric space.

PROPOSITION 2.2 ([8]). *Let M be a submanifold of a Riemannian symmetric space N, then M is a reflective submanifold if and only if M and $M^\perp$ are totally geodesic submanifolds.*

Next, we give a necessary condition for that a totally geodesic submanifold in a Hermitian symmetric space is totally real.

LEMMA 2.2 ([4]). *Let $N = U/L$ be a compact Hermitian symmetric space and M be a totally geodesic submanifold of N. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $M = G/K$. Then M is a totally real submanifold if and only if* $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{l}'$, *namely,* $\langle [\mathfrak{m}, \mathfrak{m}], \mathfrak{so}(2) \rangle = 0$, *where the Lie algebra $\mathfrak{l}$ of L is $\mathfrak{l} = \mathfrak{so}(2) + \mathfrak{l}'$.*

THEOREM 2.3 ([4]). *Let N be a Hermitian symmetric space and M an irreducible non-Hermitian symmetric space. If M is a totally geodesic submanifold in N, then M is totally real in N. In particular* $\dim M \leq (1/2) \dim N$.

2.3. Maximality of totally geodesic submanifolds in compact symmetric spaces. In the subsection we refer to some known results of maximal totally geodesic submanifolds in compact symmetric spaces. In our classification we consider only the case where the ambient symmetric space is simply connected because of Lemma 2.4.

DEFINITION 2.7 ([9]). Let M and N be Riemannian symmetric spaces. A smooth map $f : M \to N$ is called a *morphism* if f satisfies $f \circ s_p = s_{f(p)} \circ f$ for any $p \in M$.

LEMMA 2.3 ([11]). *Let M and N be Riemannian symmetric spaces. Then a morphism $f : M \to N$ satisfies the following conditions*:

(1) *The image $f(M)$ is a totally geodesic submanifold of N.*

(2) *For any $q \in f(M)$, $f^{-1}(q)$ is a totally geodesic submanifold of M.*

(3) *$f : M \to f(M)$ is a submersion.*

LEMMA 2.4. *Let $\widetilde{N}$ and $\widetilde{M}$ be compact connected Riemannian symmetric spaces and let $f : \widetilde{N} \to N$ be a covering morphism. If $\widetilde{M}$ is a maximal totally geodesic submanifold of $\widetilde{N}$, then $f(\widetilde{M})$ is a maximal totally geodesic submanifold of N. And if M is a maximal totally geodesic submanifold of N, then each connected component of $f^{-1}(M)$ is a maximal totally geodesic submanifold of $\widetilde{N}$.*

PROOF. If $f(\widetilde{M})$ is not maximal, there exists a totally geodesic submanifold X of N such that $f(\widetilde{M}) \subsetneqq X \subsetneqq N$. Then each connected component of $f^{-1}(X)$ is a totally geodesic submanifold of $\widetilde{N}$ by Lemma 2.3 (2). Hence it contradicts the maximality of $\widetilde{M}$ which is contained in $f^{-1}(X)$.

If $f^{-1}(M)$ is not maximal, there exists a totally geodesic submanifold $\widetilde{X}$ of $\widetilde{N}$ such that $f^{-1}(M) \subsetneqq \widetilde{X} \subsetneqq \widetilde{N}$. Then $f(\widetilde{X})$ is a totally geodesic submanifold of N by Lemma 2.3 (1). Hence it contradicts the maximality of M which is contained in $f(\widetilde{X})$. □

LEMMA 2.5. *Let $N = U/L$ be a compact simply connected irreducible symmetric space. Then L is a maximal connected Lie subgroup of U.*

PROOF. Since N is simply connected, L is connected. If L is not maximal, then there is a connected Lie subgroup H of U such that $L \subsetneqq H \subsetneqq U$. Let $\mathfrak{h}$, $\mathfrak{l}$ and $\mathfrak{u}$ be the Lie algebras of H, L and U respectively. Then we have:

$$\mathfrak{l} \subsetneqq \mathfrak{h} \subsetneqq \mathfrak{u} .$$

Let B be the killing form of $\mathfrak{u}$, then we obtain the orthogonal decomposition of $\mathfrak{u}$ with respect to B:

$$\mathfrak{u} = \mathfrak{h} + \mathfrak{h}^{\perp} .$$

And we also obtain the orthogonal decomposition of $\mathfrak{u}$ with respect to $B|_{\mathfrak{h}\times\mathfrak{h}}$:

$$\mathfrak{h} = \mathfrak{l} + \mathfrak{l}^{\perp} .$$

The fact $\mathfrak{h} \neq \mathfrak{l}$ follows $\mathfrak{l}^{\perp} \neq \{0\}$. Let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition, then $\mathfrak{p} = \mathfrak{h}^{\perp} + \mathfrak{l}^{\perp}$. Also, we obtain $[\mathfrak{l}, \mathfrak{l}^{\perp}] \subset \mathfrak{h} \cap \mathfrak{p} = \mathfrak{l}^{\perp}$. By the assumption that $\mathfrak{l}$ acts on $\mathfrak{p}$ irreducibly, this is a contradiction. □

3. Known results

3.1. Maximal totally geodesic submanifolds of maximal rank in compact symmetric spaces associated with normal real form. In this subsection we refer to the result in [7] which makes use of the result in [1].

DEFINITION 3.1. A compact symmetric space $N = U/L$ is associated with *normal real form* if $r(N) = r(U)$.

THEOREM 3.1 ([7]). *Let $N = U/L$ be a compact symmetric space associated with normal real form, and U' be a maximal Lie subgroup of maximal rank in U. Then $U'/U' \cap L$ is a maximal totally geodesic submanifold of maximal rank in N. Conversely, any maximal totally geodesic submanifold of maximal rank in N is obtained in this manner. The totally geodesic submanifold $U'/U' \cap L$ mentioned above is also a compact symmetric space associated with normal real form or locally isomorphic to the product of a compact symmetric space associated with normal real form and S^1.*

THEOREM 3.2 ([7]). *A necessary and sufficient condition that a totally geodesic submanifold U in a compact connected simple Lie group is maximal is that U is a Cartan embedding or a maximal Lie subgroup.*

COROLLARY 3.3 ([7]). *A necessary and sufficient condition that a totally geodesic submanifold U of maximal rank in a compact connected simple Lie group is maximal is that U is a Cartan embedding of a compact symmetric space corresponding to a normal real form or a maximal Lie subgroup of maximal rank.*

3.2. Maximal totally geodesic submanifolds in compact symmetric spaces of rank one. There is the classification of totally geodesic submanifolds in compact symmetric spaces of rank one, which was given by J. A. Wolf.

THEOREM 3.4 ([12]). *Let N be a compact symmetric space of rank one. If M is a totally geodesic submanifold of N, then M is one of the followings*:

(1) $N = S^n$, $M = S^r (1 \leq r \leq n)$

(2) $N = \mathbf{R}P^n$, $M = \mathbf{R}P^r (1 \leq r \leq n)$
(3) $N = \mathbf{C}P^n$, $M = \mathbf{R}P^r, \mathbf{C}P^r (1 \leq r \leq n)$
(4) $N = \mathbf{H}P^n$, $M = \mathbf{R}P^r, \mathbf{C}P^r, \mathbf{H}P^r, (1 \leq r \leq n)$
(5) $N = \mathbf{O}P^2$, $M = \mathbf{R}P^r, \mathbf{C}P^r, \mathbf{H}P^r (1 \leq r \leq 2), \mathbf{O}P^1$

REMARK 3.1. Here, we note that $\mathbf{R}P^1 = S^1$, $\mathbf{C}P^1 = S^2$, $\mathbf{H}P^1 = S^4$ and $\mathbf{O}P^1 = S^8$.

We give the following Table 1 by [4].

TABLE 1. Polars and the corresponding meridians in compact symmetric spaces of rank one

M	M^+	M^-
S^n	{a pole}	S^n
$\mathbf{R}P^n$	$\mathbf{R}P^{n-1}$	S^1
$\mathbf{C}P^n$	$\mathbf{C}P^{n-1}$	S^2
$\mathbf{H}P^n$	$\mathbf{H}P^{n-1}$	S^4
$\mathbf{O}P^2$	S^8	S^8

4. A global extension of Chen-Nagano's method

In this section, firstly we introduce Chen-Nagano's method, then we extend this method to a global one.

PROPOSITION 4.1 ([4]). *Let M and N be compact irreducible symmetric spaces. If M^+ and N^+ are polars of M and N respectively and M^- and N^- are the corresponding meridians respectively, then the pairs of a polar and the corresponding meridian of $M \times N$ are $(N^+, M \times N^-)$, $(M^+, M^- \times N)$ and $(M^+ \times N^+, M^- \times N^-)$.*

By Proposition 4.1, we have Table 2 for products of compact symmetric spaces of rank one.

LEMMA 4.1. *Let $M \cdot N$ denote $\{M \times N\}/\mathbf{Z}_2$. Then the pairs of a polar and the corresponding meridian $\big((M \cdot N)^+, (M \cdot N)^-\big)$ are $(M^+ \cdot N^+, M^- \cdot N^-)$ and $(C_M(o_M, p_M) \cdot C_N(o_N, p_N), C_M^\perp(o_M, p_M) \cdot C_N^\perp(o_N, p_N))$, where o_M and o_N are origins of M and N and p_M and p_N are poles of o_M and o_N, respectively.*

PROOF. It follows Theorem 4.1 immediately. □

COROLLARY 4.2. *The pairs of a polar and the corresponding meridian of $S^n \cdot S^m$ are* ({the pole}, $S^n \cdot S^m$) *and* $(S^{n-1} \cdot S^{m-1}, S^1 \cdot S^1)$.

The following theorem is very useful for the classification.

THEOREM 4.3 ([4]). *Let o_N and o_M be the origins of N and M respectively. Let c_N (resp. c_M) be a closed geodesic in N (resp. M) and let p_N (resp. p_M) be the antipodal point*

TABLE 2. Polars and the corresponding meridians in products of compact symmetric spaces of rank one

M	M^+	M^-	Remark
$S^n \times S^m$	$\{(p, p')\}$ $\{(o, p')\}$ $\{(p, o')\}$	$S^n \times S^m$ $S^n \times S^m$ $S^n \times S^m$	A point o (resp. o') is a origin of S^n (resp. S^m). A point p is a pole of o. A point p' is a pole of o'.
$S^n \times \mathbf{R}P^m$	$\mathbf{R}P^{m-1} \times \{o\}$ $\{(p, o')\}$ $\mathbf{R}P^{m-1} \times \{p\}$	$S^n \times S^1$ $S^n \times \mathbf{R}P^m$ $S^n \times S^1$	A point o (resp. o') is a origin of S^n (resp. $\mathbf{R}P^m$). A point p is a pole of o.
$S^n \times \mathbf{C}P^m$	$\mathbf{C}P^{m-1} \times \{o\}$ $\{(p, o')\}$ $\mathbf{C}P^{m-1} \times \{p\}$	$S^n \times S^2$ $S^n \times \mathbf{C}P^m$ $S^n \times S^2$	A point o (resp. o') is a origin of S^n (resp. $\mathbf{C}P^m$). A point p is a pole of o.
$S^n \times \mathbf{H}P^m$	$\mathbf{H}P^{m-1} \times \{o\}$ $\{(p, o')\}$ $\mathbf{H}P^{m-1} \times \{p\}$	$S^n \times S^4$ $S^n \times \mathbf{H}P^m$ $S^n \times S^4$	A point o (resp. o') is a origin of S^n (resp. $\mathbf{H}P^m$). A point p is a pole of o.
$S^n \times \mathbf{O}P^2$	$S^8 \times \{o\}$ $\{(p, o')\}$ $S^8 \times \{p\}$	$S^n \times S^8$ $S^n \times \mathbf{O}P^2$ $S^n \times S^8$	A point o (resp. o') is a origin of S^n (resp. $\mathbf{O}P^2$). A point p is a pole of o.
$\mathbf{R}P^n \times \mathbf{R}P^m$	$\mathbf{R}P^{m-1} \times \{o\}$ $\mathbf{R}P^{n-1} \times \{o'\}$ $\mathbf{R}P^{n-1} \times \mathbf{R}P^{m-1}$	$\mathbf{R}P^n \times S^1$ $S^1 \times \mathbf{R}P^m$ $S^1 \times S^1$	A point o is a origin of $\mathbf{R}P^n$. A point o' is a origin of $\mathbf{R}P^m$.
$\mathbf{R}P^n \times \mathbf{C}P^m$	$\mathbf{C}P^{m-1} \times \{o\}$ $\mathbf{R}P^{n-1} \times \{o'\}$ $\mathbf{R}P^{n-1} \times \mathbf{C}P^{m-1}$	$\mathbf{R}P^n \times S^2$ $S^1 \times \mathbf{C}P^m$ $S^1 \times S^2$	A point o is a origin of $\mathbf{R}P^n$. A point o' is a origin of $\mathbf{C}P^m$.
$\mathbf{R}P^n \times \mathbf{H}P^m$	$\mathbf{H}P^{m-1} \times \{o\}$ $\mathbf{R}P^{n-1} \times \{o'\}$ $\mathbf{R}P^{n-1} \times \mathbf{H}P^{m-1}$	$\mathbf{R}P^n \times S^4$ $S^1 \times \mathbf{H}P^m$ $S^1 \times S^4$	A point o is a origin of $\mathbf{R}P^n$. A point o' is a origin of $\mathbf{H}P^m$.
$\mathbf{R}P^n \times \mathbf{O}P^2$	$S^8 \times \{o\}$ $\mathbf{R}P^{n-1} \times \{o'\}$ $\mathbf{R}P^{n-1} \times S^8$	$\mathbf{R}P^n \times S^8$ $S^1 \times \mathbf{O}P^2$ $S^1 \times S^8$	A point o is a origin of $\mathbf{R}P^n$. A point o' is a origin of $\mathbf{O}P^2$.
$\mathbf{C}P^n \times \mathbf{C}P^m$	$\mathbf{C}P^{m-1} \times \{o\}$ $\mathbf{C}P^{n-1} \times \{o'\}$ $\mathbf{C}P^{n-1} \times \mathbf{C}P^{m-1}$	$\mathbf{C}P^n \times S^2$ $S^2 \times \mathbf{C}P^m$ $S^2 \times S^2$	A point o is a origin of $\mathbf{C}P^n$. A point o' is a origin of $\mathbf{C}P^m$.
$\mathbf{C}P^n \times \mathbf{H}P^m$	$\mathbf{H}P^{m-1} \times \{o\}$ $\mathbf{C}P^{n-1} \times \{o'\}$ $\mathbf{C}P^{n-1} \times \mathbf{H}P^{m-1}$	$\mathbf{C}P^n \times S^4$ $S^2 \times \mathbf{H}P^m$ $S^2 \times S^4$	A point o is a origin of $\mathbf{C}P^n$. A point o' is a origin of $\mathbf{H}P^m$.
$\mathbf{C}P^n \times \mathbf{O}P^2$	$S^8 \times \{o\}$ $\mathbf{C}P^{n-1} \times \{o'\}$ $\mathbf{C}P^{n-1} \times S^8$	$\mathbf{C}P^n \times S^8$ $S^2 \times \mathbf{O}P^2$ $S^2 \times S^8$	A point o is a origin of $\mathbf{C}P^n$. A point o' is a origin of $\mathbf{O}P^2$.
$\mathbf{H}P^n \times \mathbf{H}P^m$	$\mathbf{H}P^{m-1} \times \{o\}$ $\mathbf{H}P^{n-1} \times \{o'\}$ $\mathbf{H}P^{n-1} \times \mathbf{H}P^{m-1}$	$\mathbf{H}P^n \times S^4$ $S^4 \times \mathbf{H}P^m$ $S^4 \times S^4$	A point o is a origin of $\mathbf{H}P^n$. A point o' is a origin of $\mathbf{H}P^m$.
$\mathbf{H}P^n \times \mathbf{O}P^2$	$S^8 \times \{o\}$ $\mathbf{H}P^{n-1} \times \{o'\}$ $\mathbf{H}P^{n-1} \times S^8$	$\mathbf{H}P^n \times S^8$ $S^4 \times \mathbf{O}P^2$ $S^4 \times S^8$	A point o is a origin of $\mathbf{H}P^n$. A point o' is a origin of $\mathbf{O}P^2$.
$\mathbf{O}P^2 \times \mathbf{O}P^2$	$S^8 \times \{o\}$ $S^8 \times \{o'\}$ $S^8 \times S^8$	$\mathbf{O}P^2 \times S^8$ $S^8 \times \mathbf{O}P^2$ $S^8 \times S^8$	A point o is a origin of $\mathbf{O}P^2$. A point o' is a origin of $\mathbf{O}P^2$.

of o_N (resp. o_M) in c_N (resp. c_M). Let $f : M \to N$ be a totally geodesic immersion such that $f(o_M) = o_N$ (so $f(p_M) = p_N$). Then f induces the following totally geodesic immersions:

$$f^+ : M^+(p_M) \to N^+(p_N)$$

$$f^- : M^-(p_M) \to N^-(p_N)\,.$$

Theorem 4.3 implies that a necessary condition for that M is a totally geodesic submanifold in N.

THEOREM 4.4 ([4]). *Let M and N be compact Riemannian symmetric spaces with $r(M) = r(N)$ and let $f : M \to N$ be a totally geodesic imbedding. We denote by $P(M)$ and $P(N)$ the sets of pairs of a polar and the corresponding meridian of M and N respectively. Then $f^{\pm}$ induced by f give rise to a mapping $P(f) : P(M) \to P(N)$ and $P(f)$ is a surjection.*

COROLLARY 4.5. *Let $N = U/L$ be a compact irreducible symmetric space of rank two with a pole and let N^+ be a polar which is not a pole. If $f : S^n \cdot S^m \to N$ is a totally geodesic imbedding, then $f_i^{\pm}$ $(i = 1, 2)$ induced by f give rise to totally geodesic imbeddings*:

$$f_1^{\pm} : (\{\text{the pole}\}, S^n \cdot S^m) \to (\{\text{a pole}\}, N)$$

$$f_2^{\pm} : (S^{n-1} \cdot S^{m-1}, S^1 \cdot S^1) \to (N^+, N^-)\,.$$

PROOF. It follows Theorem 4.4 immediately. □

LEMMA 4.2. *Let $N = U/L$ be a compact irreducible symmetric space whose rank is greater than two and N has no pole. If $f : S^n \cdot S^m \to N$ $(2 \le n, 0 \le m)$ is a totally geodesic imbedding, then $f_i^{\pm}$ $(i = 1, 2)$ are the following*:

$$f_1^{\pm} : (\{\text{the pole}\}, S^n \cdot S^m) \to (N_i^+, N_i^-)$$

$$f_2^{\pm} : (S^{n-1} \cdot S^{m-1}, S^1 \cdot S^1) \to (N_j^+, N_j^-)\,,$$

where (N_i^+, N_i^-) and (N_j^+, N_j^-) are pairs of a polar and the meridian in N. In particular, $S^n \cdot S^m$ is not maximal in N.

PROOF. It follows Theorem 4.4 immediately. □

LEMMA 4.3. *Let o_M and o_N be the origins of M and N respectively. Let p_M and p_N their poles be respectively. We assume $r(M) = r(N)$. If a totally geodesic imbedding $f : M \to N$ satisfies $f(o_M) = o_N$ and $f(p_M) = p_N$, then f induces totally geodesic imbedding f_c and $f_c^{\perp}$* :

$$f_c : C_M(o_M, p_M) \to C_N(o_N, p_N)$$

$$f_c^{\perp} : C_M^{\perp}(o_M, p_M) \to C_N^{\perp}(o_N, p_N),$$

where $C_M^{\perp}(o_M, p_M)$ and $C_N^{\perp}(o_N, p_N)$ denote the orthogonal complements to $C_M(o_M, p_M)$ in M and that to $C_N(o_N, p_N)$ in N respectively.

PROOF. It follows Theorem 4.3 and Theorem 4.4 immediately. □

PROPOSITION 4.6. *Let $N = S^p \times S^q (1 \leq p, 2 \leq q)$. If M is a maximal totally geodesic submanifold of N, then M is isomorphic to one of $S^p \times S^{q-1}$, $S^{p-1} \times S^q$ and $\triangle S^p (p = q)$, where $\triangle S^p$ denotes the diagonal of $S^p \times S^p$.*

PROOF. Clearly, $S^p \times S^{q-1}$ and $S^{p-1} \times S^q$ are maximal totally geodesic submanifolds of $S^p \times S^q$. When $p = q$, we assume that there is a compact symmetric space N satisfies,

$$\triangle S^p \subset N \subset S^p \times S^p .$$

Now, by $r(S^p) = 1$ and Theorem 4.3, N is of rank one and has a pole. Hence $N = S^m$ for some natural number $m \geq p$. Since $S^m \to S^p \times S^p$ is a totally geodesic imbedding, by Theorem 4.3 the pole of S^m coincides with the furthest pole of $S^p \times S^p$. Hence the centrosome S^{m-1} of S^m is a totally geodesic submanifold of centrosome $S^{p-1} \times S^{p-1}$ of $S^p \times S^p$. Thus we have the totally geodesic imbedding $S^{m-1} \to S^{p-1} \times S^{p-1}$. By the similar discussion we obtain $S^{m-p+1} \subset S^1 \times S^1$. Since the pole of S^{m-p+1} is the furthest pole of $S^1 \times S^1$, S^{m-p+1} is isomorphic to $\triangle S^1$. Namely $m = p$. □

COROLLARY 4.7. *Let $N = S^p \cdot S^q (1 \leq p, 2 \leq q)$. If M is a maximal totally geodesic submanifold in N, then M is isomorphic to one of $S^p \cdot S^{q-1}$, $S^{p-1} \cdot S^q$ and $\mathbf{R}P^p (p = q)$.*

We obtain another proof of Theorem 4.8 by using Proposition 4.6 and Corollary 4.7.

THEOREM 4.8 ([3]). *Any maximal totally geodesic submanifold of $G_2^o(\mathbf{R}^{n+2}) (n \geq 3)$ is isomorphic to one of $G_2^o(\mathbf{R}^{n+1})$, $\mathbf{C}P^{[\frac{n}{2}]}$ and $S^p \cdot S^q (p + q = n)$.*

We have Table 3 which gives the list all polars and the corresponding meridians in compact irreducible symmetric spaces of rank two by [9].

5. Maximal subspaces in compact symmetric spaces of rank two

PROPOSITION 5.1. *Any maximal totally geodesic submanifold M of $AI(3)$ is isomorphic to $\mathbf{R}P^2$ or $S^1 \cdot S^2$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

There is no compact symmetric space of rank two whose dimension is less than $5 = \dim AI(5)$.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

By Theorem 4.3 a necessary condition for that $S^n \times S^m$ is a totally geodesic submanifold of $AI(3)$ is $m = n = 1$. Now, $S^1 \times S^1$ is isomorphic to $S^1 \cdot S^1$. Since $S^1 \cdot S^1$ is a totally

TABLE 3. Polars and the corresponding meridians in compact irreducible symmetric spaces of rank two

M	M^+	M^-	dim M
$AI(3)$	$\mathbf{R}P^2$	$T \cdot S^2$	5
$AI(3)/\mathbf{Z}_3$	$\mathbf{R}P^2$	$T/\mathbf{Z}_3 \cdot S^2$	5
$AII(3)$	$\mathbf{H}P^2$	$T \cdot S^5$	14
$AII(3)/\mathbf{Z}_3$	$\mathbf{H}P^2$	$T/\mathbf{Z}_3 \cdot S^5$	14
$SU(3)$	$\mathbf{C}P^2$	$T \cdot S^3$	8
$SU(3)/\mathbf{Z}_3$	$\mathbf{C}P^2$	$T/\mathbf{Z}_3 \cdot S^3$	8
EIV	$\mathbf{O}P^2$	$T \cdot S^9$	26
$EIV/\mathbf{Z}_3$	$\mathbf{O}P^2$	$T/\mathbf{Z}_3 \cdot S^9$	26
$G_2^o(\mathbf{R}^{n+2})$ $(2 \leq n)$	{ a pole } $G_2^o(\mathbf{R}^n)$	$G_2^o(\mathbf{R}^{n+2})$ $G_2^o(\mathbf{R}^4)$	$2n$
$G_2(\mathbf{R}^{n+2})$ $(2 \leq n)$	$G_2(\mathbf{R}^n)$ $S^1 \times \mathbf{R}P^{n-1}$	$G_2(\mathbf{R}^4)$ $S^1 \times \mathbf{R}P^{n-1}$	$2n$
$Sp(2)$	{ a pole } S^4	$Sp(2)$ $Sp(1) \times Sp(1)$	10
$Sp(2)/\mathbf{Z}_2$	$\mathbf{R}P^4$ $G_2(\mathbf{R}^5)$	$Sp(1) \cdot Sp(1)$ $S^1 \times \mathbf{R}P^3$	10
$G_2(\mathbf{C}^{n+2})$ $(2 \leq n)$	$G_2(\mathbf{C}^n)$ $S^2 \times \mathbf{C}P^{n-1}$	$G_2(\mathbf{C}^4)$ $S^2 \times \mathbf{C}P^{n-1}$	$4n$
$G_2(\mathbf{H}^{n+2})$ $(3 \leq n)$	$G_2(\mathbf{H}^n)$ $S^4 \times \mathbf{H}P^{n-1}$	$G_2(\mathbf{H}^4)$ $S^4 \times \mathbf{H}P^{n-1}$	$8n$
$G_2(\mathbf{H}^4)$	{ a pole } $S^4 \times S^4$	$G_2(\mathbf{H}^4)$ $S^4 \times S^4$	16
$G_2(\mathbf{H}^4)/\mathbf{Z}_2$	$Sp(2)/\mathbf{Z}_2$ $S^4 \cdot S^4$	$S^1 \times \mathbf{R}P^5$ $S^4 \cdot S^4$	16
$DIII(5)$	$G_2(\mathbf{C}^5)$ $\mathbf{C}P^4$	$S^2 \times \mathbf{C}P^3$ $G_2^o(\mathbf{R}^8)$	20
GI	$S^2 \cdot S^2$	$S^2 \cdot S^2$	8
G_2	GI	$SO(4)$	14
$EIII$	$G_2^o(\mathbf{R}^{10})$ $DIII(5)$	$G_2^o(\mathbf{R}^{10})$ $S^2 \times \mathbf{C}P^5$	32

geodesic submanifold of $S^1 \cdot S^2$, which is the meridian of $AI(3)$, $S^1 \times S^1$ is not maximal. For the cases of $M = \mathbf{R}P^n \times \mathbf{R}P^m (m, n \geq 2)$ and $M = \mathbf{R}P^n \times S^m$ $(n \geq 2, m \geq 1)$, we obtain the conclusion that there is no totally geodesic submanifold of $AI(3)$, because of the dimensions and Theorem 4.3.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

By Theorem 4.3 a necessary condition for that $S^n \cdot S^m$ is a totally geodesic submanifold of $AI(3)$ is $n = 1$ and $1 \leq m \leq 2$. Then $S^1 \cdot S^2$ is isomorphic to the meridian of $AI(3)$. $S^1 \cdot S^2$ is maximal. since in the remaining case, case 4, $r(M) = 1$.

Case 4. M is a compact symmetric space of rank one.

By the comparison of dimension, $\mathbf{O}P^2$ is not a totally geodesic submanifold of $AI(3)$. By Theorem 4.3 a necessary condition for that S^n is a totally geodesic submanifold of $AI(3)$ is $n = 2$. Then S^2 is a totally geodesic submanifold of $S^1 \cdot S^2$. Thus S^2 is not maximal. From Table 1 a polar of $\mathbf{C}P^n$ is $\mathbf{C}P^{n-1}$. When $n \geq 2$, $\mathbf{C}P^{n-1}$ is not a totally geodesic submanifold of $\mathbf{R}P^2$. Thus $\mathbf{C}P^n$ is not in $AI(3)$. Since $\mathbf{C}P^n$ is a totally geodesic submanifold of $\mathbf{H}P^n$, $\mathbf{H}P^n$ is not in $AI(3)$. By the above discussion, when $n \geq 4$, $\mathbf{R}P^n$ is not a totally geodesic submanifold of $AI(3)$. In the case of $\mathbf{R}P^3$, the isometry group of $\mathbf{R}P^3$ is $SO(4)$ and if $\mathbf{R}P^3$ is a totally geodesic submanifold of $AI(3)$, then $SO(4)$ is a Lie subgroup of $SU(3)$ locally which is an isometry group of $AI(3)$. This is a contradiction since a maximal Lie subgroup of $SU(3)$ is isomorphic to $T \cdot SU(2)$. Hence $\mathbf{R}P^3$ is not in $AI(3)$. $\mathbf{R}P^2$ is isomorphic to a polar of $AI(3)$. $\mathbf{R}P^2$ is maximal, since $\mathbf{R}P^2$ is not contained in $S^1 \cdot S^2$ by Corollary 4.7. □

For the other cases we can argue in a similar fashion. So we only refer to the following three steps:

(i) Pick up the possible totally geodesic submanifolds of N by Theorem 4.3

(ii) Investigate whether each totally geodesic submanifold in (i) is really totally geodesic submanifold or not.

(iii) Investigate whether each totally geodesic submanifold in (ii) is maximal or not.

PROPOSITION 5.2. *Any maximal totally geodesic submanifold M of $SU(3)$ is isomorphic to one of $AI(3)$, $\mathbf{C}P^2$, $\mathbf{R}P^3$ and $S^1 \cdot S^3$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to $AI(3)$.

(ii) From [5] $AI(3)$ is a totally geodesic submanifold of $SU(3)$.

(iii) $AI(3)$ is maximal in $SU(3)$, because there is no other irreducible compact symmetric space of rank two in $SU(3)$.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one

(i) There is no such M in $SU(3)$.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^1 \cdot S^3$.

(ii) From Table 3 $S^1 \cdot S^3$ is isomorphic to the meridian $T \cdot S^3$ of $SU(3)$.

(iii) From [1] $T \cdot S^3 \cong T \cdot SU(2)$ is a maximal Lie subgroup of $SU(3)$. Thus $S^1 \cdot S^3$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of S^3, $\mathbf{R}P^3$ and $\mathbf{C}P^n (2 \leq n \leq 3)$.

(ii) Since S^3 is a totally geodesic submanifold of $S^1 \cdot S^3$ which is the meridian of $SU(3)$. In the case of $\mathbf{R}P^3$, $\mathbf{R}P^3$ is isomorphic to $SO(3)$ and since $SO(3)$ is a Lie subgroup of $SU(3)$, $\mathbf{R}P^3$ is a totally geodesic submanifold of $SU(3)$. In the case of $\mathbf{C}P^3$, a isometry group of $\mathbf{C}P^3$ is $SU(4)$. $SU(4)$ is not a totally geodesic submanifold of $SU(3) \times SU(3)$ which is the isometry group of $SU(3)$. Thus $\mathbf{C}P^3$ is not in $SU(3)$. From Table 3 $\mathbf{C}P^2$ is isomorphic to the meridian of $SU(3)$.

(iii) By (ii) S^3 is not maximal. By Proposition 5.1 $\mathbf{C}P^2$ and $\mathbf{R}P^3$ are not totally geodesic submanifold of $AI(3)$. Also, by Corollary 4.7 $\mathbf{C}P^2$ and $\mathbf{R}P^3$ are not in $S^1 \cdot S^3$. Thus $\mathbf{C}P^2$ and $\mathbf{R}P^3$ are maximal. □

PROPOSITION 5.3. *Any maximal totally geodesic submanifold M of $AII(3)$ is isomorphic to one of $SU(3)$, $\mathbf{C}P^3$, $\mathbf{H}P^2$ and $S^1 \cdot S^5$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $AI(3)$, $AI(3)/\mathbf{Z}_3$, $SU(3)$ and $SU(3)/\mathbf{Z}_3$.

(ii) By Theorem 2.1 $AI(3)/\mathbf{Z}_3$ and $SU(3)/\mathbf{Z}_3$ are not totally geodesic submanifolds of $AII(3)$. From [5] $SU(3)$ is a totally geodesic submanifold of $AII(3)$. Also by Proposition 5.2 $AI(3)$ is a totally geodesic submanifold of $AII(3)$.

(iii) By Proposition 5.2 $AI(3)$ is a totally geodesic submanifold of $SU(3)$, thus $SU(3)$ is an irreducible maximal totally geodesic submanifold of rank two of $AII(3)$.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to $S^1 \times S^1$

(ii) $S^1 \times S^1$ is isomorphic to $S^1 \cdot S^1$, also $S^1 \cdot S^1$ is a totally geodesic submanifold of $S^1 \cdot S^5$ which is the meridian of $AII(3)$.

(iii) By Lemma 4.2 $S^1 \times S^1$ is not maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^1 \cdot S^5$.

(ii) $S^1 \cdot S^5$ is isomorphic to the meridian of $AII(3)$.

(iii) By Proposition 5.2 $S^1 \cdot S^5$ is not a totally geodesic submanifold of $SU(3)$. Thus $S^1 \cdot S^5$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of S^5, $\mathbf{R}P^n$, $\mathbf{C}P^n$ and $\mathbf{H}P^n (2 \leq n \leq 3)$.

(ii) In the case of $\mathbf{H}P^3$, if $\mathbf{H}P^3$ is a totally geodesic submanifold of $AII(3)$, then the isometry group $Sp(4)$ of $\mathbf{H}P^3$ is a Lie subgroup of the isometry group $SU(6)$ of $AII(3)$. This is a contradiction because $Sp(4)$ is not a Lie subgroup of $SU(6)$. $\mathbf{H}P^2$ is isomorphic to the polar of $AII(3)$. $\mathbf{C}P^3$ is the orthogonal complement to $SU(3)$, i.e., a reflective submanifold. Thus $\mathbf{C}P^3$ is a totally geodesic submanifold of $AII(3)$. Also $\mathbf{R}P^3$ is a totally geodesic submanifold of $AII(3)$.

(iii) By Corollary 4.7 $\mathbf{H}P^2$ and $\mathbf{C}P^3$ are not totally geodesic submanifolds of $S^1 \cdot S^5$. Also, by Proposition 5.2 $\mathbf{H}P^2$ and $\mathbf{C}P^3$ are not in $SU(3)$. Thus $\mathbf{H}P^2$ and $\mathbf{C}P^3$ are maximal. □

PROPOSITION 5.4. *Any maximal totally geodesic submanifold M of EIV is isomorphic to one of $AII(3)$, $\mathbf{H}P^3$, $S^1 \cdot S^9$ and $\mathbf{O}P^2$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $AI(3)$, $AI(3)/\mathbf{Z}_3$, $SU(3)$, $SU(3)/\mathbf{Z}_3$, $AII(3)$ and $AII(3)/\mathbf{Z}_3$.

(ii) By Theorem 2.1 $AI(3)/\mathbf{Z}_3$, $SU(3)/\mathbf{Z}_3$ and $AII(3)/\mathbf{Z}_3$ are not totally geodesic submanifolds of EIV. From [5] $AII(3)$ is a totally geodesic submanifold of EIV. Also by Proposition 5.2 and Proposition 5.3 $AI(3)$ and $SU(3)$ are totally geodesic submanifolds of EIV.

(iii) By Proposition 5.3 $AII(3)$ is maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to $S^1 \times S^1$.

(ii) $S^1 \times S^1$ is a totally geodesic submanifold of $S^1 \cdot S^9$ which is the meridian of EIV, since $S^1 \times S^1$ is isomorphic to $S^1 \cdot S^1$.

(iii) By Lemma 4.2 $S^1 \times S^1$ is not maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^1 \cdot S^9$.

(ii) $S^1 \cdot S^9$ is isomorphic to the meridian of EIV.

(iii) By Proposition 5.3 $S^1 \cdot S^9$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of S^9, $\mathbf{O}P^2$, $\mathbf{R}P^n$, $\mathbf{C}P^n$ and $\mathbf{H}P^n (2 \leq n \leq 3)$.

(ii) S^9 is a totally geodesic submanifold of $S^1 \cdot S^9$. $\mathbf{O}P^2$ is isomorphic to the polar of EIV. $\mathbf{H}P^3$ is a totally geodesic submanifold of EIV, since $\mathbf{H}P^3$ is the orthogonal complement to $AII(3)$ in EIV. Thus $\mathbf{R}P^3$ and $\mathbf{C}P^3$ are totally geodesic submanifolds of EIV.

(iii) Since $\mathbf{H}P^3$ is a totally geodesic submanifold of EIV, $\mathbf{R}P^3$ and $\mathbf{C}P^3$ are not maximal. By Corollary 4.7, $\mathbf{H}P^3$ and $\mathbf{O}P^2$ are not totally geodesic submanifolds of $S^1 \cdot S^9$. Also by Proposition 5.3, $\mathbf{H}P^3$ and $\mathbf{O}P^2$ are not in $AII(3)$. Hence both $\mathbf{H}P^3$ and $\mathbf{O}P^2$ are maximal. □

PROPOSITION 5.5. *Any maximal totally geodesic submanifold M of $Sp(2)$ is isomorphic to one of $G_2^o(\mathbf{R}^5)$, S^4, $S^1 \cdot S^3$ and $S^3 \times S^3$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to $G_2^o(\mathbf{R}^5)$ or $G_2(\mathbf{R}^5)$.

(ii) By Theorem 2.1 $G_2(\mathbf{R}^5)$ is not a totally geodesic submanifold of $Sp(2)$. $G_2^o(\mathbf{R}^5)$ is a totally geodesic submanifold of $Sp(2)$, since $G_2^o(\mathbf{R}^5)$ is isomorphic to the centrosome of $Sp(2)$ (see [5]).

(iii) $G_2^o(\mathbf{R}^5)$ is maximal in $Sp(2)$, because there is no other irreducible compact symmetric space of rank two in $Sp(2)$.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^3 \times S^3$, $S^n \times \mathbf{R}P^2$ and $S^n \times \mathbf{C}P^2 (1 \leq n \leq 3)$.

(ii) $S^3 \times S^3$ is isomorphic to the meridian of $Sp(2)$. If $S^1 \times \mathbf{R}P^2$ is a totally geodesic submanifold of $Sp(2)$, then the isometry group $SO(2) \times SO(3)$ of $S^1 \times \mathbf{R}P^2$ is a Lie subgroup of the isometry group $Sp(2) \times Sp(2)$ of $Sp(2)$. This is a contradiction because $SO(2) \times SO(3)$ is not a Lie subgroup of $Sp(2) \times Sp(2)$. Thus $S^n \times \mathbf{R}P^m$ and $S^n \times \mathbf{C}P^m (1 \leq n \leq 3, m = 2)$ are not totally geodesic submanifolds of $Sp(2)$.

(iii) By Theorem 4.8 $S^3 \times S^3$ is not a totally geodesic submanifold of $G_2^o(\mathbf{R}^5)$. Thus $S^3 \times S^3$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^1 \cdot S^m (2 \leq m \leq 5)$.

(ii) Now, $S^1 \cdot S^3$ is a totally geodesic submanifold of $S^1 \cdot S^4$ and $S^1 \cdot S^3$ is isomorphic to $T \cdot SU(2)$. This is a contradiction because $T \cdot SU(2)$ is a maximal Lie subgroup of $Sp(2)$. Therefore $S^1 \cdot S^4$ is not a totally geodesic submanifold of $Sp(2)$ ([1]).

(iii) Since $T \cdot SU(2)$ is a maximal Lie subgroup of $Sp(2)$, $S^1 \cdot S^3$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $S^n (1 \leq n \leq 5)$, $\mathbf{R}P^2$ and $\mathbf{C}P^2$.

(ii) If S^5 is a totally geodesic submanifold of $Sp(2)$, the isometry group of $SO(6)$ of S^5 is a Lie subgroup of the isometry group $Sp(2) \times Sp(2)$ of $Sp(2)$. This is a contradiction because $r(SO(6)) = 3 \leq r(Sp(2)) = 2$. Thus S^5 is not a totally geodesic submanifold of $Sp(2)$. S^4 is isomorphic to the polar of $Sp(2)$. By the similar discussion if $\mathbf{R}P^2$ is a totally geodesic submanifold of $Sp(2)$, $SO(3)$ is a Lie subgroup of $Sp(2)$. This is a contradiction. Therefore $\mathbf{R}P^2$ and $\mathbf{C}P^2$ are not totally geodesic submanifolds of $Sp(2)$.

(iii) By Theorem 4.8, Proposition 4.6 and Corollary 4.7 S^4 is not a totally geodesic submanifold of $G_2^o(\mathbf{R}^5)$, $S^1 \cdot S^3$ and $S^3 \times S^3$. Thus S^4 is maximal. □

PROPOSITION 5.6. *Any maximal totally geodesic submanifold M of $G_2(\mathbf{H}^4)$ is isomorphic to one of* $Sp(2)$, $\mathbf{H}P^2$, $S^1 \cdot S^5$, $S^4 \times S^4$ *and* $G_2(\mathbf{C}^4)$.

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $Sp(2)$, $G_2(\mathbf{H}^3)$ and $G_2^o(\mathbf{R}^{n+2})(2 \leq n \leq 4)$.

(ii) From [5] $Sp(2)$ is isomorphic to a centrosome of $G_2(\mathbf{H}^4)$. Since $G_2(\mathbf{H}^3)$ is a compact symmetric space of rank one, we will discuss it in the case 4. $G_2^o(\mathbf{R}^6)$ is isomorphic to $G_2(\mathbf{C}^4)$. Since $G_2(\mathbf{C}^4)$ is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$, so is $G_2^o(\mathbf{R}^6)$.

(iii) By Proposition 5.5 $G_2^o(\mathbf{R}^6)$ is not a totally geodesic submanifold of $Sp(2)$. Also by Theorem 4.8 $Sp(2)$ is not in $G_2^o(\mathbf{R}^6)$. Thus $G_2^o(\mathbf{R}^6)$ and $Sp(2)$ are maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^n \times S^m$, $S^n \times \mathbf{R}P^m$, $S^n \times \mathbf{C}P^m$ and $S^n \times \mathbf{H}P^m$.

(ii) If $S^1 \times \mathbf{R}P^2$ is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$, by Theorem 4.3 and Theorem 4.4, we obtain

$$f_1^{\pm} : (\{\text{the pole}\}, S^1 \cdot \mathbf{R}P^2) \to (\{\text{the pole}\}, G_2(\mathbf{H}^4))$$

$$f_2^{\pm} : (S^1, S^1 \times S^1) \to (S^4 \times S^4, S^4 \times S^4)\,.$$

Then a pole of $S^1 \times \mathbf{R}P^2$ corresponds to the pole of $G_2(\mathbf{H}^4)$, by Lemma 4.3 a centrosome of $S^1 \times \mathbf{R}P^2$ is a totally geodesic submanifold of the centrosome of $G_2(\mathbf{H}^4)$, the orthogonal complement of centrosome of $S^1 \times \mathbf{R}P^2$ is a totally geodesic submanifold of the orthogonal complement of the centrosome of $G_2(\mathbf{H}^4)$. This is to say:

$$C(o, p) \amalg C(o', p') \subset Sp(2)\,, \quad C^{\perp}(o, p) \amalg C^{\perp}(o', p') \subset S^1 \cdot S^5\,,$$

where $C(o, p) = \{\text{a point}\}$, $C(o', p') = \{\text{a point}\}$, $C^{\perp}(o, p) = S^1$, $C^{\perp}(o', p') = \mathbf{R}P^2$. This is a contradiction, by Corollary 4.7. Thus $S^n \times \mathbf{R}P^m$, $S^n \times \mathbf{C}P^m$ and $S^n \times \mathbf{H}P^m$ are not totally geodesic submanifolds of $G_2(\mathbf{H}^4)$. $S^4 \times S^4$ is isomorphic to one of a polar and the corresponding meridian of $G_2(\mathbf{H}^4)$.

(iii) By Theorem 4.8 and Proposition 5.5, $S^4 \times S^4$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^1 \cdot S^m (1 \leq m \leq 5)$.

(ii) From [5] $S^1 \cdot S^5$ is isomorphic to the orthogonal complement of the centrosome of $G_2(\mathbf{H}^4)$.

(iii) By Theorem 4.8, Proposition 4.6, Corollary 4.7 and Proposition 5.5, $S^1 \cdot S^5$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $S^n (1 \leq n \leq 4)$, $\mathbf{R}P^2$, $\mathbf{C}P^2$ and $\mathbf{H}P^2$.

(ii) $\mathbf{H}P^2$ is isomorphic to $G_2(\mathbf{H}^3)$ and a totally geodesic submanifold of $G_2(\mathbf{H}^4)$. Thus $\mathbf{R}P^2$ and $\mathbf{C}P^2$ are not maximal. S^4 is a totally geodesic submanifold of $S^4 \times S^4$ which is the meridian of $G_2(\mathbf{H}^4)$.

(iii) By Lemma 4.2 S^4 is not maximal. Also by Theorem 4.8, Proposition 4.6, Corollary 4.7 and Proposition 5.5, $\mathbf{H}P^2$ is maximal. □

From now on, we use Theorem 4.4 in the step (i).

PROPOSITION 5.7. *Any maximal totally geodesic submanifold M of $G_2(\mathbf{C}^5)$ is isomorphic to one of $G_2(\mathbf{C}^4)$, $G_2(\mathbf{R}^5)$, $S^2 \times \mathbf{C}P^2$ and $\mathbf{C}P^3$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $G_2^o(\mathbf{R}^{n+2}) (2 \leq n \leq 4)$, $G_2(\mathbf{R}^{n+2}) (2 \leq n \leq 5)$ and $G_2(\mathbf{C}^4)$.

(ii) Clearly, $G_2(\mathbf{C}^4)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$. $G_2^o(\mathbf{R}^6)$ is isomorphic to $G_2(\mathbf{C}^4)$. If $G_2(\mathbf{R}^{n+2})$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$, then $G_2(\mathbf{R}^{n+2})$ is totally real. By Theorem 2.3 $n = 2$ or 3. $G_2(\mathbf{R}^5)$ is a totally real totally geodesic submanifold of $G_2(\mathbf{C}^5)$.

(iii) By Theorem 4.8 $G_2(\mathbf{R}^5)$ is not a totally geodesic submanifold of $G_2^o(\mathbf{R}^6)$. Thus $G_2(\mathbf{R}^5)$ and $G_2(\mathbf{C}^4)$ are maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^n \times S^m (1 \leq n, m \leq 2)$, $S^n \times \mathbf{R}P^2$ and $S^n \times \mathbf{C}P^2 (1 \leq n \leq 2)$.

(ii) $S^2 \times \mathbf{C}P^2$ is isomorphic to a polar of $G_2(\mathbf{C}^5)$ from Table 3. $S^2 \times S^2$ and $S^2 \times \mathbf{R}P^2$ are totally geodesic submanifolds of $S^2 \times \mathbf{C}P^2$.

(iii) $S^2 \times S^2$ and $S^2 \times \mathbf{R}P^2$ are not maximal. By Theorem 4.8 $S^2 \times \mathbf{C}P^2$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^m \cdot S^n$ $(1 \leq m, n \leq 2)$.

(ii) By Theorem 4.8 $S^2 \cdot S^2$ is a totally geodesic submanifold of $G_2(\mathbf{C}^4) \cong G_2^o(\mathbf{R}^6)$.

(iii) By (ii), $S^2 \cdot S^2$ is not maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $S^n (1 \leq n \leq 4)$, $\mathbf{R}P^n (2 \leq n \leq 3)$ and $\mathbf{C}P^n (2 \leq n \leq 3)$.

(ii) By Theorem 4.8 S^4 is a totally geodesic submanifold of $G_2(\mathbf{C}^4) \cong G_2^o(\mathbf{R}^6)$. Also $\mathbf{R}P^3$ is a totally geodesic submanifold of $G_2(\mathbf{R}^5)$. We will show that $\mathbf{C}P^3$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$ in Proposition 5.8.

(iii) By (ii), S^4 and $\mathbf{R}P^3$ are not maximal. From Theorem 4.8 $\mathbf{C}P^3$ is not in both $G_2(\mathbf{C}^4)$ and $G_2(\mathbf{R}^5)$. Clearly $\mathbf{C}P^3$ is not in $S^2 \times \mathbf{C}P^2$. Thus $\mathbf{C}P^3$ is maximal. □

PROPOSITION 5.8. *Any maximal totally geodesic submanifold M of $G_2(\mathbf{C}^{n+2})$ is isomorphic to one of $G_2(\mathbf{C}^{n+1})$, $G_2(\mathbf{R}^{n+2})$, $\mathbf{C}P^k \times \mathbf{C}P^l (k + l = n)$ and $\mathbf{H}P^{[\frac{n}{2}]}$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $G_2(\mathbf{C}^{n+1})$, $G_2^o(\mathbf{R}^{m+2}) (3 \leq m \leq 4)$ and $G_2(\mathbf{R}^{m+2}) (2 \leq m \leq n)$.

(ii) Clearly $G_2(\mathbf{C}^{n+1})$ and $G_2(\mathbf{R}^{n+2})$ are totally geodesic submanifolds of $G_2(\mathbf{C}^{n+2})$. $G_2^o(\mathbf{R}^6)$ is isomorphic to $G_2(\mathbf{C}^4)$ which is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+1})$.

(iii) By the comparison of dimension $G_2(\mathbf{C}^{n+1})$ is not a totally geodesic submanifold of $G_2(\mathbf{R}^{n+2})$. Also if $G_2(\mathbf{R}^{n+2})$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+1})$, then Theorem 2.3 holds. This is a contradiction because the only totally real totally geodesic submanifold in $G_2(\mathbf{C}^{n+1})$ is $G_2(\mathbf{R}^{n+1})$.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^k \times S^l (1 \leq k, l \leq 2)$, $S^k \times \mathbf{R}P^l$, $S^k \times \mathbf{C}P^l$, $\mathbf{R}P^k \times \mathbf{R}P^l$, $\mathbf{R}P^k \times \mathbf{C}P^l$ and $\mathbf{C}P^k \times \mathbf{C}P^l (1 \leq k \leq 2, 2 \leq l \leq n-1)$.

(ii) If $\mathbf{R}P^k \times \mathbf{R}P^l$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$, then by Theorem 4.3 a polar $\mathbf{R}P^{k-1} \times \mathbf{R}P^{l-1}$ of $\mathbf{R}P^k \times \mathbf{R}P^l$ is a totally geodesic submanifold of $G_2(\mathbf{C}^n)$ which is a polar of $G_2(\mathbf{C}^{n+2})$. We repeat the discussion. When n is odd, $\mathbf{R}P^{k-\frac{n-3}{2}} \times \mathbf{R}P^{l-\frac{n-3}{2}}$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$. When n is even, $\mathbf{R}P^{k-\frac{n}{2}+1} \times \mathbf{R}P^{l-\frac{n}{2}+1}$ is a totally geodesic submanifold of $G_2(\mathbf{C}^4)$. Thus the necessary condition is $k + l \leq n$ by Theorem 4.8 and Proposition 5.7. Similarly, the necessary condition is $k + l \leq n$ for $\mathbf{R}P^k \times \mathbf{C}P^l$ and $\mathbf{C}P^k \times \mathbf{C}P^l$.

We will show that $\mathbf{C}P^k \times \mathbf{C}P^l\,(k+l=n)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathbf{C}P^k \times \mathbf{C}P^l$ and let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of $G_2(\mathbf{C}^{n+2})$, where $\mathfrak{u} := \mathfrak{su}(n+2)$, $\mathfrak{l} := \mathbf{R} + \mathfrak{su}(2) + \mathfrak{su}(n)$, $\mathfrak{g} := \mathfrak{su}(k+1) + \mathfrak{su}(l+1)$, $\mathfrak{k} := \mathbf{R}^2 + \mathfrak{su}(k) + \mathfrak{su}(l)$,

$$\mathfrak{m} := \left\{ \left(\begin{array}{cc} 0 & -{}^t\bar{z}_1 \\ z_1 & 0 \end{array} \right) \middle| z_1 \in \mathbf{C}^k \right\} + \left\{ \left(\begin{array}{cc} 0 & -{}^t\bar{z}_2 \\ z_2 & 0 \end{array} \right) \middle| z_2 \in \mathbf{C}^l \right\},$$

and

$$\mathfrak{p} := \left\{ \left(\begin{array}{cc} 0 & -{}^t\bar{z} \\ z & 0 \end{array} \right) \middle| z \in M(n, 2 : \mathbf{C}) \right\}.$$

Here the inclusion $\mathfrak{m} \hookrightarrow \mathfrak{p}$ is:

$$\left(\left(\begin{array}{cc} 0 & -{}^t\bar{z}_1 \\ z_1 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & -{}^t\bar{z}_2 \\ z_2 & 0 \end{array} \right) \right) \mapsto \left(\begin{array}{cc|ccc|ccc} 0 & 0 & & -{}^t\bar{z}_1 & & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & & -{}^t\bar{z}_2 & \\ \hline & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ z_1 & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \hline 0 & & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & z_2 & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array} \right).$$

Then we find that $\mathfrak{m}$ satisfies $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$. Thus $\mathbf{C}P^k \times \mathbf{C}P^l\,(k+l=n)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$.

(iii) From the construction of $\mathfrak{m}$, $\mathbf{C}P^k \times \mathbf{C}P^l\,(k+l=n)$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^k \cdot S^l\,(1 \leq k, l \leq 2)$.

(ii) By Theorem 4.8 $S^2 \cdot S^2$ is a totally geodesic submanifold of $G_2^o(\mathbf{R}^6) \cong G_2(\mathbf{C}^4)$.

(iii) By Lemma 4.2 $S^2 \cdot S^2$ is not maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $\mathbf{R}P^m$, $\mathbf{C}P^m\,(2 \leq m \leq n)$ and $\mathbf{H}P^m\,(2 \leq m \leq [\frac{n}{2}])$.

(ii) By Case 2, $\mathbf{C}P^n$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$. $\mathbf{R}P^n$ is a totally geodesic submanifold of $\mathbf{C}P^n$. Now, we show that $\mathbf{H}P^{[\frac{n}{2}]}$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathbf{H}P^{[\frac{n}{2}]}$ and let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of $G_2(\mathbf{C}^{n+2})$, where $\mathfrak{u} := \mathfrak{su}(n+2)$, $\mathfrak{l} := \mathbf{R} + \mathfrak{su}(2) + \mathfrak{su}(n)$, $\mathfrak{g} := \mathfrak{sp}([\frac{n}{2}]+1)$, $\mathfrak{k} := \mathfrak{sp}(1) + \mathfrak{sp}([\frac{n}{2}])$

and

$$\mathfrak{m} := \left\{ \left(\begin{array}{cc|cc} 0 & -{}^t\overline{z} & 0 & -{}^t\overline{w} \\ z & 0 & -\overline{w} & 0 \\ \hline 0 & {}^t w & 0 & -{}^t z \\ w & 0 & \overline{z} & 0 \end{array} \right) \middle| z, w \in \mathbf{C}^m \right\}.$$

When n is even, the inclusion $\mathfrak{m} \hookrightarrow \mathfrak{p}$ is:

$$\left(\begin{array}{cc|cc} 0 & -{}^t\overline{z} & 0 & -{}^t\overline{w} \\ z & 0 & -\overline{w} & 0 \\ \hline 0 & {}^t w & 0 & -{}^t z \\ w & 0 & \overline{z} & 0 \end{array} \right) \mapsto \left(\begin{array}{cc|cc} 0 & 0 & -{}^t\overline{z} & -{}^t\overline{w} \\ 0 & 0 & {}^t w & -{}^t z \\ \hline z & -\overline{w} & 0 & 0 \\ w & \overline{z} & 0 & 0 \end{array} \right).$$

When n is odd, the inclusion $\mathfrak{m} \hookrightarrow \mathfrak{p}$ is:

$$\left(\begin{array}{cc|cc} 0 & -{}^t\overline{z} & 0 & -{}^t\overline{w} \\ z & 0 & -\overline{w} & 0 \\ \hline 0 & {}^t w & 0 & -{}^t z \\ w & 0 & \overline{z} & 0 \end{array} \right) \mapsto \left(\begin{array}{cc|ccc} 0 & 0 & -{}^t\overline{z} & -{}^t\overline{w} & 0 \\ 0 & 0 & {}^t w & -{}^t z & 0 \\ \hline z & -\overline{w} & 0 & 0 & 0 \\ w & \overline{z} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

Then we find that $\mathfrak{m}$ satisfies $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$. Thus $\mathbf{H}P^{[\frac{n}{2}]}$ is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$.

(iii) Clearly $\mathbf{H}P^{[\frac{n}{2}]}$ is not a totally geodesic submanifold of $G_2(\mathbf{C}^{n+1})$ and $\mathbf{C}P^k \times \mathbf{C}P^l (k + l = n)$. Thus $\mathbf{H}P^{[\frac{n}{2}]}$ is maximal. □

PROPOSITION 5.9. *Any maximal totally geodesic submanifold M of $DIII(5)$ is isomorphic to one of $G_2^o(\mathbf{R}^8)$, $G_2(\mathbf{C}^5)$, $SO(5)$, $S^2 \times \mathbf{C}P^3$ and $\mathbf{C}P^4$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $G_2^o(\mathbf{R}^{n+2})(2 \leq n \leq 6)$, $G_2(\mathbf{R}^{n+2})(2 \leq n \leq 3)$, $G_2(\mathbf{C}^{n+2})(2 \leq n \leq 3)$, $Sp(2)/\mathbf{Z}_2$, $G_2(\mathbf{H}^4)$ and $G_2(\mathbf{H}^4)/\mathbf{Z}_2$.

(ii) From Table 3 $G_2^o(\mathbf{R}^8)$ is isomorphic to a meridian of $DIII(5)$. $G_2(\mathbf{R}^5)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$ which is a polar of $DIII(5)$. $G_2(\mathbf{C}^5)$ is isomorphic to a polar of $DIII(5)$. From [5] $Sp(2)/\mathbf{Z}_2 \cong SO(5)$ is a totally geodesic submanifold of $DIII(5)$. If $G_2(\mathbf{H}^4)$ is a totally geodesic submanifold of $DIII(5)$, then from Proposition 5.6 $Sp(2)$ is a totally geodesic submanifold of $DIII(5)$. This is a contradiction because of (i). Therefore $G_2(\mathbf{H}^4)$ is not in $DIII(5)$. If $G_2(\mathbf{H}^4)/\mathbf{Z}_2$ is a totally geodesic submanifold of $DIII(5)$, then Theorem 2.3 holds, which is a contradiction. Thus $G_2(\mathbf{H}^4)/\mathbf{Z}_2$ is not in $DIII(5)$.

(iii) By Proposition 5.7 and Proposition 5.5 $G_2^o(\mathbf{R}^8)$ is not a totally geodesic submanifold of both $G_2(\mathbf{C}^5)$ and $SO(5)$. Similarly from Theorem 4.8 $G_2(\mathbf{C}^5)$ and $SO(5)$

are not totally geodesic submanifolds of $G_2^o(\mathbf{R}^8)$. $G_2(\mathbf{C}^5)$ and $SO(5)$ are not totally geodesic submanifolds of each other. Thus $G_2^o(\mathbf{R}^8)$, $G_2(\mathbf{C}^5)$ and $SO(5)$ are maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^n \times S^m (1 \leq n, m \leq 2)$, $S^n \times \mathbf{R}P^m$ and $S^n \times \mathbf{C}P^m (1 \leq n \leq 2, 2 \leq m \leq 3)$.

(ii) $S^2 \times S^2$ and $S^2 \times \mathbf{R}P^3$ are totally geodesic submanifolds of $S^2 \times \mathbf{C}P^3$ which is a polar of $DIII(5)$. From Table 3 $S^2 \times \mathbf{C}P^3$ is isomorphic to a polar of $DIII(5)$.

(iii) By Theorem 4.8, Proposition 5.5 and Proposition 5.7 $S^2 \times \mathbf{C}P^3$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^n \cdot S^m (1 \leq n, m \leq 3)$.

(ii) $S^3 \cdot S^3$ is a totally geodesic submanifold of $G_2^o(\mathbf{R}^8)$.

(iii) By (ii) $S^3 \cdot S^3$ is not maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $S^n (1 \leq n \leq 6)$, $\mathbf{R}P^n (2 \leq n \leq 5)$ and $\mathbf{C}P^n (2 \leq n \leq 5)$.

(ii) S^6 is a totally geodesic submanifold of $G_2^o(\mathbf{R}^8)$. If $\mathbf{R}P^n$ is a totally geodesic submanifold of $DIII(5)$, then Theorem 2.3 holds, so $2 \leq n \leq 4$. Thus for $\mathbf{C}P^n$ n must be $2 \leq n \leq 4$. $\mathbf{R}P^4$ is a totally geodesic submanifold of $\mathbf{C}P^4$ which is isomorphic to a polar of $DIII(5)$. $\mathbf{C}P^4$ is isomorphic to a polar of $DIII(5)$.

(iii) By (ii) S^6 and $\mathbf{R}P^4$ are not maximal in $DIII(5)$. Also by Theorem 4.8, Proposition 5.5, Proposition 5.7 and the fact that $\mathbf{C}P^4$ is not in $S^2 \times \mathbf{C}P^3$, $\mathbf{C}P^4$ is maximal. □

PROPOSITION 5.10. *Any maximal totally geodesic submanifold M of $EIII$ is isomorphic to one of* $G_2(\mathbf{H}^4)/\mathbf{Z}_2$, $\mathbf{O}P^2$, $S^2 \times \mathbf{C}P^5$, $DIII(5)$, $G_2^o(\mathbf{R}^{10})$ *and* $G_2(\mathbf{C}^6)$.

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $G_2(\mathbf{H}^4)/\mathbf{Z}_2$, $DIII(5)$, $G_2^o(\mathbf{R}^{n+2})(3 \leq n \leq 8)$, $G_2(\mathbf{R}^{n+2})(3 \leq n \leq 4)$ and $G_2(\mathbf{C}^{n+2})(3 \leq n \leq 4)$.

(ii) From [5] $G_2(\mathbf{H}^4)/\mathbf{Z}_2$ and $G_2(\mathbf{C}^6)$ are totally geodesic submanifolds of $EIII$. $DIII(5)$ and $G_2^o(\mathbf{R}^{10})$ are isomorphic to polars of $EIII$. $G_2(\mathbf{R}^6)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^6)$.

(iii) By (ii) $G_2(\mathbf{R}^6)$ is not maximal in $EIII$. By Theorem 4.8, Proposition 5.6, Proposition 5.8 and Proposition 5.9 $G_2(\mathbf{H}^4)/\mathbf{Z}_2$, $DIII(5)$, $G_2^o(\mathbf{R}^{10})$ and $G_2(\mathbf{C}^6)$ are maximal in $EIII$.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^n \times S^m (1 \leq n, m \leq 2)$, $S^n \times \mathbf{R}P^m$, $S^n \times \mathbf{C}P^m (1 \leq n \leq 2, 2 \leq m \leq 5)$, $\mathbf{R}P^2 \times \mathbf{R}P^2$, $\mathbf{R}P^2 \times \mathbf{C}P^2$ and $\mathbf{C}P^2 \times \mathbf{C}P^2$.

(ii) $S^2 \times \mathbf{C}P^5$ is isomorphic to a meridian of $EIII$. By Proposition 5.8 $\mathbf{C}P^2 \times \mathbf{C}P^2$ is a totally geodesic submanifold of $G_2(\mathbf{C}^6)$ which is a reflective submanifold of

$EIII$. $S^2 \times S^2$ and $S^2 \times \mathbf{R}P^5$ are totally geodesic submanifolds of $S^2 \times \mathbf{C}P^5$. Also $\mathbf{R}P^2 \times \mathbf{R}P^2$ and $\mathbf{R}P^2 \times \mathbf{C}P^2$ are totally geodesic submanifolds of $\mathbf{C}P^2 \times \mathbf{C}P^2$.

(iii) By (ii) $S^2 \times S^2$, $S^2 \times \mathbf{R}P^5$, $\mathbf{R}P^2 \times \mathbf{R}P^2$, $\mathbf{R}P^2 \times \mathbf{C}P^2$ and $\mathbf{C}P^2 \times \mathbf{C}P^2$ are not maximal in $EIII$. Also from Theorem 4.8, Proposition 5.6, Proposition 5.8 and Proposition 5.9 $S^2 \times \mathbf{C}P^5$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^m \cdot S^n (1 \leq n, m \leq 4)$.

(ii) By Theorem 4.8 $S^4 \cdot S^4$ is a totally geodesic submanifold of $G_2^o(\mathbf{R}^{10})$ which is a polar of $EIII$.

(iii) By Lemma 4.2 $S^4 \cdot S^4$ is not maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $S^n (1 \leq n \leq 8)$, $\mathbf{R}P^n, \mathbf{C}P^n (2 \leq n \leq 5)$, $\mathbf{H}P^2$ and $\mathbf{O}P^2$.

(ii) By Theorem 4.8 S^8 is a totally geodesic submanifold of $G_2^o(\mathbf{R}^{10})$ which is a polar of $EIII$. From [5] $\mathbf{O}P^2$ is a totally geodesic submanifold of $EIII$. $\mathbf{H}P^2$ is a totally geodesic submanifold of $\mathbf{O}P^2$. $\mathbf{C}P^5$ is a totally geodesic submanifold of $S^2 \times \mathbf{C}P^5$ which is a meridian of $EIII$.

(iii) By (ii), S^8, $\mathbf{R}P^5$, $\mathbf{C}P^5$ and $\mathbf{H}P^2$ are not maximal in $EIII$. Also from Theorem 4.8, Proposition 5.6, Proposition 5.8 and Proposition 5.9 $\mathbf{O}P^2$ is maximal. □

PROPOSITION 5.11. *Any maximal totally geodesic submanifold M of $G_2(\mathbf{H}^5)$ is isomorphic to one of $G_2(\mathbf{H}^4)$, $G_2(\mathbf{C}^5)$, $S^4 \times \mathbf{H}P^2$ and $\mathbf{H}P^3$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of $G_2^o(\mathbf{R}^{n+2})(2 \leq n \leq 4), G_2(\mathbf{R}^{n+2})(2 \leq n \leq 3)$, $Sp(2)$, $G_2(\mathbf{C}^{n+2})(2 \leq n \leq 3)$ and $G_2(\mathbf{H}^4)$.

(ii) Clearly $G_2(\mathbf{H}^4)$ and $G_2(\mathbf{C}^5)$ are totally geodesic submanifolds of $G_2(\mathbf{H}^5)$. By Proposition 5.6 $Sp(2)$ is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$. Since $G_2^o(\mathbf{R}^6)$ is isomorphic to $G_2(\mathbf{C}^4)$, $G_2^o(\mathbf{R}^6)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$. Also $G_2(\mathbf{R}^5)$ is a totally geodesic submanifold of $G_2(\mathbf{C}^5)$.

(iii) By (ii), $Sp(2)$, $G_2^o(\mathbf{R}^6)$ and $G_2(\mathbf{R}^5)$ are not maximal in $G_2(\mathbf{H}^5)$. From Proposition 5.6 and Proposition 5.7 $G_2(\mathbf{H}^4)$ and $G_2(\mathbf{C}^5)$ are maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to one of $S^k \times S^l (1 \leq k, l \leq 4)$, $S^k \times \mathbf{R}P^2$, $S^k \times \mathbf{C}P^2$ and $S^k \times \mathbf{H}P^2 (1 \leq k \leq 4)$.

(ii) $S^4 \times \mathbf{H}P^2$ is isomorphic to a polar of $G_2(\mathbf{H}^5)$. Clearly $S^4 \times S^4$, $S^4 \times \mathbf{R}P^2$ and $S^4 \times \mathbf{C}P^2$ are totally geodesic submanifolds of $S^4 \times \mathbf{H}P^2$.

(iii) By (ii) $S^4 \times S^4$, $S^4 \times \mathbf{R}P^2$ and $S^4 \times \mathbf{C}P^2$ are not maximal in $G_2(\mathbf{H}^5)$. From Proposition 5.6 and Proposition 5.7 $S^4 \times \mathbf{H}P^2$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^1 \cdot S^l (1 \le l \le 5)$.
(ii) $S^1 \cdot S^5$ is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$.
(iii) By Lemma 4.2 $S^1 \cdot S^5$ is not maximal.

Case 4. M is a compact symmetric space of rank one.
(i) M is isomorphic to one of $S^n (1 \le n \le 5)$, $\mathbf{R}P^n$, $\mathbf{C}P^n$ and $\mathbf{H}P^n (2 \le n \le 3)$.
(ii) By Proposition 5.6 S^5 is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$. After we will show that $\mathbf{H}P^3$ is a totally geodesic submanifold of $G_2(\mathbf{H}^5)$ in Proposition 5.12. $\mathbf{R}P^3$ and $\mathbf{C}P^3$ are totally geodesic submanifolds of $\mathbf{H}P^3$.
(iii) By (ii) S^5, $\mathbf{R}P^3$ and $\mathbf{C}P^3$ are not maximal in $G_2(\mathbf{H}^5)$. Also by Proposition 5.6 and Proposition 5.7, and the fact that $\mathbf{H}P^3$ is not in $S^4 \times \mathbf{H}P^2$, $\mathbf{H}P^3$ is maximal. $\square$

PROPOSITION 5.12. *Any maximal totally geodesic submanifold M of $G_2(\mathbf{H}^{n+2})$ is isomorphic to one of $G_2(\mathbf{H}^{n+1})$, $G_2(\mathbf{C}^{n+2})$ and $\mathbf{H}P^k \times \mathbf{H}P^l (k + l = n)$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.
(i) M is isomorphic to one of $Sp(2)$, $G_2(\mathbf{H}^{n+1})$, $G_2^o(\mathbf{R}^{m+2})(3 \le m \le 4)$ $G_2(\mathbf{R}^{m+2})(2 \le m \le n)$ and $G_2(\mathbf{C}^{m+2})(2 \le m \le n)$.
(ii) Clearly $G_2(\mathbf{H}^{n+1})$ and $G_2(\mathbf{C}^{n+2})$ are totally geodesic submanifolds of $G_2(\mathbf{H}^{n+2})$. $G_2^o(\mathbf{R}^6)$ is isomorphic to $G_2(\mathbf{C}^4)$ which is a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$. By Proposition 5.6 $Sp(2)$ is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$ which is a totally geodesic submanifold of $G_2(\mathbf{H}^{n+1})$.
(iii) By the comparison of dimension $G_2(\mathbf{H}^{n+1})$ is not a totally geodesic submanifold of $G_2(\mathbf{C}^{n+2})$. Also by Theorem 4.3 $G_2(\mathbf{C}^{n+2})$ is not a totally geodesic submanifold of $G_2(\mathbf{H}^{n+1})$. Thus $G_2(\mathbf{H}^{n+1})$ and $G_2(\mathbf{C}^{n+2})$ are maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.
(i) M is isomorphic to one of $S^k \times S^l (1 \le k, l \le 4)$, $S^k \times \mathbf{R}P^l$, $S^k \times \mathbf{C}P^l$, $S^k \times \mathbf{H}P^l (1 \le k \le 4, 2 \le l \le n-1)$, $\mathbf{R}P^k \times \mathbf{R}P^l$, $\mathbf{R}P^k \times \mathbf{C}P^l$, $\mathbf{R}P^k \times \mathbf{H}P^l$, $\mathbf{C}P^k \times \mathbf{C}P^l$, $\mathbf{C}P^k \times \mathbf{H}P^l$ and $\mathbf{H}P^k \times \mathbf{H}P^l (2 \le k \le n-1, 2 \le l \le n-1)$.
(ii) If $\mathbf{R}P^k \times \mathbf{R}P^l$ is a totally geodesic submanifold of $G_2(\mathbf{H}^{n+2})$, then by Theorem 4.3 a polar $\mathbf{R}P^{k-1} \times \mathbf{R}P^{l-1}$ of $\mathbf{R}P^k \times \mathbf{R}P^l$ is a totally geodesic submanifold of $G_2(\mathbf{H}^n)$ which is a polar of $G_2(\mathbf{H}^{n+2})$. We repeat the discussion. When n is odd, $\mathbf{R}P^{k-\frac{n-3}{2}} \times \mathbf{R}P^{l-\frac{n-3}{2}}$ is a totally geodesic submanifold of $G_2(\mathbf{H}^5)$. When n is even, $\mathbf{R}P^{k-\frac{n}{2}+1} \times \mathbf{R}P^{l-\frac{n}{2}+1}$ is a totally geodesic submanifold of $G_2(\mathbf{H}^4)$. Thus the necessary condition is $k+l \le n$ by Proposition 5.6 and Proposition 5.11. Similarly, the necessary condition is $k+l \le n$ for $\mathbf{R}P^k \times \mathbf{C}P^l$, $\mathbf{C}P^k \times \mathbf{C}P^l$ and the others. Now we show that $\mathbf{H}P^k \times \mathbf{H}P^l (k+l = n)$ is a totally geodesic submanifold of $G_2(\mathbf{H}^{n+2})$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the canonical decomposition of $\mathbf{H}P^k \times \mathbf{H}P^l$ and let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of $G_2(\mathbf{H}^{n+2})$, where $\mathfrak{u} := \mathfrak{sp}(n+2)$,

$\mathfrak{l} := \mathfrak{sp}(2)+\mathfrak{sp}(n)$, $\mathfrak{g} := \mathfrak{sp}(k+1)+\mathfrak{sp}(l+1)$, $\mathfrak{k} := \mathfrak{sp}(1)+\mathfrak{sp}(k)+\mathfrak{sp}(1)+\mathfrak{sp}(l)$,

$$\mathfrak{m} := \left\{ \left(\begin{array}{cc|cc} 0 & -{}^t\bar{x} & 0 & -{}^t\bar{y} \\ x & 0 & -\bar{y} & 0 \\ \hline 0 & {}^t y & 0 & -{}^t x \\ y & 0 & \bar{x} & 0 \end{array} \right) \middle| x, y \in \mathbf{C}^k \right\} + \left\{ \left(\begin{array}{cc|cc} 0 & -{}^t\bar{z} & 0 & -{}^t\bar{w} \\ z & 0 & -\bar{w} & 0 \\ \hline 0 & {}^t w & 0 & -{}^t z \\ w & 0 & \bar{z} & 0 \end{array} \right) \middle| z, w \in \mathbf{C}^l \right\}$$

and

$$\mathfrak{p} := \left\{ \left(\begin{array}{cc|cc} 0 & -{}^t\overline{C} & 0 & -{}^t\overline{D} \\ C & 0 & -\overline{D} & 0 \\ \hline 0 & {}^t D & 0 & -{}^t C \\ D & 0 & \overline{C} & 0 \end{array} \right) \middle| C, D \in M(2, n : \mathbf{C}) \right\}.$$

Here the inclusion $\mathfrak{m} \hookrightarrow \mathfrak{p}$ is:

$$\left(\begin{array}{cc|cc} 0 & -{}^t\bar{x} & 0 & -{}^t\bar{y} \\ x & 0 & -\bar{y} & 0 \\ \hline 0 & {}^t y & 0 & -{}^t x \\ y & 0 & \bar{x} & 0 \end{array} \right) + \left(\begin{array}{cc|cc} 0 & -{}^t\bar{z} & 0 & -{}^t\bar{w} \\ z & 0 & -\bar{w} & 0 \\ \hline 0 & {}^t w & 0 & -{}^t z \\ w & 0 & \bar{z} & 0 \end{array} \right) \mapsto$$

$$\left(\begin{array}{cc|cc|cc|cc} 0 & 0 & -{}^t\bar{x} & 0 & 0 & 0 & -{}^t\bar{y} & 0 \\ 0 & 0 & 0 & -{}^t\bar{z} & 0 & 0 & 0 & -{}^t\bar{w} \\ \hline x & 0 & 0 & 0 & -\bar{y} & 0 & 0 & 0 \\ 0 & z & 0 & 0 & 0 & -\bar{w} & 0 & 0 \\ \hline 0 & 0 & {}^t y & 0 & 0 & 0 & -{}^t x & 0 \\ 0 & 0 & 0 & {}^t w & 0 & 0 & 0 & -{}^t z \\ \hline y & 0 & 0 & 0 & \bar{x} & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 & \bar{z} & 0 & 0 \end{array} \right).$$

Then we find that $\mathfrak{m}$ satisfies $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] \subset \mathfrak{m}$. Thus $\mathbf{H}P^k \times \mathbf{H}P^l (k+l=n)$ is a totally geodesic submanifold of $G_2(\mathbf{H}^{n+2})$. $S^4 \times S^4$, $S^4 \times \mathbf{R}P^{n-1}$, $S^4 \times \mathbf{C}P^{n-1}$, $S^4 \times \mathbf{H}P^{n-1}$, $\mathbf{R}P^k \times \mathbf{R}P^l$, $\mathbf{R}P^k \times \mathbf{C}P^l$, $\mathbf{R}P^k \times \mathbf{H}P^l$, $\mathbf{C}P^k \times \mathbf{C}P^l$ and $\mathbf{C}P^k \times \mathbf{H}P^l (k+l=n)$ are totally geodesic submanifolds of $\mathbf{H}P^k \times \mathbf{H}P^l$.

(iii) By Proposition 5.8 and by the construction of $\mathfrak{m}$, $\mathbf{H}P^k \times \mathbf{H}P^l (k+l=n)$ is maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^k \cdot S^l (1 \leq k, l \leq 2)$.

(ii) By Theorem 4.8 $S^2 \cdot S^2$ is a totally geodesic submanifold of $G_2^o(\mathbf{R}^6) \cong G_2(\mathbf{C}^4)$.

(iii) By Lemma 4.2 $S^2 \cdot S^2$ is not maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $\mathbf{R}P^m$, $\mathbf{C}P^m (2 \leq m \leq n)$ and $\mathbf{H}P^m (2 \leq m \leq n)$.

(ii) By Case 2, $\mathbf{H}P^n$ is a totally geodesic submanifold of $G_2(\mathbf{H}^{n+2})$. $\mathbf{R}P^n$ and $\mathbf{C}P^n$ are totally geodesic submanifolds of $\mathbf{H}P^n$.

(iii) By (ii) in Case 2, $\mathbf{H}P^n$ is maximal. □

PROPOSITION 5.13. *Any maximal totally geodesic submanifold M of GI is isomorphic to one of $AI(3)$, $\mathbf{C}P^2$ and $S^2 \cdot S^2$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to $AI(3)$ or $AI(3)/\mathbf{Z}_3$.

(ii) Let $\mathfrak{u} = \mathfrak{l} + \mathfrak{p}$ be the canonical decomposition of GI, where $\mathfrak{u} = \mathfrak{g}_2$, $\mathfrak{l} = \mathfrak{so}(4)$ and $\mathfrak{p} \cong T_oGI$. We take a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Then we have the restricted root decomposition of $\mathfrak{g}_2$ with respect to $\mathfrak{a}$:

$$\mathfrak{u} = \sum_{\alpha \in R^+(GI)} \mathfrak{l}_\alpha \oplus \mathfrak{a} \oplus \sum_{\alpha \in R^+(GI)} \mathfrak{p}_\alpha\,,$$

where $R^+(GI) = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$ is a positive restricted root system of GI. We take a subset D^+ of $R^+(GI)$:

$$D^+ = \{\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}\,.$$

Put

$$\mathfrak{g} := \sum_{\alpha \in D^+} \mathfrak{l}_\alpha \oplus \mathfrak{a} \oplus \sum_{\alpha \in D^+} \mathfrak{p}_\alpha\,.$$

Then $\mathfrak{g}$ is isomorphic to $\mathfrak{su}(3)$ and $\mathfrak{g} \cap \mathfrak{l} \cong \mathfrak{so}(3)$. Thus $(\mathfrak{g}, \mathfrak{g} \cap \mathfrak{l})$ is a symmetric pair and $\mathfrak{g}/\mathfrak{g} \cap \mathfrak{l}$ is locally isomorphic to $AI(3)$. Let $R^+(AI(3)) = \{\beta_1, \beta_2, \beta_1 + \beta_2\}$ be a positive restricted root system of $AI(3)$. Then we have $D^+ \cong R^+(AI(3))$. Clearly the unit lattice $\mathfrak{a}_K$ coincides with $\mathfrak{a}_L$. Therefore M is globally isomorphic to $AI(3)$.

(iii) By (ii), $AI(3)$ is maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to $S^1 \times S^1$.

(ii) $S^1 \times S^1$ is isomorphic to $S^1 \cdot S^1$ and $S^1 \cdot S^1$ is a totally geodesic submanifold of $S^2 \cdot S^2$ which is a polar of GI.

(iii) By Lemma 4.2 $S^1 \times S^1$ is not maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^n \cdot S^m$ $(1 \leq m, n \leq 2)$.

(ii) From Table 3 $S^2 \cdot S^2$ is isomorphic to a polar of GI.

(iii) By Proposition 5.1 $S^2 \cdot S^2$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to one of $S^n (1 \leq n \leq 2)$, $\mathbf{R}P^n (2 \leq n \leq 3)$ and $\mathbf{C}P^2$.

(ii) S^2 is a totally geodesic submanifold in a polar $S^2 \cdot S^2$ of GI. Now $f : \mathbf{R}P^3 \to GI$ be a totally geodesic imbedding. $SO(4)$-action on $\mathbf{R}P^3$ is the restriction of G_2-action on GI. In particular, at a point $o \in \mathbf{R}P^3$ the isotropy subgroup $S(O(1) \times O(3))$ at o acts on $T_o\mathbf{R}P^3$ by the restriction of the isotropy action of $SO(4)$ on

T_oGI. The canonical decomposition of GI is $\mathfrak{g}_2 = \mathfrak{so}(4) + \mathfrak{p}$. Now, the highest weight of the isotropy representation of $\mathfrak{so}(4) \cong \mathfrak{su}(2) + \mathfrak{su}(2)$ on $\mathfrak{p}$ is $\varpi_1(A_1) + 3\varpi_1(A_1')$, here we denote the former $\mathfrak{su}(2)$ by A_1 and the latter $\mathfrak{su}(2)$ by A_1'. When we restrict this representation to $\mathfrak{so}(3)$, we obtain the decomposition $\mathfrak{p} = V_{4\varpi_1(A_1)} + V_{2\varpi_1(A_1)}$, where $V_{4\varpi_1(A_1)}$ (resp. $V_{2\varpi_1(A_1)}$) is isomorphic to $T_oAI(3)$ (resp. $T_o\mathbf{R}P^3$) as $\mathfrak{so}(3)$-module. Since $AI(3)$ is not a reflective submanifold, $\mathbf{R}P^3$ is not a totally geodesic submanifold of GI. We take a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}_2$. Then we have the root decomposition of $\mathfrak{g}_2$ with respect to $\mathfrak{h}$:

$$\mathfrak{g}_2 = \mathfrak{h} \oplus \sum_{\alpha \in R^+(G_2)} \mathfrak{g}_\alpha ,$$

where $R^+(G_2) = R^+(GI)$. We take the subset D^+ of $R^+(G_2)$:

$$D^+ = \{\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\} .$$

Put

$$\mathfrak{g} := \mathfrak{h} \oplus \sum_{\alpha \in D^+} \mathfrak{g}_\alpha$$

and

$$\mathfrak{g}' := \mathfrak{h} \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{3\alpha_1+2\alpha_2} .$$

Then we have $\mathfrak{g} \cong \mathfrak{su}(3)$, $\mathfrak{g}' \cong \mathfrak{so}(4)$ and $\mathfrak{g} \cap \mathfrak{g}' \cong \mathfrak{u}(2)$. Thus $(\mathfrak{g}, \mathfrak{g} \cap \mathfrak{g}')$ is a symmetric pair and $\mathfrak{g}/\mathfrak{g} \cap \mathfrak{g}'$ is isomorphic to $\mathbf{C}P^2$.

(iii) By Proposition 5.1 and Corollary 4.7, $\mathbf{C}P^2$ is maximal. □

PROPOSITION 5.14. *Any maximal totally geodesic submanifold M of G_2 is isomorphic to one of GI, $SU(3)$ and $S^3 \cdot S^3$.*

PROOF. Case 1. M is an irreducible compact symmetric space of rank two.

(i) M is isomorphic to one of GI, $AI(3)$, $AI(3)/\mathbf{Z}_3$,$SU(3)$ and $SU(3)/\mathbf{Z}_3$.

(ii) From Table 3 GI is isomorphic to a polar of G_2. By Case 4 in Proposition 5.13, $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in D^+} \mathfrak{g}_\alpha$ is locally isomorphic to $SU(3)$ or $SU(3)/\mathbf{Z}_3$. Also the unit lattice $\mathfrak{a}_G$ coincides with $\mathfrak{a}_U$ by the similar discussion in Case 4 in Proposition 5.13. Thus G is isomorphic to $SU(3)$. By Proposition 5.2 $AI(3)$ (resp. $AI(3)/\mathbf{Z}_3$) is a totally geodesic submanifold of $SU(3)$ (resp. $SU(3)/\mathbf{Z}_3$).

(iii) By Proposition 5.2 and Proposition 5.13, $SU(3)$ and GI are maximal.

Case 2. $M = M_1 \times M_2$, M_1 and M_2 are compact symmetric spaces of rank one.

(i) M is isomorphic to $S^1 \times S^1$.

(ii) $S^1 \times S^1$ is isomorphic to $S^1 \cdot S^1$ and $S^1 \cdot S^1$ is a totally geodesic submanifold of $SO(4) \cong S^3 \cdot S^3$.

(iii) By Lemma 4.2 $S^1 \times S^1$ is not maximal.

Case 3. $M = S^m \cdot S^n$ $(m, n \geq 1)$.

(i) M is isomorphic to $S^n \cdot S^m (1 \leq n, m \leq 3)$.

(ii) $S^3 \cdot S^3$ is isomorphic to $SO(4)$.

(iii) By Lemma 2.5 $S^3 \cdot S^3$ is maximal.

Case 4. M is a compact symmetric space of rank one.

(i) M is isomorphic to $S^n (1 \leq n \leq 3)$, $\mathbf{R}P^n$ and $\mathbf{C}P^n (2 \leq n \leq 3)$.

(ii) S^3 is a totally geodesic submanifold of $SO(4)$. By Corollary 4.7 $\mathbf{R}P^3$ is a totally geodesic submanifold of $S^3 \cdot S^3$. If $\mathbf{C}P^3$ is a totally geodesic submanifold of G_2, then the isometry group $SU(4)$ of $\mathbf{C}P^3$ is a Lie subgroup of the isometry group $G_2 \times G_2$ of G_2. This is a contradiction because $SU(3)$ is a maximal Lie subgroup of G_2([1]). Thus $\mathbf{C}P^3$ is not in G_2. From Proposition 5.13 $\mathbf{C}P^2$ is a totally geodesic submanifold of GI.

(iii) By (ii), there is no maximal compact symmetric space of rank one. □

THEOREM 5.15. *All the maximal totally geodesic submanifolds in compact simply connected irreducible symmetric spaces of rank two are given by the following Table* 4.

TABLE 4. Maximal totally geodesic submanifolds in compact symmetric spaces of rank two

N	*Maximal totally geodesic submanifolds in N*
$AI(3)$	$\mathbf{R}P^2$, $S^1 \cdot S^2$
$SU(3)$	$AI(3)$, $SO(3)$,$\mathbf{C}P^2$, $S^1 \cdot S^3$
$AII(3)$	$SU(3)$, $\mathbf{C}P^3$, $\mathbf{H}P^2$, $S^1 \cdot S^5$
EIV	$AII(3)$, $\mathbf{H}P^3$, $S^1 \cdot S^9$, $\mathbf{O}P^2$
$G_2^o(\mathbf{R}^{n+2})$ $(n \geq 3)$	$G_2^o(\mathbf{R}^{n+1})$, $S^p \cdot S^q$ $(p+q=n)$, $\mathbf{C}P^{[\frac{n}{2}]}$
$Sp(2)$	$G_2^o(\mathbf{R}^5)$, $S^1 \cdot S^3$, $S^3 \times S^3$, S^4
$G_2(\mathbf{H}^4)$	$Sp(2)$, $\mathbf{H}P^2$, $S^1 \cdot S^5$, $S^4 \times S^4$, $G_2(\mathbf{C}^4)$
GI	$AI(3)$, $\mathbf{C}P^2$, $S^2 \cdot S^2$
G_2	GI, $SU(3)$, $S^3 \cdot S^3$
$G_2(\mathbf{C}^{n+2})$ $(n \geq 3)$	$G_2(\mathbf{C}^{n+1})$, $G_2(\mathbf{R}^{n+2})$, $\mathbf{C}P^k \times \mathbf{C}P^l$ $(k+l=n)$, $\mathbf{H}P^{[\frac{n}{2}]}$
$G_2(\mathbf{H}^{n+2})$ $(n \geq 3)$	$G_2(\mathbf{H}^{n+1})$, $G_2(\mathbf{C}^{n+2})$, $\mathbf{H}P^k \times \mathbf{H}P^l$ $(k+l=n)$
$DIII(5)$	$G_2^o(\mathbf{R}^8)$, $G_2(\mathbf{C}^5)$, $SO(5)$, $S^2 \times \mathbf{C}P^3$, $\mathbf{C}P^4$
$EIII$	$G_2(\mathbf{H}^4)/\mathbf{Z}_2$, $\mathbf{O}P^2$, $S^2 \times \mathbf{C}P^5$, $DIII(5)$, $G_2(\mathbf{C}^6)$, $G_2^o(\mathbf{R}^{10})$

References

[1] A. BOREL and J. SIEBENTHAL, Les sous-groupes fermes connexes de rang maximal des groupes de Lie clos, Comm. Math. Helv. **23** (1949/50), 200–221.

[2] N. BOURBAKI, *Lie Groups and Lie Algebras* 4-6, Springer-Verlag, New York, 2002.

[3] B. Y. CHEN and T. NAGANO, Totally geodesic submanifold of symmetric spaces I, Duke Math. J. **44** (1977), 745–755.

[4] B. Y. CHEN and T. NAGANO, Totally geodesic submanifold of symmetric spaces II, Duke Math. J. **45** (1978), 405–425.

[5] B. Y. CHEN and T. NAGANO, A Riemannian geometric invariant and its applications to a problem of Borel and Serre, Trans. Amer. Math. Soc. **308** (1988), 273–297.

[6] S. HELGASON, *Differential Geometry, Lie Group, and Symmetric Spaces*, Academic Press, New York, 1978.

[7] O. IKAWA and H. TASAKI, Totally geodesic submanifold of maximal rank in symmetric spaces, Japan. J. Math. **26** no. 1 (2000), 1–29.

[8] D. S. P. LEUNG, On the classification of reflective submanifolds of Riemannian symmetric spaces, Indiana. Univ. Math. J, **24** (1974), 327–339; Errata, Indiana. Univ. Math. J **24** (1975), 1199.

[9] T. NAGANO, The Involutions of Compact Symmetric Spaces, Tokyo J. Math. **11** (1988), 57–79.

[10] T. NAGANO, The Involutions of Compact Symmetric Spaces II, Tokyo J. Math. **15** (1992), 39–82.

[11] T. NAGANO, Geometry theory of symmetric space, RIMS Kokyuroku **1206**, (2001), 55–82.

[12] J. A. WOLF, Elliptic Spaces in Grassmann manifolds, Illinois. J. Math. **7** (1963), 447–462.

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