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On a Genus of a Closed Surface Containing a Brunnian Link

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Abstract. Let L be an n-component Brunnian link and F a genus g closed surface containing L. Then, we show that $g > (n+3)/3$.

1. Introduction

An n-component link $L = C_1 \cup \cdots \cup C_n$ ($n \geq 3$) in the 3-sphere S^3 is said to be *Brunnian* if it is non-trivial but $L - C_i$ is trivial for all i ([B]). Kazuaki Kobayashi observed that the Borromean rings are contained in a genus 3 Heegaard surface of S^3, and asked whether it is contained in a genus 2 Heegaard surface of S^3. In this article, we answer Kobayashi's question in the following theorem.

THEOREM 1. *Let L be an n-component Brunnian link and F a genus g closed surface containing L. Then, $g > (n+3)/3$ holds.*

Theorem 1 shows that the Borromean rings can not be contained in a genus 2 closed surface.

It seems from the proof that the estimation in Theorem 1 is very rough. The author would expect the following.

CONJECTURE 1. *Let L be an n-component Brunnian link and F a genus g closed surface containing L. Then, $g \geq n$ holds.*

We note that the inequality of Conjecture 1 is best possible since any n-component link can be contained in a genus n closed surface, which is constructed from peripheral tori of the link by $n-1$ tubings.

2. Proof

LEMMA 1. *Let $L = C_1 \cup \cdots \cup C_n$ be an n-component Brunnian link. Then, for any component C_i of L, there exists an essential tangle decomposing sphere S_i for L such that S_i intersects L only in C_i.*

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PROOF. Without loss of generality, it is sufficient to show this lemma only for $i = 1$. Since $L - C_1$ is a trivial link, there exists a splitting sphere S for $L - C_1$. We assume that S intersects C_1 minimally among all splitting spheres for $L - C_1$. Then, S − intN(L) is incompressible and ∂-incompressible in S^3 − intN(L), namely, S is an essential tangle decomposing sphere for L. □

By Lemma 1, the Borromean rings admits at least three essential tangle decompositions. In fact, it was shown in Theorem 4 of [O] that the Borromean rings admits exactly three essential tangle decompositions.

PROOF OF THEOREM 1. Let $L = C_1 \cup \cdots \cup C_n$ be an n-component Brunnian link and F be a genus g closed surface containing L. If $g \leq (n+3)/3$, then there exists a component of $F - L$ which is an open disk, say D, an open annulus, say A, or an open pair of pants, say P.

If an open disk D exists, then without loss of generality, let $\partial(D \cup C_1) = C_1$. Thus C_1 is trivial in the complement of $C_2 \cup \cdots \cup C_n$. Then, since $L - C_1$ is trivial by the Brunnian property of L, L is also trivial. This contradicts that L is Brunnian.

If an open annulus A exists, then without loss of generality, let $\partial(A \cup C_1 \cup C_2) = C_1 \cup C_2$. Thus C_1 is parallel to C_2 in the complement of $C_3 \cup \cdots \cup C_n$. Then, since $L - C_1$ is trivial by the Brunnian property, L is also trivial. This contradicts that L is Brunnian.

If an open pair of pants P exists, then without loss of generality, there are two possibilities;

CASE 1. C_2 bounds a punctured torus $P' = P \cup C_1 \cup C_2$ in F.

CASE 2. $C_1 \cup C_2 \cup C_3$ bounds a pair of pants $P'' = P \cup C_1 \cup C_2 \cup C_3$ in F.

In Case 1, we note that $P' - L$ is incompressible in $S^3 - L$, otherwise at least one of C_1 and C_2 bounds a disk D in the complement of the rest. This contradicts that L is Brunnian. By Lemma 1, there exists an essential tangle decomposing sphere S for L such that S intersects L in only C_1. We assume that S intersects P' minimally up to isotopy of S in the pair (S^3, L). Then, $S \cap P'$ consists of essential loops in P' which are disjoint from C_2. Let α be an innermost loop of $S \cap P'$ in S and δ be the corresponding innermost disk in S. By compressing P' along δ, we obtain a disk bounded by C_2 in $S^3 - L$. This contradicts that L is Brunnian.

In Case 2, a trivial link $L - (C_2 \cup C_3)$ is obtained from a trivial link $L - C_1$ by a band sum along a band $b \subset P''$. By 8.11 Corollary of [S], the band b is trivial, i.e. there exists a 2-sphere containing $L - C_1$ and b. Hence, L is trivial and contradicts that L is Brunnian. □

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