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Hilbert-Schmidt Hankel Operators and Berezin Iteration

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Abstract. Let H be a reproducing kernel Hilbert space contained in a wider space $L^2(X, \mu)$. We study the Hilbert-Schmidt property of Hankel operators H_g on H with bounded symbol g by analyzing the behavior of the iterated Berezin transform. We determine symbol classes $\mathcal{S}$ such that for $g \in \mathcal{S}$ the Hilbert-Schmidt property of H_g implies that $H_{\bar{g}}$ is a Hilbert-Schmidt operator as well and there is a norm estimate of the form $\|H_{\bar{g}}\|_{\mathrm{HS}} \leq C \cdot \|H_g\|_{\mathrm{HS}}$. Finally, applications to the case of Bergman spaces over strictly pseudo convex domains in $\mathbf{C}^n$, the Fock space, the pluri-harmonic Fock space and spaces of holomorphic functions on a quadric are given.

1. Introduction

Let X be a set with a measure μ and H be a closed subspace of $L^2(X, \mu)$. For any bounded measurable function g on X and the orthogonal projection P from $L^2(X, \mu)$ onto H the Hankel operator H_g resp. the Toeplitz operator T_g on H are define by:

$$H_g f := (I - P)(fg) \quad \text{and} \quad T_g := P(fg)\,. \tag{1.1}$$

Among a variety of examples the operators (1.1) have been treated intensively in the case of Bergman and Hardy spaces and spaces of harmonic or pluri-harmonic functions. The study of Toeplitz operators T_g or algebras generated by those require an analysis of the Hankel operators H_g and $H_{\bar{g}}$. In particular, the compactness or Schatten-p-properties of H_g and $H_{\bar{g}}$ are of importance to obtain spectral results and to determine Fredholmness of T_g, c.f. [10], [18], [21], [23], [24]. For a reproducing kernel Hilbert space H a general *symbol calculus* was introduced by Berezin [8], [9] which can be regarded as an *inverse quantization* and frequently has been applied to the analysis of the operators (1.1). In particular, the *Berezin symbol* $\tilde{g}$ of T_g was used to introduce the notion of *mean oscillation* MO(g) of g. At least for Bergman spaces over bounded symmetric domains or the Segal-Bargmann space there are characterizations in terms of the function MO(g) for H_g and $H_{\bar{g}}$ to belong to the ideals of Schatten-p-class or

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compact operators, c.f. [7], [10], [23]. As a matter of fact the assignment $g \mapsto \mathrm{MO}(g)$ is invariant under complex conjugation such that these characterizations hold for H_g and $H_{\bar{g}}$ simultaneously. In [24] the compactness of H_g and $H_{\bar{g}}$ was proved in the case of Bergman spaces over strictly pseudo convex domains Ω in $\mathbf{C}^n$ and smooth symbols g on Ω continuous up to the boundary. An analog theorem for the case of weighted harmonic Bergman spaces over the unit ball in $\mathbf{R}^n$ can be found in [22]. Schatten-p-class properties of the Hankel operators do not follow automatically, c.f. [22], [25]. On the one hand it was observed in [10], [21] (resp. [4]) that for the Segal-Bargmann space H and bounded symbol g the operator H_g is compact (resp. Hilbert-Schmidt) if and only if $H_{\bar{g}}$ is compact (resp. Hilbert-Schmidt). On the other hand, the existence of non-constant bounded holomorphic functions implies that such a result in general can not be true for Bergman spaces over bounded domains $X \subset \mathbf{C}^n$, c.f. [25]. Let $\mathcal{L}^2(H, H^\perp)$ denote the Hilbert-Schmidt operators from H to its orthogonal complement $H^\perp$ in $L^2(X, \mu)$ and with norm $\|\cdot\|_{\mathrm{HS}}$. Here, we determine spaces $\mathcal{S}$ of bounded measurable symbols such that:

(P) *For $g \in \mathcal{S}$ and $H_g \in \mathcal{L}^2(H, H^\perp)$ it follows that $H_{\bar{g}} \in \mathcal{L}^2(H, H^\perp)$ and there is a constant $C > 0$ with $\|H_{\bar{g}}\|_{HS} \leq C\|H_g\|_{HS}$.*

Following ideas in [4], we express $\|H_g\|_{\mathrm{HS}}$ by integral conditions on g and $\tilde{g}$. No further assumptions on X are required besides the existence of a reproducing kernel K. For a finite measure μ property (P) holds with $\mathcal{S} := L^2(X, V)$ and $C = 1$ where the *Berezin measure* V is defined by $dV(z) = K(z, z)d\mu(z)$ (c.f. Proposition 4.1).

There is a natural metric d on X induced by K and equivalent to the Bergman distance in the case of Bergman spaces H over bounded domains $X \subset \mathbf{C}^n$. We assume that a priori there is a second metric $\mathbf{d}$ on X related to d and turning the space $C(X)$ of continuous functions on X equipped with the compact-open topology into a Fréchet space. For symbols $g \in L^\infty(X)$ such that $\|H_g\|_{\mathrm{HS}} < \infty$ the following can be said about the sequence of iterated Berezin transforms. Theorem I is essential in the proof of the Theorems II and III.

THEOREM I. *The sequence $(B^j g)_{j \in \mathbf{N}} \subset C(X, \mathbf{d})$, where B denotes the Berezin transform, has cluster points $h \in C(X, \mathbf{d})$ with $Bh = h$.*

We observe that $\mathcal{S} := L^2(X, V)$ is an *invariant space* for the Berezin transform. Moreover, for any symbol $g \in \mathcal{S}$ the invariance $g = \tilde{g}$ implies that $g \equiv 0$ (see example 3.1). In fact this observation can be used to obtain a defining property for $\mathcal{S}$ in (P):

THEOREM II. *Let $\mathcal{S}_0 \subset L^\infty(X)$ such that*:
(i) *$\mathcal{S}_0$ is asymptotically invariant under the Berezin transform* (*c.f. Definition* 3.2).
(ii) *For $h \in \mathcal{S}_0$ the equality $h = \tilde{h}$ implies that $H_{\bar{h}} = 0$.*
Then (P) *holds with $\mathcal{S} := \mathcal{S}_0$ and $C := 2$.*

In the case of the Segal-Bargmann space H_{h} assumptions (i) and (ii) of Theorem II are fulfilled with $\mathcal{S}_0 := L^\infty(\mathbf{C}^n)$. Here $\mathcal{S}_0$ is invariant under complex conjugation and (P) holds in a symmetric way, c.f. [4] (for invariance under Berezin transform [1], [15]). In our analysis iteration of the Berezin transform B plays a crucial role. Let $\Omega \subset \mathbf{C}^n$ be a *strictly pseudo convex domain* with C^3-boundary and $H = H^2(\Omega, \mu)$ a weighted Bergman space over Ω with $K(x, x) > 0$. For $f \in C(\overline{\Omega})$ the sequence of iterated Berezin transforms converges uniformly on the closure $\overline{\Omega}$ to a unique *fix point* $f_0 \in C(\overline{\Omega})$ of B preserving the boundary values of f, c.f. [2]. Let $C_0(\Omega)$ denote the space of continuous functions on $\overline{\Omega}$ vanishing at the boundary.

THEOREM III. $\mathcal{S}_0 := C_0(\Omega)$ *fulfills the condition* (i) *and* (ii) *of Theorem* II.

To give an example of a non-symmetric situation we consider the Banach algebra:

$$\mathcal{A}_{\mathrm{ah}}(\Omega) := \left\{ f \in C(\overline{\Omega}) : \; f_{|\Omega} \; \text{ is anti-holomorphic} \right\}$$

and set $\mathcal{S}_0 := C_0(\Omega) \oplus \mathcal{A}_{\mathrm{ah}}(\Omega)$. This choice again leads to a solution of (P) whereas the symbol space $\mathcal{S}_{0,c} := \{\bar{g} : g \in \mathcal{S}_0\}$ in general does not. This can be seen by the fact that there are no non-zero Hilbert-Schmidt Hankel operators on the Bergman space of the open unit ball in $\mathbf{C}^n$ with anti-holomorphic symbols when $n \geq 2$, c.f. [25]. We examine the *pluri-harmonic Fock space* H_{ph} on $\mathbf{C}^n$. With $g \in L^\infty(\mathbf{C}^n)$ and the pluri-harmonic Hankel operator H_g^{ph} it holds $\|H_{\bar{g}}^{\mathrm{ph}}\|_{\mathrm{HS}} \leq \sqrt{2} \cdot \|H_g^{\mathrm{ph}}\|_{\mathrm{HS}}$ and the Hilbert-Schmidt property of the corresponding Hankel operators H_g^{h} on the *Fock space* H_{h} and H_g^{ph} on H_{ph} are related. As an application of Theorem II we show that $H_g^h \in \mathcal{L}^2(H_h, H_h{}^\perp)$ implies that H_g^{ph} and $H_{\bar{g}}^{\mathrm{ph}}$ are of Hilbert-Schmidt type as well and

$$\max\left\{ \|H_g^{\mathrm{ph}}\|_{\mathrm{HS}}, \|H_{\bar{g}}^{\mathrm{ph}}\|_{\mathrm{HS}} \right\} \leq \sqrt{5 \cdot \min\left\{ \|H_g^{\mathrm{h}}\|_{\mathrm{HS}}^2, \|H_{\bar{g}}^{\mathrm{h}}\|_{\mathrm{HS}}^2 \right\}}. \tag{1.2}$$

It was remarked in [20] that H_{h} arises naturally by *pairing of polarizations* from the real and Käler polarization on the cotangent bundle $T^*(\mathbf{R}^n) \cong \mathbf{C}^n$. The Euclidean space $\mathbf{R}^n$ can be replaced with the n-dimensional *sphere* $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$ or the *complex projective space* $P\mathbf{C}^n$. Then this method leads to a family of reproducing kernel Hilbert spaces of holomorphic functions on a quadric in $\mathbf{C}^{n+1}$ resp. on a space of $(n+1) \times (n+1)$ complex matrices parametrized by two real parameters. Several aspects of the analysis on these spaces are treated in [5]. In the final part of this paper we are interested in the asymptotic behavior of the *Berezin measure* in these examples. As an application of the general theory we determine a class of Hilbert-Schmidt Hankel operators in the sphere and complex projective space case.

2. Preliminaries

Let $L^2(X, \mu)$ denote the classes of μ-square integrable functions on a measure space $(X, \mathcal{F}, \mu)$. We write $\langle \cdot, \cdot \rangle$ (resp. $\| \cdot \|$) for the inner product (resp. norm) of $L^2(X, \mu)$.

A linear space H of μ-square integrable functions on X is said to be closed in $L^2(X, \mu)$ iff the canonical projection $p : H \to L^2(X, \mu)$ is injective with closed range and H is identified with $p(H)$. We write $P : L^2(X, \mu) \to H$ and $Q := I - P$ for the orthogonal projection onto H and its orthogonal complement $H^{\perp}$ respectively. Assume, that H admits a *reproducing kernel* function, i.e. there is a $\mathcal{F}\otimes\mathcal{F}$-measurable function $K : X\times X \to \mathbf{C}$ such that $X \ni x \mapsto K(x,x) \in (0,\infty)$ is measurable and for all $x, y \in X$:

(i) $K(\cdot, x) \in H$,

(ii) $\overline{K(x, y)} = K(y, x)$,

(iii) *Reproducing property*: For all $f \in H$ it holds $f(x) = \langle f, K(\cdot, x)\rangle$.

By (i) and for any $x \in X$ the *normalized kernel* is given by

$$k_x := K(\cdot, x)\cdot \|K(\cdot, x)\|^{-1} \in H \tag{2.1}$$

where by (i), (iii): $\|K(\cdot, x)\| = K(x, x)^{\frac{1}{2}} > 0$. We define a *symbol space*:

$$\mathcal{T}(X) := \{f : L^2(X,\mu) : \ f k_x \in L^2(X,\mu), \ \forall\, x \in X\}.$$

DEFINITION 2.1 (Berezin transform). For $f \in \mathcal{T}(X)$ the Berezin transform (BT) $\tilde{f} : X \to \mathbf{C}$ is defined by:

$$\tilde{f}(\lambda) := \langle f k_\lambda, k_\lambda\rangle. \tag{2.2}$$

Naturally (2.2) extends to operators on H such that $\tilde{f}$ and $\widetilde{T_f}$ coincide and it can be regarded as an *inverse quantization*. If T_f is bounded $\tilde{f}$ clearly is bounded by $\|T_f\|$. On functions (BT) is an integral operator with positive kernel and commutes with the complex conjugation: $\bar{\tilde{f}} = \tilde{\bar{f}}$. We write M_g for the multiplication with a symbol g and $\mathcal{L}(V, W)$ for the continuous operators between topological vector spaces V and W. We also use the shorter notation $\mathcal{L}(V) := \mathcal{L}(V, V)$.

DEFINITION 2.2 (Hankel and Toeplitz operators). For $g \in L^\infty(X)$ the Hankel operator H_g and the Toeplitz operator T_g with symbols g are given by $H_g := QM_g \in \mathcal{L}(H, H^{\perp})$ and $T_g := PM_g \in \mathcal{L}(H)$.

Definition 2.2 can be generalized to classes of unbounded symbols. Then H_g and T_g will be unbounded in general. On X we consider the *Berezin measure* V:

$$dV(x) := K(x, x)d\mu(x). \tag{2.3}$$

There is a *trace formula* for positive operators on H which leads to a characterization of the Hilbert-Schmidt Hankel operators by an integral condition with respect to V. We write $\|\cdot\|_{\mathrm{HS}}$ for the *Hilbert-Schmidt norm*.

LEMMA 2.1. *Let g be a measurable function on X such that $M_g P$ is a bounded operator on $L^2(X,\mu)$, then* (a) *and* (b) *below are equivalent*:

(a) $H_g : H \to H^{\perp}$ *is a Hilbert-Schmidt operator* (*we write* $H_g \in \mathcal{L}^2(H, H^{\perp})$).

(b) $I := \int_X \|H_g k_x\|^2 dV(x) < \infty$.

If (a) *and* (b) *are valid, then* $\sqrt{I} = \|H_g\|_{HS}$.

PROOF. Fix an orthonormal basis (ONB) $[e_j : j \in \mathbf{N}_0]$ in H. Because QM_gP is bounded, there is $T \in \mathcal{L}(H)$ such that $(QM_gP)^*(QM_gP) = T^*T$ on H. Hence

$$I = \int_X \|H_g k_x\|^2 dV(x) = \int_X \langle TK(\cdot, x), TK(\cdot, x)\rangle d\mu(x)\,.$$

From (i)–(iii) we obtain for all $x \in X$:

$$TK(\cdot, x) = \sum_{j=0}^{\infty} \langle TK(\cdot, x), e_j\rangle e_j = \sum_{j=0}^{\infty} \overline{[T^*e_j](x)}\, e_j\,. \tag{2.4}$$

By inserting (2.4) into the integral above and using the monotone convergence theorem together with $\|T^*\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$ one obtains that:

$$I = \int_X \sum_{j=0}^{\infty} \big|[T^*e_j](x)\big|^2 d\mu(x) = \sum_{j=0}^{\infty} \|T^*e_j\|^2 = \sum_{j=0}^{\infty} \|H_g e_j\|^2\,.$$

Hence the equivalence of (a) and (b) and $\sqrt{I} = \|H_g\|_{HS}$ are proved. □

REMARK 2.1. The analogous result of Lemma 2.1 holds if we replace H_g by the Toeplitz operator T_g in (a) and (b) above. Note that $T_g^* = T_{\bar{g}}$ in Lemma 2.1.

By a further decomposition of the integral expression in Lemma 2.1 (b), the Berezin symbol of g naturally appears.

LEMMA 2.2. *For $g \in L^\infty(X)$ and with I defined as in Lemma* 2.1 (b), *it holds*:

$$I = \int_X \Big\{\big\|P[\bar{g}k_z] - \overline{\tilde{g}(z)}k_z\big\|^2 + \big|g(z) - \tilde{g}(z)\big|^2\Big\} dV(z)\,. \tag{2.5}$$

The right hand side of (2.5) *is finite if and only if the left hand side is finite.*

PROOF. By *Fubini's theorem* and using (2.3):

$$\begin{aligned} I &= \int_X \big\|H_g K(\cdot, \lambda)\big\|^2 d\mu(\lambda) \qquad (2.6)\\ &= \int_X \int_X \big|g(z)K(z, \lambda) - P\big[gK(\cdot, \lambda)\big](z)\big|^2 d\mu(z) d\mu(\lambda)\\ &= \int_X \int_X \big|\overline{g(z)}K(\lambda, z) - P\big[\bar{g}K(\cdot, z)\big](\lambda)\big|^2 d\mu(\lambda) d\mu(z)\,. \end{aligned}$$

In the last equality we have used (ii) as well as

$$\overline{P\big[gK(\cdot, \lambda)\big](z)} = P\big[\bar{g}K(\cdot, z)\big](\lambda) \tag{2.7}$$

which can be deduced from (i)–(iii) by a straightforward calculation. Using $\tilde{\bar{g}} = \bar{\tilde{g}}$ we have:

$$\big\langle P\big[\bar{g}K(\cdot, z)\big], K(\cdot, z)\big\rangle = \big\langle \bar{g}K(\cdot, z), K(\cdot, z)\big\rangle = \big\langle \overline{\tilde{g}(z)}K(\cdot, z), K(\cdot, z)\big\rangle$$

which can be written as $\langle P\big[\bar{g}K(\cdot,z)\big] - \overline{\tilde{g}(z)}K(\cdot,z), K(\cdot,z)\rangle = 0$. From the *Pythagorean theorem* we obtain for the inner integral on the right hand side of (2.6) and fixed $z \in X$:

$$\begin{aligned}
&\int_X \big|\overline{g(z)}K(\lambda,z) - P\big[\bar{g}K(\cdot,z)\big](\lambda)\big|^2 d\mu(\lambda) \\
&\quad = \int_X \Big|\Big\{\overline{g(z)} - \overline{\tilde{g}(z)}\Big\}K(\lambda,z) - \Big\{P\big[\bar{g}K(\cdot,z)\big](\lambda) - \overline{\tilde{g}(z)}K(\lambda,z)\Big\}\Big|^2 d\mu(\lambda) \\
&\quad = \int_X \Big|\Big\{\overline{g(z)} - \overline{\tilde{g}(z)}\Big\}K(\lambda,z)\Big|^2 d\mu(\lambda) \\
&\qquad + \int_X \Big|P\big[\bar{g}K(\cdot,z)\big](\lambda) - \overline{\tilde{g}(z)}K(\lambda,z)\Big|^2 d\mu(\lambda) \\
&\quad = K(z,z)\Big\{\big|\overline{g(z)} - \overline{\tilde{g}(z)}\big|^2 + \big\|P[\bar{g}k_z] - \overline{\tilde{g}(z)}k_z\big\|^2\Big\}.
\end{aligned}$$

Finally, by inserting this expression into (2.6) the assertion follows. □

COROLLARY 2.1. *Let $g \in L^\infty(X)$ such that $H_g \in \mathcal{L}^2(H, H^\perp)$, then $g - \tilde{g} \in L^2(X, V)$.*

PROOF. Lemma 2.1 (b) holds and the assertion directly follows from Lemma 2.2. □

In order to derive some further decomposition of the integral I we prove:

LEMMA 2.3. *Let $g \in L^\infty(X)$, then*:

$$I_1 := \int_X \big\|P[\bar{g}k_z] - \overline{\tilde{g}(z)}k_z\big\|^2 dV(z) = \int_X \big\|P[gk_\lambda] - \tilde{g}k_\lambda\big\|^2 dV(\lambda)\,.$$

The right hand side is finite if and only if the left hand side is finite.

PROOF. By using *Fubini's theorem* and (2.7) again one concludes that:

$$\begin{aligned}
I_1 &= \int_X \int_X \big|P[\bar{g}K(\cdot,z)](\lambda) - \overline{\tilde{g}(z)}K(\lambda,z)\big|^2 d\mu(\lambda)d\mu(z) \\
&= \int_X \int_X \big|P[gK(\cdot,\lambda)](z) - \tilde{g}(z)K(z,\lambda)\big|^2 d\mu(z)d\mu(\lambda) \\
&= \int_X \big\|P[gk_\lambda] - \tilde{g}k_\lambda\big\|^2 dV(\lambda)\,.
\end{aligned}$$
□

Combinings Lemmas 2.1, 2.2 and 2.3 we can prove a *decomposition formula* for the Hilbert-Schmidt norm of Hankel operators:

PROPOSITION 2.4. *Let $g \in L^\infty(X)$ such that H_g is a Hilbert-Schmidt operator. Then $H_{\tilde{g}}$, $T_{g-\tilde{g}}$ and $H_{g-\tilde{g}}$ are of Hilbert-Schmidt type as well and:*

$$\|H_g\|_{HS}^2 = \|T_{g-\tilde{g}}\|_{HS}^2 + \|H_{\tilde{g}}\|_{HS}^2 + \|g - \tilde{g}\|_{L^2(X,V)}^2\,. \tag{2.8}$$

PROOF. From Lemma 2.1, 2.2 and 2.3 we have:

$$\begin{aligned}\|H_g\|^2_{\mathrm{HS}} - \|g-\tilde{g}\|^2_{L^2(X,V)} &= \int_X \big\|P\big[\bar{g}k_z\big] - \overline{\tilde{g}(z)}k_z\big\|^2 dV(z)\\ &= \int_X \big\|P\big[gk_\lambda\big] - \tilde{g}k_\lambda\big\|^2 dV(\lambda)\,.\end{aligned}$$

After decomposing the integrand into an orthogonal sum:

$$\big\|P\big[gk_\lambda\big] - \tilde{g}k_\lambda\big\|^2 = \big\|T_{g-\tilde{g}}k_\lambda\big\|^2 + \big\|H_{\tilde{g}}k_\lambda\big\|^2$$

and using Lemma 2.1 and Remark 2.1 we conclude that:

$$\begin{aligned}\|H_g\|^2_{\mathrm{HS}} - \|g-\tilde{g}\|^2_{L^2(X,V)} &= \int_X \Big\{\big\|T_{g-\tilde{g}}k_\lambda\big\|^2 + \big\|H_{\tilde{g}}k_\lambda\big\|^2\Big\} dV(\lambda)\\ &= \big\|T_{g-\tilde{g}}\big\|^2_{\mathrm{HS}} + \big\|H_{\tilde{g}}\big\|^2_{\mathrm{HS}}\,.\end{aligned}$$

□

3. Iteration of the Berezin transform

For $\lambda \in X$ we consider the *rank one projection* $P_\lambda := \langle\cdot, k_\lambda\rangle k_\lambda$ on $L^2(X,\mu)$ where k_λ denotes the normalized kernel (2.1), c.f. [11], [12]. With this notation the Berezin transform $\tilde{f}$ of a symbols $f \in L^\infty(X)$ can be expressed as an *operator trace*:

$$\tilde{f}(\lambda) = \langle fk_\lambda, k_\lambda\rangle = \langle M_f P_\lambda k_\lambda, k_\lambda\rangle = \mathrm{trace}(M_f P_\lambda)\,. \tag{3.1}$$

In particular, it was observed in [11], [12] that $\tilde{f}$ has some Lipschitz property. Recall that the *trace norm* $\|\cdot\|_{\mathrm{trace}}$ is defined by $\|A\|_{\mathrm{trace}} := \mathrm{trace}\sqrt{A^*A}$ where $\sqrt{A^*A}$ is the unique square root of A^*A. By a standard estimate it follows from (3.1):

$$\big|\tilde{f}(\lambda_1) - \tilde{f}(\lambda_2)\big| \le \|f\|_\infty \big\|P_{\lambda_1} - P_{\lambda_2}\big\|_{\mathrm{trace}}\,. \tag{3.2}$$

Motivated by (3.2) we consider the function $d : X \times X \to \mathbf{R}$ given by:

$$d(\lambda_1,\lambda_2) := \big\|P_{\lambda_1} - P_{\lambda_2}\big\|_{\mathrm{trace}}\,.$$

The following formula was proved in [11], THEOREM 1 and the case of any reproducing kernel Hilbert space $H \subset L^2(X,\mu)$ of the type we are considering here:

PROPOSITION 3.1 ([11]). *For $a, b \in X$ it holds*:

$$d(a,b) = 2\Big\{1 - \big|\langle k_a, k_b\rangle\big|^2\Big\}^{\frac{1}{2}} = 2\Big\{1 - \frac{\big|K(a,b)\big|^2}{K(a,a)K(b,b)}\Big\}^{\frac{1}{2}}\,. \tag{3.3}$$

COROLLARY 3.1. *d is a metric if for $a, b \in X$ there is $h \in H$ with $h(a) = 0 \neq h(b)$.*

PROOF. We only show that $[d(a,b) = 0] \Rightarrow [a = b]$. (3.3) vanishes iff $|\langle k_a, k_b\rangle| = 1$ and by the Cauchy-Schwartz inequality together with $\|k_a\| = \|k_b\| = 1$ it follows that $k_a =$

$\lambda \cdot k_b$ where $|\lambda| = 1$. For $a \neq b$ let $h \in H$ with $h(a) = 0 \neq h(b)$. Applying the reproducing property of K and $K(b,b) > 0$ we obtain the contradiction $0 = h(b) \cdot \bar{\lambda} \cdot K(b,b)^{-\frac{1}{2}}$. □

Hence d is a metric in the case where H is "big enough". From now on we assume that H satisfies the condition of Corollary 3.1 such that (X, d) becomes a metric space. In our applications X a priori will be a metric space carrying a second metric $\mathbf{d}$ and we also assume this in general. Both metrics $\mathbf{d}$ and d should be related through the assumption that the embedding

$$\big(X, \mathbf{d}\big) \hookrightarrow \big(X, d\big) \tag{3.4}$$

is continuous, c.f. Corollary 3.2. Further, let $(X, \mathbf{d})$ fulfill (P1)–(P3):

- (P1) $(X, \mathbf{d})$ is *hemi-compact*, i.e. there is a *fundamental sequence* $(K_n)_{n\in\mathbf{N}}$ of *compact sets* in $(X, \mathbf{d})$ such that $K_n \subset K_{n+1}$ and $X = \bigcup_{n\in\mathbf{N}} K_n$.
- (P2) $(X, \mathbf{d})$ is a *k-space*, i.e. a functions f on $(X, \mathbf{d})$ is continuous if and only if its restriction to any compact subset $K \subset X$ is continuous.
- (P3) All open set in $(X, \mathbf{d})$ have strictly positive volume with respect to μ.

COROLLARY 3.2. *Let $K : X \times X \to \mathbf{C}$ be continuous in the product topology with respect to the metric $\mathbf{d}$ on X. Then there is a continuous embedding* (3.4).

We remark that the assumption of Corollary 3.2 typically holds for reproducing kernel Hilbert spaces $H := \mathcal{N} \cap L^2(X, \mu)$ where $\mathcal{N}$ is nuclear in the F-space $C(X, \mathbf{d})$. In the case of a bounded domain $X \subset \mathbf{C}^n$ and with the usual Bergman space H over X, the function d induces the Euclidean topology $\mathbf{d}(a, b) := |a - b|$. Some relation between d and the *Bergman distance* are discussed in [19].

LEMMA 3.1. *Let $f \in L^\infty(X)$, then $\tilde{f}$ is continuous in the topology of* $(X, \mathbf{d})$.

PROOF. By (3.4) both d and $\mathbf{d}$ induce the same topology on compact sets $K \subset (X, \mathbf{d})$. From (3.2) we conclude that the restriction of $\tilde{f}$ to K is continuous with respect to $\mathbf{d}$ and from $(P2)$ it follows that $\tilde{f} \in C(X, \mathbf{d})$. □

Let us also write $Bf := \tilde{f}$ for the Berezin transform, when it is considered as an operator. From (3.2) it follows that B can be regarded as bounded operator:

$$B : L^\infty(X) \to BC(X, d), \quad \text{and} \quad \|B\| \leq 1$$

where $\mathrm{BC}(X, d)$ (resp. $\mathrm{BC}(X, \mathbf{d})$) are the bounded functions in $C(X, d)$ (resp. in $C(X, \mathbf{d})$) equipped with the sup-norm. From (3.4) one has continuous embeddings:

$$C\big(X, d\big) \hookrightarrow C\big(X, \mathbf{d}\big) \quad \text{and} \quad BC\big(X, d\big) \hookrightarrow BC\big(X, \mathbf{d}\big). \tag{3.5}$$

Here $C(X, \mathbf{d})$ is a *Fréchet space* (F-space) with respect to the *compact-open topology* by assumptions (P1) and (P2) on the metric $\mathbf{d}$.

LEMMA 3.2. *Let* $(g_n)_n \subset BC(X, \mathbf{d})$ *be a norm-bounded sequence converging in* $C(X, \mathbf{d})$ *to* $g \in BC(X, \mathbf{d})$. *Then it follows that* $\lim_{n\to\infty} Bg_n = Bg$ *in* $C(X, \mathbf{d})$ *and* $Bg \in BC(X, \mathbf{d})$.

PROOF. Fix $c > 0$ such that $\|g_n\|_\infty \le c$ for all $n \in \mathbf{N}$ and let $T \subset (X, \mathbf{d})$ be compact. For $n \in \mathbf{N}$ and $x \in X$ one has:

$$\big|\big[Bg_n - Bg\big](x)\big| \le \int_X \big|g_n - g\big| \, \frac{|K(\cdot, x)|^2}{K(x,x)} d\mu =: (*)\,.$$

Let $(K_m)_m$ denote the sequence of compact sets in (P1) and fix $m \in \mathbf{N}$, then:

$$(*) \le \sup_{K_m} |g_n - g| + 2c \int_{X\setminus K_m} \frac{|K(\cdot, x)|^2}{K(x,x)} d\mu =: C_{n,m}(x)\,.$$

For fixed $x \in T$ and $m \to \infty$ the sequence $(q_m)_m \subset C(X, \mathbf{d})$ given by:

$$q_m(x) := \int_{X\setminus K_m} \frac{|K(\cdot, x)|^2}{K(x,x)} d\mu = \widetilde{\chi_{X\setminus K_m}}(x)$$

is monotonely decreasing to 0. By *Dini's Lemma* the convergence is uniform on T. For any $\varepsilon > 0$ fix $m_0 \in \mathbf{N}$ with $\sup_{x\in T} |q_m(x)| \le \varepsilon$ for all $m \ge m_0$. Finally, we can choose $n_0 \in \mathbf{N}$ with $\sup_{K_{m_0}} |g_n - g| < \varepsilon$ for $n \ge n_0$. Uniformly on T this leads to $C_{m_0,n}(x) \le \varepsilon(1 + 2c)$ for $n \ge n_0$. Because g is bounded it follows that $Bg \in \mathrm{BC}(X, \mathbf{d})$. □

DEFINITION 3.1. (Iterated Berezin transform). For $f \in L^\infty(X)$ we define the Berezin transforms inductively by:

$$f^{(0)} := f \quad \text{and} \quad f^{(j+1)} := \widetilde{f^{(j)}}, \qquad j \ge 0\,.$$

COROLLARY 3.3. *Let* $g \in L^\infty(X)$ *such that* H_g *is a Hilbert-Schmidt operator, then all the operators* $H_{g^{(m)}}$ *for* $m \in \mathbf{N}$ *are Hilbert-Schmidt operators with*:

$$\big\|H_{g^{(m)}}\big\|_{HS} \le \big\|H_g\big\|_{HS}\,. \tag{3.6}$$

Moreover:

$$\sum_{j=0}^{\infty} \big\|g^{(j)} - g^{(j+1)}\big\|^2_{L^2(X,V)} \le \big\|H_g\big\|^2_{HS} < \infty\,. \tag{3.7}$$

PROOF. Both, (3.6) and (3.7) follow by iteration of (2.8). □

For $\mathcal{S} \subset C(X, \mathbf{d})$ we write $\mathrm{Fix}\,(\mathcal{S}) := \{f \in \mathcal{S} : Bf = f\}$ for the *fix points* of B in $\mathcal{S}$. Further, let $\overline{\mathcal{S}}$ be the closure of $\mathcal{S}$ in the F-space $C(X, \mathbf{d})$. For $g \in L^\infty(X)$, we define

$$\mathcal{S}_g := \big\{g^{(j)} : j \in \mathbf{N}\big\} \subset C(X, \mathbf{d}) \tag{3.8}$$

for the *B-invariant space* of *iterated Berezin transforms* of g. Combining Corollary 3.3 with general properties of B we can prove:

PROPOSITION 3.2. *Let $g \in L^\infty(X)$ such that the Hankel operator H_g is of Hilbert-Schmidt type, then Fix $(\overline{\mathcal{S}_g}) \neq \emptyset$. Moreover, $\overline{\mathcal{S}_g} \setminus \mathcal{S}_g \subset Fix(\overline{\mathcal{S}_g})$.*

PROOF. For any $k \in \mathbf{N}$ it is clear that $\|g^{(k)}\|_\infty \leq \|g\|_\infty$ and with $\lambda_1, \lambda_2 \in X$ it holds:

$$\left|g^{(k)}(\lambda_1) - g^{(k)}(\lambda_2)\right| \leq \|g\|_\infty d\big(\lambda_1, \lambda_2\big)\,.$$

This shows that $\mathcal{S}_g \subset C(X, \mathbf{d})$ is *bounded* and *equi-continuous*. Hence there is a subsequence $(g^{(m_k)})_k$ which is uniformly compact convergent to some $h \in \overline{\mathcal{S}_g}$. We show next that $h \in$ Fix $(\overline{\mathcal{S}_g})$. First let us note that by Lemma 3.2:

$$\lim_{k\to\infty} \widetilde{g^{(m_k)}}(x) = \tilde{h}(x) \tag{3.9}$$

where the convergence in (3.9) is uniformly compact on $(X, \mathbf{d})$. From our assumption on H_g and (3.7) we conclude that $\lim_{k\to\infty} \left\|g^{(m_k)} - \widetilde{g^{(m_k)}}\right\|_{L^2(X,V)} = 0$. Hence there is $A \subset X$ with $V(X \setminus A) = 0$ and a subsequence of $(g^{(m_k)})_k$ (which we denote by $(g^{(m_k)})_k$ again) such that for all $x \in A$:

$$\lim_{k\to\infty} \left\{g^{(m_k)}(x) - \widetilde{g^{(m_k)}}(x)\right\} = 0\,. \tag{3.10}$$

By the definition of h, (3.9) and (3.10) it follows for $x \in A$ that:

$$h(x) = \lim_{k\to\infty} g^{(m_k)}(x) = \lim_{k\to\infty} \widetilde{g^{(m_k)}}(x) = \tilde{h}(x)\,. \tag{3.11}$$

Because of $K(x, x) > 0$ for all $x \in X$ we obtain that $\mu(X \setminus A) = 0$ and by (P3) the complement $X \setminus A$ cannot contain an open subset of $(X, \mathbf{d})$. Thus A must be dense in $(X, \mathbf{d})$. Finally, the continuity of h together with (3.11) imply that $h \in$ Fix $(\overline{\mathcal{S}_g})$.

The second assertion follows by the same argument and the fact that the functions in the complement $\overline{\mathcal{S}_g} \setminus \mathcal{S}_g$ are limit points of a subsequences of $(g^{(k)})_k \subset C(X, \mathbf{d})$. □

We remark that in contrary to Fix$(\overline{\mathcal{S}_g})$ the set $\overline{\mathcal{S}_g} \setminus \mathcal{S}_g$ might be empty.

DEFINITION 3.2. We call a subspace $\mathcal{S} \subset L^\infty(X)$ asymptotically invariant under B iff for any $f \in \mathcal{S}$ the inclusion $\overline{\mathcal{S}_f} \subset \mathcal{S}$ holds.

By our results above it follows that symbols of Hilbert-Schmidt Hankel operators generate spaces asymptotically invariant under B:

COROLLARY 3.3. *Let $g \in L^\infty(X)$ such that H_g is a Hilbert-Schmidt operator, then $\overline{\mathcal{S}_g}$ is asymptotically invariant under B.*

PROOF. Let $f \in \overline{\mathcal{S}_g}$ be arbitrary. For $f \in \mathcal{S}_g$ it is clear that $\overline{\mathcal{S}_f} \subset \overline{\mathcal{S}_g}$. In the case where $f \in \overline{\mathcal{S}_g} \setminus \mathcal{S}_g \subset$ Fix $(\overline{\mathcal{S}_g})$ it follows that $\overline{\mathcal{S}_f} = \mathcal{S}_f = \{f\} \subset \overline{\mathcal{S}_g}$. □

Further examples of spaces asymptotically invariant under B are obviously given by the fix point set Fix $(\mathcal{S})$ of any subspace $\mathcal{S} \subset L^{\infty}(X)$ or by the "*eventually fix points*":

$$\{f \in \mathcal{S} : \exists \;\; j \in \mathbf{N} \;\text{ such that }\;\; f^{(j)} = f^{(j+1)}\}.$$

EXAMPLE 3.1. Let μ be a finite measure on X and fix $g \in L^2(X, V)$. By a straightforward calculation one obtains that:

$$\int_{X^3} \frac{1}{K(y,y)} \frac{|k_u(\lambda)|^2}{K(u,u)} \frac{|k_\lambda(y)|^2}{K(\lambda,\lambda)} dV(y)dV(\lambda)dV(u) = \mu(X) < \infty .$$

By Tonelli's theorem, the function:

$$L(u,y) := \frac{1}{K(y,y)} \int_X |k_u(\lambda)|^2 |k_\lambda(y)|^2 d\mu(\lambda)$$

is finite for a.e. $(u, y) \in X^2$ with respect to the product measure $V \otimes V$. Moreover,

$$\begin{aligned} \|\tilde{g}\|^2_{L^2(X,V)} &= \int_{X^3} g(u)\overline{g(y)} \left|k_\lambda(u)\right|^2 \left|k_\lambda(y)\right|^2 d\mu(u)d\mu(y)dV(\lambda) \\ &= \int_{X^3} g(u)\overline{g(y)} \left|k_u(\lambda)\right|^2 \left|k_\lambda(y)\right|^2 d\mu(\lambda)dV(u)d\mu(y) \\ &= \int_{X\times X} g(u)\overline{g(y)} L(u,y) dV\otimes V(u,y) . \end{aligned}$$

By Cauchy-Schwartz inequality and $\int_X L(u,y)dV(u) = \int_X L(u,y)dV(y) = 1$:

$$\|\tilde{g}\|^2_{L^2(X,V)} \leq \|g\|^2_{L^2(X,V)} . \tag{3.12}$$

Equality in (3.12) only holds if $G_1(u,y) := g(u)$ and $G_2(u,y) := g(y)$ are linear dependent showing that g is constant. By an easy consequence of Remark 2.1 together with $T_1 = id$ the measure V cannot be finite whenever H is infinite dimensional. In this case $g \equiv 0$ and there are no non-trivial functions in $L^2(X, V)$ invariant under B.

4. Hilbert-Schmidt Hankel operators

We apply our previous results to prove Theorem II of the Introduction:

PROPOSITION 4.1. *For $g \in L^2(X, V)$, the operator H_g is of Hilbert-Schmidt type and*:

$$\|H_g\|_{HS} = \|H_{\tilde{g}}\|_{HS} \leq \|g\|_{L^2(X,V)} . \tag{4.1}$$

PROOF. For $f \in L^2(X, \mu)$ it follows from $|[Pf](u)|^2 \leq \|Pf\|^2 \cdot K(u,u)$ that:

$$\|M_g Pf\|^2 \leq \|Pf\|^2 \int_X |g(u)|^2 K(u,u)d\mu(u) \leq \|f\|^2 \|g\|^2_{L^2(X,V)} .$$

Hence $M_g P$ is a bounded operator on $L^2(X,\mu)$ and by Lemma 2.1 it is sufficient to prove Lemma 2.1, (b).

$$\begin{aligned}\|H_g\|_{\mathrm{HS}}^2 &\leq \int_X \|gK(\cdot,x)\|^2 d\mu(x)\\ &= \int_X |g(\lambda)|^2 \int_X \big|K(\lambda,x)|^2 d\mu(x)d\mu(\lambda) = \|g\|^2_{L^2(X,V)} < \infty\,.\end{aligned}$$

By Remark 2.1 and using the same calculation it also follows that the Toeplitz operator T_g is a Hilbert-Schmidt operator. From $T_{|g|^2} = H_g^* H_g + T_{\bar g} T_g$ we derive that $T_{|g|^2}$, $T_{\bar g}T_g$ and $H_g^*H_g$ are of *trace class*. Hence

$$\begin{aligned}\|H_g\|_{\mathrm{HS}}^2 &= \mathrm{trace}\big(T_{|g|^2} - T_{\bar g}T_g\big)\\ &= \mathrm{trace}\big(T_{|g|^2}\big) - \mathrm{trace}\big(T_{\bar g}T_g\big)\\ &= \mathrm{trace}\big(T_{|g|^2}\big) - \mathrm{trace}\big(T_g T_{\bar g}\big) = \mathrm{trace}\big(H_{\bar g}^* H_{\bar g}\big) = \|H_{\bar g}\|_{\mathrm{HS}}^2\,. \qquad \square\end{aligned}$$

LEMMA 4.1. *Let* $(g_m)_m \in L^\infty(X)$ *be a bounded sequence and point wise convergent to* g. *Then* $(H_{g_m})_m$ *converges to* H_g *in the strong operator topology.*

PROOF. Let $f \in H$, then by Lebesgue's convergence theorem it follows that:

$$\big\|H_{g_m-g} f\big\|^2 \leq \textstyle\int_X \big|g_m - g\big|^2 \big|f\big|^2 d\mu \xrightarrow{m\to\infty} 0\,. \qquad \square$$

Let $\mathcal{N}_{\mathrm{sym}} := \{h \in L^\infty(X) : H_{\bar h} = 0\}$ be the kernel of the *symbol map* $h \mapsto H_{\bar h}$. Then we consider the space $\mathcal{S}$ of symbols defined by:

$$\mathcal{S} := \big\{g \in L^\infty(X) : \overline{S}_g \cap \mathcal{N}_{\mathrm{sym}} \neq \emptyset\big\}\,. \tag{4.2}$$

THEOREM 4.1. *Let* $g \in \mathcal{S}$ *such that* H_g *is a Hilbert-Schmidt operator*, *then* $H_{\bar g}$ *is a Hilbert-Schmidt operator as well and* $\|H_{\bar g}\|_{HS} \leq 2\|H_g\|_{HS}$.

PROOF. Because H_g is a Hilbert-Schmidt operator and by applying Corollary 3.3 it follows that $g^{(m-1)} - g^{(m)} \in L^2(X,V)$ for all $m \in \mathbf{N}$. Hence one concludes that:

$$g - g^{(m)} = \big\{g - g^{(1)}\big\} + \cdots + \big\{g^{(m-1)} - g^{(m)}\big\} \in L^2(X,V)\,.$$

By Proposition 4.1 and Corollary 3.3 again one has for all $m \in \mathbf{N}$:

$$\big\|H_{\bar g - \bar g^{(m)}}\big\|_{\mathrm{HS}} = \big\|H_{g-g^{(m)}}\big\|_{\mathrm{HS}} \leq \big\|H_g\big\|_{\mathrm{HS}} + \big\|H_{g^{(m)}}\big\|_{\mathrm{HS}} \leq 2\cdot\big\|H_g\big\|_{\mathrm{HS}}\,. \tag{4.3}$$

Choose $h \in \overline{S}_g \cap \mathcal{N}_{\mathrm{sym}} \neq \emptyset$ and assume that h belongs to $\mathcal{S}_g$. Then there is $i_0 \in \mathbf{N}$ such that $h = g^{(i_0)}$ and for $i \geq i_0$ it follows from (3.6) that: $0 \leq \|H_{\bar g^{(i)}}\|_{\mathrm{HS}} \leq \|H_{\bar h}\|_{\mathrm{HS}} = 0$ showing that $H_{\bar g^{(i)}} = 0$. In particular, for $f \in H$:

$$\lim_{i\to\infty} \|H_{\bar g^{(i)}} f\| = 0\,. \tag{4.4}$$

For $h \in \overline{\mathcal{S}_g} \setminus \mathcal{S}_g$ there is a sequence $(m_k)_k \subset \mathbf{N}$ such that $\lim_{k\to\infty} g^{(m_k)} = h$ with respect to the Fréchet topology of $C(X, \mathbf{d})$. Because of $\|g^{(m_k)}\|_\infty \leq \|g\|_\infty$ and Lemma 4.1 we obtain for $f \in H$ that:

$$\lim_{k\to\infty} \left\| H_{\bar{g}^{(m_k)}} f \right\| = \left\| H_{\bar{h}} f \right\| = 0 \,. \tag{4.5}$$

Let $[e_j : j \in \mathbf{N}]$ be an ONB of H and fix $l \in \mathbf{N}$. Then by (4.3) we conclude:

$$\begin{aligned} \sum_{j=1}^{l} \left\| H_{\bar{g}} e_j \right\|^2 &= \lim_{k\to\infty} \sum_{j=1}^{l} \left\| H_{\bar{g}-\bar{g}^{(m_k)}} e_j \right\|^2 \\ &\leq \limsup_{k\to\infty} \left\| H_{\bar{g}-\bar{g}^{(m_k)}} \right\|_{\mathrm{HS}}^2 \leq 4 \left\| H_g \right\|_{\mathrm{HS}}^2 . \end{aligned}$$

in both cases (4.4) and (4.5). For $l \to \infty$ the assertion follows. □

PROPOSITION 4.2. *Let $\mathcal{S}_0 \subset L^\infty(X)$ be asymptotically invariant under Berezin transform such that Fix$(\mathcal{S}_0) \subset \mathcal{N}_{sym}$. Then $\mathcal{S}$ in Theorem* 4.1 *can be replaced by $\mathcal{S}_0$.*

PROOF. Fix $g \in \mathcal{S}_0$ and let $\|H_g\|_{\mathrm{HS}} < \infty$. It is sufficient to show $g \in \mathcal{S}$ defined in (4.2). By assumption it follows that $\overline{\mathcal{S}_g} \subset \mathcal{S}_0$. Moreover, as a consequence of Proposition 3.2 one obtains that $\emptyset \neq \mathrm{Fix}\left(\overline{\mathcal{S}_g}\right) \subset \mathrm{Fix}\left(\mathcal{S}_0\right) \subset \mathcal{N}_{\mathrm{sym}}$. Hence $\overline{\mathcal{S}_g} \cap \mathcal{N}_{\mathrm{sym}} \neq \emptyset$ and $g \in \mathcal{S}$. □

Let $\mathcal{K}$ be the ideal of compact operators on H and denote by $\sigma_e(T)$ the *essential spectrum* of $T \in \mathcal{L}(H)$. For the following result and with the reproducing kernel K we assume that the assignment

$$X \ni x \mapsto K(x, x) \in (0, \infty) \tag{4.6}$$

is continuous. Then we can prove (c.f. [10], [21]):

PROPOSITION 4.3. *Let $\mu(K) < \infty$ for all compact $K \subset X$ and $g \in L^\infty(X)$ such that H_g and $H_{\bar{g}}$ are compact. With a sequence $(K_m)_m$ of compact sets as in* (P1) *it follows that:*

$$\sigma_e\big(T_g\big) \subset \bigcap_{m\in\mathbf{N}} \mathit{closure}\ g\big(X \setminus K_m\big) \,. \tag{4.7}$$

If $T_{g-g^{(m)}}$ is compact, we can replace g by $g^{(m)}$ on the right hand side of (4.7).

PROOF. Suppose that $\lambda \notin \mathit{closure} g(X \setminus K_m)$ for $m \in \mathbf{N}$, then consider h defined by:

$$h(z) := \begin{cases} \{g(z) - \lambda\}^{-1} & \text{if } z \in X \setminus K_m \\ 1 & \text{else} . \end{cases}$$

The function h clearly is bounded and it can be easily verified that:

(a) $T_h T_{g-\lambda} = I + T_{(g-\lambda)h-1} - H^*_{\bar{h}} H_g$,

(b) $T_{g-\lambda} T_h = I + T_{(g-\lambda)h-1} - H^*_{\bar{g}} H_h$.

By (4.6) it is clear that $z \mapsto K(z,z)$ and $f := (g-\lambda)h - 1$ are bounded on K_m and because of $\mu(K_m) < \infty$ we have:

$$\|f\|^2_{L^2(X,V)} = \int_{K_m} |f(z)|^2 K(z,z)\,d\mu(z) < \infty\,.$$

Hence, T_f is of Hilbert-Schmidt type and so it is compact. By our assumptions on H_g and $H_{\bar{g}}$ both (a) and (b) show that $T_{g-\lambda} \in [\mathcal{L}(H)/\mathcal{K}]^{-1}$ and $\lambda \notin \sigma_e(T_g)$. The second assertion is an immediate consequence of $\sigma_e(T_g) = \sigma_e(T_{g^{(m)}})$. □

5. Examples and Applications

Various aspects of the Berezin symbol have been studied c.f. [2], [4], [10], [15] and most recently [11], [12]. Below we apply some of these results to obtain examples of our assumptions in THEOREM 4.1. In particular, we prove THEOREM III and (1.2) of the introduction. All spaces X appearing in this section are metric with (P1)–(P3).

5.1. Bergman spaces over bounded domains. Let $\Omega \subset \mathbf{C}^n$ be a bounded domain with a measure μ. By $H := H^2(\Omega,\mu)$ we denote the Bergman space of all holomorphic μ-square integrable functions on Ω. We assume that the *point evaluations* on H are continuous and the reproducing kernel K is strictly positive on the diagonal. The following is due to J. Arazy and M. Englis (c.f. [2], THEOREM 2.3.):

THEOREM 5.1 ([2]). *Let Ω be either a bounded domain in the complex plane with C^1-boundary, or a strictly pseudo convex domain in $\mathbf{C}^n$ with C^3-boundary, then*

(a) *B maps $C(\overline{\Omega})$ into itself and preserves the boundary values.*
(b) *For any $f \in C(\overline{\Omega})$, the sequence $(f^{(\mathrm{k})})_k$ of iterated Berezin transforms converges uniformly on $\overline{\Omega}$ to a function $g \in C(\overline{\Omega})$ satisfying $Bg = g$ and $g_{|\partial\Omega} = f_{|\partial\Omega}$.*
(c) *For any $\Phi \in C(\partial\Omega)$ there exists a unique $g \in C(\overline{\Omega})$ satisfying $Bg = g$ and $g_{|\partial\Omega} = \Phi$. The function g is called B-Poisson extension of Φ.*

Let $\Omega \subset \mathbf{C}^n$ be as in Theorem 5.1 and denote by $C_0(\Omega)$ the continuous functions on $\overline{\Omega}$ vanishing at the boundary. From (b) and the uniqueness result in (c) we conclude that:

COROLLARY 5.1. *Let $g \in \mathcal{S}_0 := C_0(\Omega)$. Then $(g^{(k)})_k$ converges to* 0 *uniformly on $\overline{\Omega}$. In particular, $\mathcal{S}_0$ fulfills the assumptions of Proposition* 4.2.

PROOF. The first assertion directly follows from THEOREM 5.1 and $\overline{\mathcal{S}_g} = \mathcal{S}_g \cup \{0\} \subset \mathcal{S}_0$ shows that $\mathcal{S}_0$ is asymptotically invariant under B. Moreover, by the uniqueness result in Theorem 5.1 it is clear that $\mathrm{Fix}(\mathcal{S}_0) = \{0\} \subset \mathcal{N}_{\mathrm{sym}}$. □

Note that $\mathcal{S}_0$ is symmetric under complex conjugation. In order to give an example for a non-symmetric situation we consider:

$$\mathcal{A}_{\mathrm{ah}}(\Omega) := \left\{ f \in C(\overline{\Omega}) : \ f_{|\Omega} \ \text{ is anti-holomorphic} \right\}$$

and set $\mathcal{S}_1 := C_0(\Omega) \oplus \mathcal{A}_{\text{ah}}(\Omega)$. With $f \in C_0(\Omega)$ and $h \in \mathcal{A}_{\text{ah}}(\Omega)$ consider $g = f + h \in \mathcal{S}_1$. Because of $Bh = h$ and $h = g$ on $\partial\Omega$ we conclude from Theorem 5.1 (b) and (c) that the sequence $(g^{(k)})_k$ is uniformly convergent on $\overline{\Omega}$ to h. Hence $\mathcal{S}_1$ is asymptotically invariant under Berezin transform. Moreover, $\text{Fix}(\mathcal{S}_1) = \mathcal{A}_{\text{ah}}(\Omega) \subset \mathcal{N}_{\text{sym}}$ and the assumptions of Proposition 4.2 hold.

THEOREM 5.2. *Let $g_1 \in \mathcal{S}_0 := C_0(\Omega)$ and $g_2 \in \mathcal{S}_1 := C_0(\Omega) \oplus \mathcal{A}_{ah}(\Omega)$, then*

(a) *$H_{g_1} \in \mathcal{L}^2(H, H^{\perp})$ if and only if $H_{\bar{g}_1} \in \mathcal{L}^2(H, H^{\perp})$.*

(b) *$H_{g_2} \in \mathcal{L}^2(H, H^{\perp})$ implies that $H_{\bar{g}_2} \in \mathcal{L}^2(H, H^{\perp})$.*

(c) *For $h \in \{g_1, \bar{g}_1, \bar{g}_2\}$ there is a norm estimate: $\|H_h\|_{HS} \leq 2 \cdot \|H_{\bar{h}}\|_{HS}$.*

Let B_n be the unit ball in $\mathbf{C}^n$ with $n \geq 2$. It was observed in [25] that there is no non-zero Hankel operator $H_g \in \mathcal{L}^2(H, H^{\perp})$ with anti-holomorphic symbol. Hence, in general H_{g_2} in Theorem 5.2 is not of Hilbert-Schmidt type in the case of $H_{\bar{g}_2} = 0$. Let v be the Lebesgue measure on B_n, $(n \in \mathbf{N})$ and define for $\alpha \in \mathbf{R}$ the measure μ_α by

$$d\mu_\alpha(z) = c_\alpha K(z,z)^{1-\frac{\alpha}{n+1}} dv(z)\,, \qquad c_\alpha > 0$$

where K denotes the *reproducing kernel* of the unweighted Bergman space $H^2(B_n, v)$. It is known that μ_α is finite if and only if $\alpha > n$ and in this case we choose c_α with $\mu_\alpha(B_n) = 1$. For $\alpha > n$ and in the case of the weighted Bergman space H^2_α of holomorphic functions in $L^2(B_n, \mu_\alpha)$ we want to add some remarks on compact Hankel operators. Let A be a finite sum of finite products of Toeplitz operators on H^2_α, then it was proved in [14] that A is compact if and only if its Berezin symbol vanishes at the boundary of B_n. The following Lemma corresponds to LEMMA 2.1 in the compact case:

LEMMA 5.1. *Let $g \in L^\infty(B_n)$ and $R \in \{H_g, T_g\}$ defined on H^2_α where $\alpha > n$. With the normalized reproducing kernel function k_λ in* (2.1) *it holds*:

(a) *R is compact if and only if $\|Rk_\lambda\| \to 0$ as $\lambda \to \partial B_n$.*

(b) *For $\lambda \to \partial B_n$ the sequence $(k_\lambda)_\lambda$ tends to* 0 *weakly in $L^2(B_n, \mu_\alpha)$.*

PROOF. Because R is compact if and only if R^*R is compact (a) follows from our remark above together with:

- $\|T_g k_\lambda\|^2 = \widetilde{T_g^* T_g}(\lambda) = \widetilde{T_{\bar{g}} T_g}(\lambda)$,
- $\|H_g k_\lambda\|^2 = \widetilde{H_g^* H_g}(\lambda) = (T_{|g|^2} - T_{\bar{g}} T_g)\tilde{}(\lambda)$.

To prove (b) let $h \in L^2(B_n, \mu_\alpha)$ and $\varepsilon > 0$. Choose a continuous function r on B_n having compact support such that $\|r - h\| \leq \varepsilon$. It follows that:

$$\left|\langle h, k_\lambda\rangle\right| \leq \left|\langle h - r, k_\lambda\rangle\right| + \left|\langle 1, T_{\bar{r}} k_\lambda\rangle\right| \leq \varepsilon + \left\|T_{\bar{r}} k_\lambda\right\|.$$

By Proposition 4.1 the Toeplitz operator $T_{\bar{r}}$ is compact and (b) follows from (a). □

As an application of Theorem 5.1 we remark (c.f. [24] in the case $\alpha := n + 1$ and [7]):

COROLLARY 5.2. *For $g \in C(\overline{B_n})$ both H_g and $T_{g-\tilde{g}}$ are compact on H^2_α where $\alpha > n$.*

PROOF. For all $\lambda \in B_n$ it follows with a straightforward calculation that:

$$\left\| P\big[gk_\lambda\big] - \tilde{g}(\lambda)k_\lambda \right\|^2 + \left\| H_g k_\lambda \right\|^2 = \widetilde{|g|^2}(\lambda) - \left|\tilde{g}(\lambda)\right|^2 . \tag{5.1}$$

By Theorem 5.1 both sides of (5.1) vanish at ∂B_n showing that $\lim_{\lambda \to \partial B_n} \left\| H_g k_\lambda \right\| = 0$. Similarly, for $\lambda \to \partial B_n$ one has the convergence:

$$0 \le \left\| T_{g-\tilde{g}} k_\lambda \right\|^2 \le \left\| (g - \tilde{g})k_\lambda \right\|^2 = \widetilde{\left\{ \left| g - \tilde{g} \right|^2 \right\}}(\lambda) \to 0 .$$

Finally, we can apply Lemma 5.1. □

An application of Theorem 5.1 leads to the results below. In case $\alpha = n + 1$ Theorem 5.3 and Corollary 5.3 have been originally proved in [13] by using different methods.

THEOREM 5.3. *Let $f \in C(\overline{B_n})$, then $\sigma_e(T_f) = f(\partial B_n)$.*

PROOF. The inclusion $\sigma_e(T_f) \subset f(\partial B_n)$ follows from Proposition 4.3 and Corollary 5.2. Conversely, let $\lambda = f(x_0) \in f(\partial B_n)$. By Theorem 5.1 (a) and for $R \in \mathcal{L}(H^2_\alpha)$ it holds:

$$0 \le \left\| RT_{f-\lambda} k_x \right\|^2 \le \left\| R \right\|^2 \left\| T_{f-\lambda} k_x \right\|^2 \le \left\| R \right\|^2 \widetilde{\left\{ |f - \lambda|^2 \right\}}(x) \xrightarrow{x \to x_0} 0 .$$

For $\lambda \notin \sigma_e(T_f)$ one can choose $R + \mathcal{K}$ to be a left-inverse of $T_{f-\lambda} + \mathcal{K}$ in the *Calkin algebra*. Then there is $K \in \mathcal{K}$ such that $\lim_{x \to x_0} \left\| (I - K)k_x \right\| = 0$ in contradiction to Lemma 5.1, (b) and $\|k_x\| = 1$ for all x. Hence $\lambda \in \sigma_e(T_f)$. □

COROLLARY 5.3. *For $f \in C(\overline{B_n})$ the operator T_f is Fredholm if and only if $0 \notin f(\partial B_n)$.*

5.2. Pluri-harmonic Fock space. For $n \in \mathbf{N}$ and with the usual Lebesgue measure v consider the normalized Gaussian measure μ on $\mathbf{C}^n$ defined by:

$$d\mu(z) := \pi^{-n} \exp(-|z|^2) dv(z) . \tag{5.2}$$

The space H_h of all entire and μ-square integrable functions is called *Fock space* or *Segal-Bargmann space*. It is known that H_h is a reproducing kernel Hilbert space with kernel function $K(x, y) = \exp(x \cdot \bar{y})$ for $x, y \in \mathbf{C}^n$ where $x \cdot y := x_1 y_1 + \cdots + x_n y_n$ and $|y|^2 := y \cdot \bar{y}$. We also consider the space $H_{\mathrm{ah}} := \{\bar{f} : f \in H_{\mathrm{h}}\}$ of anti-holomorphic functions and we denote by P_{h} (resp. P_{ah}) the orthogonal projection from $L^2(\mathbf{C}^n, \mu)$ onto H_{h} (resp. H_{ah}). For $f \in L^2(\mathbf{C}^n, \mu)$ note that:

$$P_{\mathrm{h}} \bar{f} = \overline{P_{\mathrm{ah}} f} . \tag{5.3}$$

Considered on functions the Berezin transforms corresponding to both spaces H_{h} and H_{ah} coincide and we denote it by B. For $g \in L^\infty(\mathbf{C}^n)$ one has:

$$\big[Bg\big](u) := \int_{\mathbf{C}^n} g(x)\exp\big(x\cdot\bar{u} + u\cdot\bar{x} - |u|^2\big)d\mu(x)\,. \tag{5.4}$$

It is readily verified that B can be regarded as a continuous convolution operator on the *Schwartz space* $\mathcal{S}(\mathbf{C}^n)$, c.f. [15]:

$$Bf = f * h \quad \text{where} \quad h := 2^n \exp\big(-|\cdot|^2\big)$$

and $f*g := (2\pi)^{-n}\int_{\mathbf{C}^n} f(y)g(\cdot - y)dv(y)$ denotes the *convolution product* on $\mathcal{S}(\mathbf{C}^n)$. Using the Fourier transform $\mathcal{F}$ on $\mathcal{S}(\mathbf{C}^n)$ and $g := \mathcal{F}h = \exp\big(-4^{-1}|\cdot|^2\big)$ it follows, that B also can be written as *pseudo-differential operator* $B = \mathcal{F}^{-1}M_g\mathcal{F}$ on $\mathcal{S}(\mathbf{C}^n)$. There is an extension of $I - B = \mathcal{F}^{-1}M_{1-g}\mathcal{F}$ to the space $\mathcal{S}'(\mathbf{C}^n)$ of tempered distributions. This observation leads to a proof of the following fact, c.f. [15]:

LEMMA 5.2. *Let $u \in \mathcal{S}'(\mathbf{C}^n)$ such that $Bu = u$, then u is a harmonic polynomial. In particular, any bounded function u which is reproduced under B must be constant.*

PROOF. The Fourier transform of $u \in \mathcal{S}'(\mathbf{C}^n)$ is denoted by $\hat{u}$. By our remarks above and with $Bu = u$ it follows that $0 = (1-g)\hat{u} = G|\cdot|^2\hat{u}$. Here the function

$$G(\xi) := \frac{1-g(\xi)}{|\xi|^2} = \frac{1-\exp\big(-4^{-1}|\xi|^2\big)}{|\xi|^2}$$

is bounded away from 0 and it can be checked that multiplication by G induces an isomorphism of $\mathcal{S}'(\mathbf{C}^n)$. Hence $0 = |\cdot|^2\hat{u}$ which is equivalent to the Laplace equation $\Delta u = 0$. Our assertion follows from a well-known extension of *Liouville's theorem*. □

As an immediate consequence it follows that, c.f. [4]:

COROLLARY 5.4. *Let $\mathcal{S}_0 := L^\infty(\mathbf{C}^n)$, then $Fix(\mathcal{S}_0) = \mathbf{C}$. In particular, the assumptions of Proposition 4.2 are fulfilled and for $g \in L^\infty(\mathbf{C}^n)$ it holds $\|H_g^h\|_{HS} \leq 2\cdot\|H_{\bar{g}}^h\|_{HS}$.*

With a symbol $g \in L^\infty(\mathbf{C}^n)$ and $(P, H) \in \{(P_{\text{h}}, H_{\text{h}}), (P_{\text{ah}}, H_{\text{ah}})\}$ we consider the *Hankel* and *Toeplitz operators*:

$$\big(I - P\big)M_g \in \mathcal{L}\big(H, H^\perp\big) \quad \text{and} \quad PM_g \in \mathcal{L}(H)$$

and denote them by H_g^{h}, H_g^{ah} resp. T_g^{h} and T_g^{ah}. As a consequence of (5.3) we remark that:

LEMMA 5.3. *Let $g \in L^\infty(\mathbf{C}^n)$, then*:

(i) $\|T_g^h\|_{HS} = \|T_{\bar{g}}^{ah}\|_{HS}$,

(ii) $\|H_g^h\|_{HS} = \|H_{\bar{g}}^{ah}\|_{HS}$

where both sides of (i) *resp.* (ii) *may be simultaneously infinite.*

PROOF. We only prove (ii). Let $[e_j : j \in \mathbf{N}_0]$ be an ONB of H_{h}, then an ONB of H_{ah} is given by $[\bar{e}_j : j \in \mathbf{N}_0]$. Now, it follows by (5.3) that:

$$\|H_g^{\mathrm{h}} e_j\|^2 = \|g e_j\|^2 - \|P_{\mathrm{h}} g e_j\|^2 = \|\bar{g}\bar{e}_j\|^2 - \|P_{\mathrm{ah}}\bar{g}\bar{e}_j\|^2 = \|H_{\bar{g}}^{\mathrm{ah}}\bar{e}_j\|^2 .$$

Summing up this equality over $j \in \mathbf{N}_0$ yields the desired result. □

DEFINITION 5.1. The *pluri-harmonic Fock space* H_{ph} consists of all $f \in C^2(\mathbf{C}^n) \cap L^2(\mathbf{C}^n, \mu)$ such that $\frac{\partial^2 f}{\partial z_j \partial \bar{z}_k} = 0$ for all $j, k = 1, \ldots, n$.

According to [18] it holds $H_{\mathrm{ph}} = H_{\mathrm{h}} \oplus \{H_{\mathrm{ah}} \ominus \mathbf{C}\}$ and any $f \in H_{\mathrm{ph}}$ can be written as:

$$f = h + \bar{r}, \quad \text{with } r(0) = 0 \tag{5.5}$$

where h and r are holomorphic. With $g \in L^\infty(\mathbf{C}^n)$ and the orthogonal projection P_{ph} from $L^2(\mathbf{C}^n, \mu)$ onto H_{ph} we define the *pluri-harmonic Hankel operator* by:

$$H_g^{\mathrm{ph}} := \big(I - P_{\mathrm{ph}}\big) M_g : H_{\mathrm{ph}} \to H_{\mathrm{ph}}{}^{\perp} .$$

For $f \in H_{\mathrm{h}}$ it can be checked by a straightforward calculation that:

(a) $\|H_g^{\mathrm{ph}} f\|^2 = \|H_g^{\mathrm{h}} f\|^2 - \|P_{\mathrm{ah}} g f\|^2 + \big|\langle g, \bar{f}\rangle\big|^2$,

(b) $\|H_g^{\mathrm{ph}} \bar{f}\|^2 = \|H_g^{\mathrm{ah}} \bar{f}\|^2 - \|P_{\mathrm{h}} g \bar{f}\|^2 + \big|\langle g, f\rangle\big|^2$.

As an application of Corollary 5.4 and Lemma 5.3 we can prove for $g \in L^\infty(\mathbf{C}^n)$:

THEOREM 5.4. *$H_g^{ph} \in \mathcal{L}^2(H_{ph}, H_{ph}^{\perp})$ iff $H_{\bar{g}}^{ph} \in \mathcal{L}^2(H_{ph}, H_{ph}^{\perp})$ and $\|H_{\bar{g}}^{ph}\|_{HS} \leq \sqrt{2} \cdot \|H_g^{ph}\|_{HS}$. Moreover, $H_g^h \in \mathcal{L}^2(H_h, H_h^{\perp})$ is sufficient for $H_g^{ph}, H_{\bar{g}}^{ph} \in \mathcal{L}^2(H_{ph}, H_{ph}^{\perp})$ and*

$$\max\Big\{\|H_g^{ph}\|_{HS}, \|H_{\bar{g}}^{ph}\|_{HS}\Big\} \leq \sqrt{5 \cdot \min\Big\{\|H_g^h\|_{HS}^2, \|H_{\bar{g}}^h\|_{HS}^2\Big\}} .$$

PROOF. With an ONB $[e_0 := 1, e_j : j \in \mathbf{N}]$ of H_{h}, the system $[e_0, e_j, \bar{e}_j : j \in \mathbf{N}]$ defines an ONB of H_{ph}. Applying (a), (b) and Lemma 5.3 above it follows that:

$$\begin{aligned}
\big\|H_g^{\mathrm{ph}}\big\|_{\mathrm{HS}}^2 + \big\|H_g^{\mathrm{ph}} 1\big\|^2 &= \sum_{j=0}^{\infty}\Big\{\big\|H_g^{\mathrm{ph}} e_j\big\|^2 + \big\|H_g^{\mathrm{ph}}\bar{e}_j\big\|^2\Big\} \qquad (5.6)\\
&= \big\|H_g^{\mathrm{h}}\big\|_{\mathrm{HS}}^2 + \big\|H_g^{\mathrm{ah}}\big\|_{\mathrm{HS}}^2 - \sum_{j=1}^{\infty}\Big\{\big\|P_{\mathrm{ah}} g e_j\big\|^2 + \big\|P_{\mathrm{h}} g \bar{e}_j\big\|^2\Big\}\\
&= \big\|H_g^{\mathrm{h}}\big\|_{\mathrm{HS}}^2 + \big\|H_{\bar{g}}^{\mathrm{h}}\big\|_{\mathrm{HS}}^2 - \sum_{j=1}^{\infty}\Big\{\big\|P_{\mathrm{h}}\bar{g}\bar{e}_j\big\|^2 + \big\|P_{\mathrm{h}} g \bar{e}_j\big\|^2\Big\}.
\end{aligned}$$

In particular, it holds

$$\|H_{\bar{g}}^{\mathrm{ph}}\|_{\mathrm{HS}}^2 + \|H_{\bar{g}}^{\mathrm{ph}} 1\|^2 = \|H_g^{\mathrm{ph}}\|_{\mathrm{HS}}^2 + \|H_g^{\mathrm{ph}} 1\|^2 .$$

Hence H_g^{ph} is of Hilbert-Schmidt type if and only if $H_{\bar{g}}^{\mathrm{ph}}$ is of Hilbert-Schmidt type and it holds $\|H_{\bar{g}}^{\mathrm{ph}}\|_{\mathrm{HS}} \leq \sqrt{2} \cdot \|H_g^{\mathrm{ph}}\|_{\mathrm{HS}}$. Moreover, with $f \in \{g, \bar{g}\}$, Corollary 5.4 and (5.6):

$$\|H_f^{\mathrm{ph}}\|_{\mathrm{HS}}^2 \leq \|H_g^{\mathrm{h}}\|_{\mathrm{HS}}^2 + \|H_{\bar{g}}^{\mathrm{h}}\|_{\mathrm{HS}}^2 \leq 5 \cdot \min\Big\{\|H_g^{\mathrm{h}}\|_{\mathrm{HS}}^2, \|H_{\bar{g}}^{\mathrm{h}}\|_{\mathrm{HS}}^2\Big\}. \qquad \square$$

5.3. Hilbert space on quadrics. Let H be a closed subspace in $L^2(X, \mu)$ with reproducing kernel K. In our analysis on Hankel operators the *Berezin measure* V defined in (2.3) plays a crucial role. In case of the *Fock space* (or *Segal-Bargmann space*) H_{h}, c.f. section 5.2, it is readily verified that:

$$\pi^n V := \Omega_{\mathbf{C}^n} = \text{Liouville volume form} \tag{5.7}$$

where $\Omega_{\mathbf{C}^n}$ coincides with the usual Lebesgue measure on $\mathbf{C}^n \cong T^*(\mathbf{R}^n)$. In fact, H_{h} is only one example of a reproducing kernel Hilbert space which naturally arises from a more general construction method. It was remarked in [20], that H_{h} can be obtained by *pairing of polarizations* from the real and Käler polarization on the cotangent bundle $T^*(\mathbf{R}^n) \cong \mathbf{C}^n$. The *Bargmann transform* between $L^2(\mathbf{R}^n)$ and H_h can be derived via this method.

By replacing $\mathbf{R}^n$ with the n-dimensional sphere $\mathbf{S}^n$ the same construction leads to a reproducing kernel Hilbert space $H_{\mathbf{S}^n}$ of holomorphic functions on a *non-singular cone* or *quadric* $\mathbf{X}_{\mathbf{S}^n}$ in $\mathbf{C}^{n+1} \setminus \{0\}$ diffeomorphic to the punctured cotangent bundle $T_0^*(\mathbf{S}^n)$. We give the definition of $H_{\mathbf{S}^n}$ which we consider to be of interest itself and prove an asymptotic version of (5.7) in the case of $H_{\mathbf{S}^n}$. For a detailed description of *pairing of polarizations* we refer to [5] and [20]. More examples of this method are treated in [5], [6], [16] and [17].

Let $\mathbf{S}^n := \{(x_0, \dots, x_n) \in \mathbf{R}^{n+1} : |x|^2 = 1\}$ be the n-dimensional sphere with the standard Riemann metric induced from the *Euclidean metric* on $\mathbf{R}^{n+1}$. As before we write $x \cdot y := \sum x_j \cdot y_j$ and $|x|^2 := x \cdot x$ for $x, y \in \mathbf{R}^{n+1}$. The tangent bundle $T(\mathbf{S}^n)$ and the cotangent bundle $T^*(\mathbf{S}^n)$ can be identified via this metric and are realized in $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$:

$$T^*(\mathbf{S}^n) \cong T(\mathbf{S}^n) = \big\{(x, y) \in \mathbf{R}^{n+1} \times \mathbf{R}^{n+1} : |x| = 1 \text{ and } x \cdot y = 0\big\}.$$

With the punctured cotangent bundle $T_0^*(\mathbf{S}^n) := \{(x, y) \in T^*(\mathbf{S}^n) : y \neq 0\}$ we define a diffeomorphism $\tau_{\mathbf{S}^n}$ onto a *quadric* $\mathbf{X}_{\mathbf{S}^n}$ by:

$$\begin{aligned} \tau_{\mathbf{S}^n} : T_0^*(\mathbf{S}^n) &\longrightarrow \mathbf{X}_{\mathbf{S}^n} := \big\{z \in \mathbf{C}^{n+1} : z \cdot z = 0 \text{ and } z \neq 0\big\} \\ (x, y) &\mapsto z = |y|x + \sqrt{-1}y. \end{aligned} \tag{5.8}$$

The *symplectic form* $\omega_{\mathbf{S}^n}$ and the *canonical one form* $\Theta_{\mathbf{S}^n}$ on $T^*(\mathbf{S}^n)$ respectively are given by the restriction of $\sum dy_k \wedge dx_k$ and $\sum y_k \cdot dx_k$ on $\mathbf{R}^{n+1} \times \mathbf{R}^{n+1}$. Via (5.8) it can be shown that the symplectic form ω_X on $\mathbf{X}_{\mathbf{S}^n}$ is expressed as:

$$\omega_X = \sqrt{-2}\bar{\partial}\partial|z|.$$

Let $\Omega_{\mathbf{S}^n} := \frac{(-1)^{n(n-1)/2}}{n!} \cdot \omega_{\mathbf{S}^n}$ be the *Liouville volume form* on $T_0^*(\mathbf{S}^n)$. Due to the isomorphism (5.8) it can be regarded as a volume form Ω_X on $\mathbf{X}_{\mathbf{S}^n}$. Let P_X denote the restriction of holomorphic polynomials on $\mathbf{C}^{n+1}$ to $\mathbf{X}_{\mathbf{S}^n}$. On P_X we consider a family of inner products depending on two real parameters (h, N) where $h > 0$ and $N > -n$:

$$\langle p, q \rangle_{(h,N)} := \int_{\mathbf{X}_{\mathbf{S}^n}} p(z)\overline{q(z)} e^{-h|z|} |z|^N d\Omega_X \,, \qquad p, q \in P_X \,. \tag{5.9}$$

By *pairing of polarizations* the case $h := 2\sqrt{2}\pi$ and $N := n/2 - 1$ naturally appears and the measure $dm_{(h,N)} := e^{-h|z|} \cdot |z|^N d\Omega_X$ corresponds to the Gaussian measure μ in (5.2). As an analog to the Segal-Bargmann space we define:

$$H^2(\mathbf{X}_{\mathbf{S}^n}, dm_{(h,N)}) := \; L^2\text{-}\textit{closure of } P_X \textit{ w.r.t. the inner product } (5.9)\,.$$

It can be shown that $H^2(\mathbf{X}_{\mathbf{S}^n}, dm_{(h,N)})$ is a reproducing kernel Hilbert space. Moreover, its elements can be extended to holomorphic functions on the whole space $\mathbf{C}^{n+1}$. The reproducing kernel $K_{(h,N)}$ can be calculated in form of an infinite sum and involving the Gamma function. More precisely, it holds (c.f. [5]):

$$K_{(h,N)}(\lambda, \lambda) = C(h, n, N) \cdot \sum_{k=0}^{\infty} \frac{\Gamma(k + n - 1) \cdot (2k + n - 1)}{\Gamma(2k + N + n) \cdot \Gamma(k + 1)} \cdot |h\lambda|^{2k} \tag{5.10}$$

with $C(h, n, N) := \frac{h^{n+N}}{\mathrm{Vol}(\Sigma(\mathbf{S}^n)) \cdot \Gamma(n)}$ and $\Sigma(\mathbf{S}^n) := \{z \in \mathbf{X}_{\mathbf{S}^n} : |z| = 1\}$.

For $H^2(\mathbf{X}_{\mathbf{S}^n}, dm_{(h,N)})$ we prove an asymptotic property corresponding to (5.7).

PROPOSITION 5.1. *For $N > -n$ and $h > 0$ it holds*:

$$\lim_{\lambda \to \infty} |\lambda|^N \cdot \exp\big(-|h\lambda|\big) \cdot K_{(h,N)}(\lambda, \lambda) = \frac{h^n}{2^{n-1} \cdot Vol(\Sigma(\mathbf{S}^n)) \cdot \Gamma(n)} \,. \tag{5.11}$$

In particular, (5.11) *can be written as* $2^{-n+1} h^{-N} C(h, n, N)$ *and is independent of* N.

A direct computation shows, that (5.10) splits into two sums:

$$\begin{aligned} K_{(h,N)}(\lambda, \lambda) = C(h, n, N) \cdot \Big\{ & 2|h\lambda|^2 \cdot \sum_{k=0}^{\infty} I(k, n, N) \cdot |h\lambda|^{2k} \\ & + (n - 1) \cdot \sum_{k=0}^{\infty} I(k, n - 1, N - 1) \cdot |h\lambda|^{2k} \Big\} \end{aligned} \tag{5.12}$$

where

$$I(k, n, N) := \frac{\Gamma(k + n)}{\Gamma(2k + N + n + 2)\Gamma(k + 1)} \,.$$

Using the expression of the *Euler integral* $\int_0^1 t^{p-1}(1 - t)^{q-1} dt$ where $p, q > 0$ in terms of the Gamma function together with the well-known *duplication formula*:

$$\sqrt{\pi} \cdot 2^{-2k} \cdot \Gamma(2k + 1) = \Gamma\left(k + \frac{1}{2}\right) \cdot \Gamma(k + 1)$$

one easily verifies in case of $\frac{N-n}{2} > -1$ and $k \in \mathbf{N}_0$:

$$I(k,n,N) = \frac{E(n,N)}{(2k)!} \cdot \int_0^1 s^{k+n-1} \cdot (1-s)^{\frac{N-n}{2}} ds \cdot \int_0^1 t^{k-\frac{1}{2}} \cdot (1-t)^{\frac{N+n}{2}} dt \,. \tag{5.13}$$

Here $E(n,N) > 0$ is given by:

$$E(n,N) := \frac{1}{2^{N+n+1} \cdot \Gamma(\frac{N-n}{2}+1) \cdot \Gamma(\frac{N+n}{2}+1)} \,. \tag{5.14}$$

Multiplying (5.13) with x^{2k} and summing up over $k \in \mathbf{N}_0$ leads to:

$$\sum_{k=0}^{\infty} I(k,n,N) \cdot x^{2k} = \int_0^1 \int_0^1 \Phi_{n,N}(s,t) \cdot \cosh\{\sqrt{st} \cdot x\} ds dt \tag{5.15}$$

where $\Phi_{n,N} : (0,1)^2 \to \mathbf{R}^+$ is defined by:

$$\Phi_{n,N}(s,t) := E(n,N) \cdot \frac{s^{n-1}}{\sqrt{t}} \cdot (1-t)^{\frac{N+n}{2}} \cdot (1-s)^{\frac{N-n}{2}} \,. \tag{5.16}$$

In (5.15) one can replace n by $n-1$ and N by $N-1$. By using (5.12) we derive the following integral expression of $K_{(h,N)}$ on the diagonal:

COROLLARY 5.5. *For* $\frac{N-n}{2} > -1$ *and with*:

$$\Psi_{n,N}(s,t,x) := C(h,n,N) \cdot \Big\{ 2x^2 \cdot \Phi_{n,N}(s,t) + (n-1) \cdot \Phi_{n-1,N-1}(s,t) \Big\} \tag{5.17}$$

it holds:

$$K_{(h,N)}(\lambda,\lambda) = \int_0^1 \int_0^1 \Psi_{n,N}\big(s,t,|h\lambda|\big) \cdot \cosh\Big\{\sqrt{st} \cdot |h\lambda|\Big\} ds dt \,.$$

Below we analyze the asymptotic behavior of integral expressions having the form (5.15) and apply our results to the proof of Proposition 5.1.

Let $f, g : \mathbf{R}^+ \to \mathbf{R}^+$ and $k > 0$, then we write $f \sim_k g$ iff $\lim_{t\to\infty} t^k \cdot f(t)$ exists and

$$\lim_{t\to\infty} t^k \cdot \{f(t) - g(t)\} = 0 \,.$$

Given a sequence of functions $g_j : \mathbf{R}^+ \to \mathbf{R}^+$ where $j \in \mathbf{N}_0$ we write $f \sim \sum g_j$ and say that the (formal) series $\sum g_j$ *represents* f *asymptotically* for large values of t whenever:

- For all $k \in \mathbf{N}_0$: $f - \{g_0 + g_1 + \cdots + g_k\} \sim_k 0$ and
- there is a constant a_k such that $g_k \sim_k \frac{a_k}{t^k}$.

Let $\Phi : [0,1]^2 \to \mathbf{C}$ be integrable and assume that $\rho : [0,1]^2 \to \mathbf{R}_{\geq 0}$ is continuous. For any measurable subset $U \subset (0,1)^2$ we define $\mathbf{J}^U_{\rho,\Phi} : \mathbf{R}^+ \to \mathbf{C}$ with $x = (s,t)$ by:

$$\mathbf{J}^U_{\rho,\Phi}(x) := \int_U \Phi(s,t) \cdot \exp\{-\rho(s,t) \cdot x\} dx \,.$$

In our application we examine the case where

(1) $\Phi(s,t) = \Phi_{\alpha,\beta}(s,t) := (1-s)^\alpha \cdot (1-t)^\beta$ and $\alpha, \beta > -1$,

(2) $\rho(s,t) := 1 - \sqrt{st}$.

The Taylor expansion of ρ at $x_0 := (1,1)$ and of first order is given by:

$$\rho(s,t) = \tau(s,t) + \sum_{k+l>1} O\big(|1-s|^k \cdot |1-t|^l\big)$$

where $\tau(s,t) := 1 - \frac{1}{2} \cdot (s+t)$. Hence it follows that:

$$\lim_{(s,t)\to x_0} \frac{\rho(s,t)}{\tau(s,t)} = 1\,. \tag{5.18}$$

We set $U := [0,1]^2$ and determine the asymptotic behavior of $\mathbf{J}^U_{\tau,\Phi_{\alpha,\beta}}$:

$$\mathbf{J}^U_{\tau,\Phi_{\alpha,\beta}}(x) = \exp(-x) \cdot \int_0^1 (1-s)^\alpha \cdot \exp\Big(\frac{sx}{2}\Big)\, ds \cdot \int_0^1 (1-t)^\beta \cdot \exp\Big(\frac{tx}{2}\Big)\, dt\,.$$

From

$$\int_0^1 (1-s)^\alpha \cdot \exp\Big(\frac{sx}{2}\Big)\, ds = \Big(\frac{2}{x}\Big)^{\alpha+1} \exp\Big(\frac{x}{2}\Big) \cdot \int_0^{\frac{x}{2}} t^\alpha \cdot \exp(-t)dt$$

it follows that:

$$\mathbf{J}^U_{\tau,\Phi_{\alpha,\beta}}(x) \sim_{\alpha+\beta+2} \frac{2^{\alpha+\beta+2}}{x^{\alpha+\beta+2}} \cdot \Gamma(\alpha+1) \cdot \Gamma(\beta+1)\,. \tag{5.19}$$

With our notations in (1) and (2) above we prove:

LEMMA 5.4. *Let $\Psi : [0,1]^2 \to \mathbf{C}$ be continuous in a neighborhood V of $x_0 := (1,1)$ and assume that $\alpha, \beta > -1$, such that $\Psi \cdot \Phi_{\alpha,\beta}$ is integrable over V. Then*

$$\mathbf{J}^V_{\rho,\Psi\cdot\Phi_{\alpha,\beta}}(x) \sim_{\alpha+\beta+2} \Psi(x_0) \cdot \frac{2^{\alpha+\beta+2}}{x^{\alpha+\beta+2}} \cdot \Gamma(\alpha+1) \cdot \Gamma(\beta+1)\,. \tag{5.20}$$

PROOF. For $1 > \varepsilon > 0$ and with (5.18) we choose a neighborhood $W \subset V$ of x_0 such that:

$$\big[1-\varepsilon\big] \cdot \tau(s,t) \le \rho(s,t) \le \big[1+\varepsilon\big] \cdot \tau(s,t)$$

for all $(s,t) \in W$. Hence, by using $\Phi_{\alpha,\beta} \ge 0$ it follows for $x > 0$ that:

$$\mathbf{J}^W_{\tau,\Phi_{\alpha,\beta}}\big([1+\varepsilon]\cdot x\big) \le \mathbf{J}^W_{\rho,\Phi_{\alpha,\beta}}(x) \le \mathbf{J}^W_{\tau,\Phi_{\alpha,\beta}}\big([1-\varepsilon]\cdot x\big)\,. \tag{5.21}$$

With $\gamma \in \{\rho,\tau\}$ and $V_0 \in \{U, V\}$ note that $\mathbf{J}^{V_0}_{\gamma,\Phi_{\alpha,\beta}} = \mathbf{J}^{V_0\setminus W}_{\gamma,\Phi_{\alpha,\beta}} + \mathbf{J}^W_{\gamma,\Phi_{\alpha,\beta}}$ and $\mathbf{J}^{V_0\setminus W}_{\gamma,\Phi_{\alpha,\beta}}$ is of order $O(x^{-\infty})$ as $x \to \infty$. An application of (5.19) and (5.21) shows that:

$$\frac{2^{\alpha+\beta+2}\cdot\Gamma(\alpha+1)\cdot\Gamma(\beta+1)}{(1+\varepsilon)^{\alpha+\beta+2}} \le \liminf_{x\to\infty} x^{\alpha+\beta+2}\cdot\mathbf{J}^{V}_{\rho,\Phi_{\alpha,\beta}}(x)$$
$$\le \limsup_{x\to\infty} x^{\alpha+\beta+2}\cdot\mathbf{J}^{V}_{\rho,\Phi_{\alpha,\beta}}(x) \le \frac{2^{\alpha+\beta+2}\cdot\Gamma(\alpha+1)\cdot\Gamma(\beta+1)}{(1-\varepsilon)^{\alpha+\beta+2}}.$$

Because $\varepsilon > 0$ was arbitrary, it follows for any neighborhood V of x_0:

$$\lim_{x\to\infty} x^{\alpha+\beta+2}\cdot\mathbf{J}^{V}_{\rho,\Phi_{\alpha,\beta}}(x) = 2^{\alpha+\beta+2}\cdot\Gamma(\alpha+1)\cdot\Gamma(\beta+1)\,. \tag{5.22}$$

By the continuity of Ψ we can assume that $|\Psi(s,t)-\Psi(x_0)| < \varepsilon$ for all $(s,t)\in W$. Moreover, by (5.22) there is $c > 0$ such that $|x^{\alpha+\beta+2}\cdot\mathbf{J}^{W}_{\rho,\Phi_{\alpha,\beta}}(x)| \le c$ for all $x > 0$. Hence

$$\left|x^{\alpha+\beta+2}\cdot\mathbf{J}^{W}_{\rho,\Psi\cdot\Phi_{\alpha,\beta}}(x) - x^{\alpha+\beta+2}\cdot\Psi(x_0)\cdot\mathbf{J}^{W}_{\rho,\Phi_{\alpha,\beta}}(x)\right| \le c\cdot\varepsilon\,.$$

Finally, (5.22) where V is replace by W and $\mathbf{J}^{V\setminus W}_{\rho,\Psi\cdot\Phi_{\alpha,\beta}}, \mathbf{J}^{V\setminus W}_{\rho,\Phi_{\alpha,\beta}} \in O(x^{-\infty})$ as $x\to\infty$ prove (5.20). □

COROLLARY 5.6. *Let V be a neighborhood of $x_0 := (1,1)$ and assume that $\Psi\in C^k(V)$. With $\alpha,\beta > -1$ is follows in generalization of* (5.20):

$$\mathbf{J}^{V}_{\rho,\Psi\cdot\Phi_{\alpha,\beta}}(x) - \sum_{|\gamma|<k}\frac{(-1)^{|\gamma|}}{\gamma!}\cdot\frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0)\cdot\mathbf{J}^{V}_{\rho,\Phi_{\alpha+\gamma_1,\beta+\gamma_2}}(x) \sim_{\alpha+\beta+k+2} G_k(x)$$

where the asymptotic of $\mathbf{J}^{V}_{\rho,\Phi_{\alpha+\gamma_1,\beta+\gamma_2}}$ is given in (5.22) *and*

$$G_k(x) := (-1)^k\cdot\frac{2^{\alpha+\beta+k+2}}{x^{\alpha+\beta+k+2}}\cdot\sum_{|\gamma|=k}\frac{1}{\gamma!}\cdot\frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0)\cdot\Gamma(\alpha+\gamma_1+1)\cdot\Gamma(\beta+\gamma_2+1)\,.$$

PROOF. By multiplying the *Taylor expansion* of Ψ at $x_0 = (1,1)$ with $\Phi_{\alpha,\beta}$ one obtains for y in a neighborhood of x_0 that:

$$F(y) := \Psi(y)\cdot\Phi_{\alpha,\beta}(y) - \sum_{|\gamma|<k}\frac{(-1)^{|\gamma|}}{\gamma!}\cdot\frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0)\cdot\Phi_{\alpha+\gamma_1,\beta+\gamma_2}(y)$$
$$= (-1)^k\cdot\sum_{|\gamma|=k}\frac{\Psi_\gamma(y)}{\gamma!}\cdot\Phi_{\alpha+\gamma_1,\beta+\gamma_2}(y)\,.$$

where $\Psi_\gamma(y) := k\cdot\int_0^1(1-t)^{k-1}\cdot\frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0+t\cdot[y-x_0])dt$ and $\Psi_\gamma(x_0) = \frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0)$. Lemma 5.4 shows for a neighborhood V of x_0 that $\mathbf{J}^{V}_{\rho,F}(x) \sim_{\alpha+\beta+k+2} G_k(x)$. □

In particular, for $\Psi\in C^\infty(V)$ we have proved $x^{\alpha+\beta+2}\cdot\mathbf{J}^{V}_{\rho,\Psi\cdot\Phi_{\alpha,\beta}} \sim \sum g_j$ where the functions g_j are given by:

$$g_j(x) := (-1)^j\cdot x^{\alpha+\beta+2}\cdot\sum_{|\gamma|=j}\frac{1}{\gamma!}\cdot\frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0)\cdot\mathbf{J}^{V}_{\rho,\Phi_{\alpha+\gamma_1,\beta+\gamma_2}}(x)\,.$$

Lemma 5.5 follows by straightforward arguments. We omit the proof.

LEMMA 5.5. *Let* $a_k, b_k > 0$ *such that* $\alpha(t) := \sum_{k\geq 0} a_k t^k$ *converges on* $\mathbf{R}$. *If* $\lim_{k\to\infty} \frac{a_k}{b_k} = 1$ *then* $\beta(t) := \sum_{k\geq 0} b_k t^k$ *converges on* $\mathbf{R}$ *and* $\lim_{t\to\infty} \alpha(t)\cdot\beta^{-1}(t) = 1$.

In Proposition 5.2 we apply Corollary 5.6 to (5.15) which holds for $\frac{N-n}{2} > -1$:

PROPOSITION 5.2. *Let* $\beta := \frac{N+n}{2} > \alpha := \frac{N-n}{2} > -1$, *then it holds*:

$$x^{N+2}\cdot\exp\big(-x\big)\cdot\sum_{k=0}^{\infty} I(k,n,N)\cdot x^{2k} \sim \sum g_j \tag{5.23}$$

where V *is a neighborhood of* $(1,1)$ *and with* $g_j : \mathbf{R}^+ \to \mathbf{R}^+$ *of order* $O(x^{-j})$ *as* $x\to\infty$*:*

$$g_j(x) = (-1)^j\cdot\frac{E(n,N)}{2}\cdot x^{\alpha+\beta+2}\cdot\sum_{|\gamma|=j}\binom{-\frac{1}{2}}{\gamma_1}\cdot\binom{n-1}{\gamma_2}\cdot\mathbf{J}^V_{\rho,\Phi_{\alpha+\gamma_1,\beta+\gamma_2}}(x)\,. \tag{5.24}$$

PROOF. It follows from (5.15) and the notation in (5.16) that:

$$x^{N+2}\cdot\exp(-x)\cdot\sum_{k=0}^{\infty} I(k,n,N)\cdot x^{2k} = x^{\alpha+\beta+2}\cdot\mathbf{J}_{\rho,\Psi\cdot\Phi_{\alpha,\beta}}(x) + O\big(x^{-\infty}\big) \tag{5.25}$$

where $\Psi(s,t) := \frac{E(n,N)}{2}\cdot s^{n-1}\cdot t^{-\frac{1}{2}}$. In particular, it holds with $\gamma := (\gamma_1,\gamma_2)\in\mathbf{N}_0^2$:

$$\frac{1}{\gamma!}\cdot\frac{\partial^{|\gamma|}\Psi}{\partial x^{\gamma}}(x_0) = \frac{E(n,N)}{2}\cdot\binom{-\frac{1}{2}}{\gamma_1}\cdot\binom{n-1}{\gamma_2}.$$

Finally, we can apply our remark above. □

REMARK 5.1. The integral expression (5.15) of the left hand side in (5.25) is not unique. It can be checked that in the case $N+\frac{1}{2} > -1$ a second integral formula is given by:

$$\begin{aligned}\exp(-x)\cdot\sum_{k=0}^{\infty} I(k,n,N)\cdot x^{2k} &= \int_0^1\int_0^1 \tilde{\Phi}_{n,N}(s,t)\cdot(1-t)^{N+\frac{1}{2}}\\ &\qquad\times\cosh\Big\{-\big(1-2\sqrt{s(1-s)t}\big)\cdot x\Big\}\,dsdt \end{aligned}\tag{5.26}$$

where

$$\tilde{\Phi}_{n,N}(s,t) := \frac{1}{\sqrt{\pi}\cdot\Gamma(N+\frac{3}{2})}\cdot\frac{s^{n-1}\cdot(1-s)^{N+1}}{\sqrt{t}}.$$

Using (5.26) instead of (5.15) in the proof of Proposition 5.2 an asymptotic expansion of the form (5.23) also can be derived for $N+\frac{1}{2} > -1$. In this case the functions g_j are given in terms of the integral expressions:

$$\mathbf{I}^W_{\alpha,\beta}(x) := \int_W \left(\frac{1}{2} - s\right)^\alpha \cdot \big(1-t\big)^\beta \cdot \exp\big\{-(1 - 2\sqrt{s(1-s)t}) \cdot x\big\} dx\,.$$

where $x = (s,t)$ and $\alpha, \beta > -1$ and W is a neighborhood of $(\frac{1}{2}, 1)$. We will not present a detailed calculation here.

According to (5.12) the kernel $K_{(h,N)}(\lambda,\lambda)$ on the diagonal can be expressed as $K_{(h,N)}(\lambda,\lambda) = F(|h\lambda|)$ with $F : \mathbf{R}^+ \to \mathbf{R}^+$. By (5.23) an asymptotic expansion of

$$x \mapsto x^N \cdot \exp\big(-x\big) \cdot F(x) \tag{5.27}$$

in terms of $\mathbf{J}^V_{\rho,\Phi_{\alpha,\beta}}$ where V is a neighborhood of $x_0 := (1,1)$ can be obtained explicitly in the case $\frac{N-n}{2} > -1$. We only calculate the $0th$*-order term* $\tilde{g}_0$ and we find that $\lim_{x\to\infty} \tilde{g}_0(x)$ is independent of N. This enables us to prove Proposition 5.1 in the case $N > -n$:

PROOF OF PROPOSITION 5.1. Let us first assume that $\frac{N-n}{2} > -1$, then it follows from (5.14) and (5.24) in the case $j = 0$ together with (5.22) and $U := [0,1]^2$ that:

$$\lim_{x\to\infty} g_0(x) = \frac{E(n,N)}{2} \cdot \lim_{x\to\infty} x^{\alpha+\beta+2} \cdot \mathbf{J}^U_{\rho,\Phi_{\alpha,\beta}}(x) = \frac{1}{2^n}\,.$$

where $\beta = \frac{N+n}{2} > \alpha = \frac{N-n}{2} > -1$. Because of (5.23) one also has:

$$\lim_{x\to\infty} x^{N+2} \cdot \exp(-x) \cdot \sum_{k=0}^\infty I(k,n,N) \cdot x^{2k} = \frac{1}{2^n}\,. \tag{5.28}$$

In the case $-n < N \leq n-2$ we choose $k_0 \in \mathbf{N}$ with $N + 2k_0 > n-2$. We define:

$$\beta(x) := \sum_{k=0}^\infty \frac{\Gamma(k+k_0+n)}{\Gamma(2k+2k_0+N+n+2) \cdot \Gamma(k+k_0+1)} \cdot x^{2k}\,.$$

According to Lemma 5.5 and the identity

$$\lim_{k\to\infty} \frac{\Gamma(k+n) \cdot \Gamma(k+k_0+1)}{\Gamma(k+1) \cdot \Gamma(k+k_0+n)} = 1$$

it follows that $\lim_{x\to\infty} \beta(x) \cdot \alpha(x)^{-1} = 1$ where $\alpha(x) := \sum_{k=0}^\infty I(k,n,N+2k_0) \cdot x^{2k}$. In particular, one obtains from (5.28) where N is replaced by $N + 2k_0$:

$$\exp(-x) \cdot \beta(x) \sim_{N+2k_0+2} \exp(-x) \cdot \alpha(x) \sim_{N+2k_0+2} \frac{1}{2^n x^{N+2k_0+2}}\,. \tag{5.29}$$

Because $\sum_{k=0}^\infty I(k,n,N) \cdot x^{2k} - x^{2k_0} \cdot \beta(x)$ is a polynomial and by applying (5.29), the asymptotic (5.28) in the case $-n < N < n-2$ is given by:

$$\lim_{x\to\infty} x^{N+2} \cdot \exp(-x) \cdot \sum_{k=0}^\infty I(k,n,N) \cdot x^{2k} = \lim_{x\to\infty} x^{N+2k_0+2} \cdot \exp(-x) \cdot \beta(x) = \frac{1}{2^n}\,.$$

Finally, (5.11) follows from (5.28) for $N > -n$ and (5.12) which shows that the $0th$-*order term* $\tilde{g}_0$ of the expansion (5.27) coincides with $2 \cdot C(h, n, N) \cdot g_0$. □

Let H_f be the Hankel operator on $H^2(\mathbf{X}_{\mathbf{S}^n}, dm_{(h,N)})$ where $h > 0$ and $N \geq -n$.

COROLLARY 5.7. *For $f \in L^2(\mathbf{X}_{\mathbf{S}^n}, \Omega_X)$ the operator H_f is Hilbert-Schmidt. Moreover, there is $c > 0$ independent from f such that $\|H_f\|_{HS} = \|H_{\bar{f}}\|_{HS} \leq c \cdot \|f\|_{L^2(\mathbf{X}_{\mathbf{S}^n}, \Omega_X)}$.*

PROOF. Apply Proposition 4.1 and Proposition 5.1 which shows that there is $c > 0$ with $\int_{\mathbf{X}_{\mathbf{S}^n}} |f(\lambda)|^2 K_{(h,N)}(\lambda, \lambda) dm_{(h,N)}(\lambda) \leq c \cdot \int_{\mathbf{X}_{\mathbf{S}^n}} |f|^2 d\Omega_X < \infty$. □

REMARK 5.2. In [5] (see also [16] and [17]) a family of reproducing kernel Hilbert spaces with kernel $K^{\mathbf{C}}_{(h,N)}$ on rank one complex matrices A and naturally arising from the complex projective space $P^n\mathbf{C}$ by *pairing of polarizations* is introduced. Here we only state the main result on the kernel asymptotic in [5]. As an analog to the quadric case one has:

PROPOSITION 5.3 ([5]). *Let $N > -n$ and $h > 0$, then*

$$\lim_{\|A\| \to \infty} K^{\mathbf{C}}_{(h,N)}(A, A) \cdot e^{-h\sqrt{\|A\|}} \cdot \|A\|^N = \frac{2^{1-2n}}{c} \cdot \frac{h^{2n}}{\Gamma(n)\Gamma(n-1)} \tag{5.30}$$

where $c > 0$ is independent of N and h. In particular, (5.30) *is independent of N.*

5.4. Problems and Remarks. (1) Is there an extension of Corollary 5.4 or Theorem 5.4 to *Schatten-p-class* ($p \neq 2$) or compact Hankel operators? (The compact case for H_{h} is treated in [10], [21]).

(2) Determine the bounded fix points of the Berezin transform in the case of the pluriharmonic Fock space or the spaces $H^2(\mathbf{X}_{\mathbf{S}^n}, dm_{(h,N)})$.

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