Resurgent transseries, mould calculus and Connes-Kreimer Hopf algebra

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Abstract: We study the resurgence structure of a formal normalization of a certain vector field to the normal form using "mould calculus" developed by J. Écalle. We also describe the resurgence structure of transseries solutions of a nonlinear ordinary differential equation.

Key words: Resurgence theory; mould calculus; Stokes phenomena; Hopf algebras; trees.

1. Introduction. This is an announcement of our forthcoming paper [10]. Let $\mathcal{X}$ be a vector field of the form

$$\mathcal{X} = x^2 \frac{\partial}{\partial x} + \sum_{j=1}^{n} A_j(x,y)\frac{\partial}{\partial y_j} \tag{1}$$

with $A_j(x,y) \in \mathbf{C}\{x,y\}$ satisfying for any i, j

$$A_j(0,y) = \lambda_j y_j \quad (\lambda_j \neq 0), \tag{2}$$

$$\partial_x \partial_{y_i} A_j(0,0) = 0. \tag{3}$$

We assume that $\lambda = (\lambda_1, \cdots, \lambda_n)$ satisfies the following non-resonance condition:

$$k\cdot\lambda \neq \lambda_j \text{ holds for any } j \text{ and } k \in \mathbf{Z}^n_{\geq 0} \setminus \{e_j\}, \tag{4}$$

where $k\cdot\lambda := k_1\lambda_1 + \cdots + k_n\lambda_n$ and e_j is the j-th unit vector. In this paper, we study the resurgence structure of the formal normalization Θ of $\mathcal{X}$ to the normal form

$$\mathcal{X}_0 = x^2 \frac{\partial}{\partial x} + \sum_{j=1}^{n} \lambda_j y_j \frac{\partial}{\partial y_j}, \tag{5}$$

which acts on $f(x,y) \in \mathbf{C}[[x,y]]$ as a formal coordinate transformation of the form

$$(\Theta f)(x,y) = f(x,\varphi(x,y)), \quad \varphi(x,y) \in \mathbf{C}[[x,y]].$$

For the purpose, we adopt "mould calculus" developed by J. Écalle (cf. [3]). To be precise, we use arborescent moulds, or arbomoulds for short, which are moulds associated with rooted forests. This problem was discussed in [4] in general settings. In [10], the singularity structure of an arbomould $\mathfrak{M}^\bullet$, which appears as coefficients of the arbomould expansion (15) of Θ, in the Borel plane is clearly described by discrete filtered sets based on the works [9], [11], [15], [18] and [19]. Further, the resurgence structure of Θ is given in the form (24) following the discussion in [17], where the same problem was treated using ordinary moulds, i.e., associated with words, when $n = 1$. (See also [13].)

As an application, we also consider the resurgence structure of transseries solutions of the following nonlinear ordinary differential equation:

$$x^2 \frac{d\Phi}{dx} = A(x,\Phi), \tag{6}$$

where $A = (A_1, \cdots, A_n)$. The Borel summability of the transseries solutions was studied in [2]. In this paper, we describe more precisely their singularity structure in the Borel plane and give their alien derivatives at the singular points by the bridge equation (25), which connects alien calculus and ordinary differential calculus.

2. The Connes-Kreimer Hopf algebra. We first recall the definition of the Connes-Kreimer Hopf algebra associated with (non-planar) rooted trees. (See [5], [7], [8], [12] and [16] for details.)

Definition 2.1. A (non-planar) rooted tree $T = (V,E)$ is a connected and simply connected set of edges and vertices in which one vertex is distinguished, where V (resp. E) is the set of vertices (resp. edges) of T. We call such a vertex the root of T. We regard a rooted tree as a directed graph by an arborescent orientation, i.e., each edge is directed away from the root. A rooted forest is a disjoint union of rooted trees. The empty graph $\emptyset$ is regarded as a rooted forest. The set of isomorphism classes of rooted trees (resp. rooted forests) as directed graphs is denoted by $\mathcal{T}$ (resp. $\mathcal{F}$).

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Let $\mathbf{D}$ be a non-empty set. We then define rooted trees decorated by $\mathbf{D}$ as follows:

Definition 2.2. A $\mathbf{D}$-decorated rooted tree is a pair $\tilde{T}=(T,\sigma)$, where $T=(V,E)$ is a rooted tree and σ is a map $\sigma: V\to\mathbf{D}$. A $\mathbf{D}$-decorated rooted forest is a disjoint union of $\mathbf{D}$-decorated rooted trees. Two $\mathbf{D}$-decorated rooted forests $\tilde{F}_1=(F_1,\sigma_1)$ and $\tilde{F}_2=(F_2,\sigma_2)$ are isomorphic if there exists an isomorphism $\phi: F_1=(V_1,E_1)\to F_2=(V_2,E_2)$ of rooted forests satisfying $\sigma_2=\sigma_1\circ\phi$. The set of isomorphism classes of $\mathbf{D}$-decorated rooted trees (resp. $\mathbf{D}$-decorated rooted forests) is denoted by $\mathcal{T}_{\mathbf{D}}$ (resp. $\mathcal{F}_{\mathbf{D}}$).

Remark 2.3. By abuse of notation, we use representatives to represent isomorphism classes of trees and forests since the names of the vertices are not important in the following discussion.

Definition 2.4. Let $\tilde{T}$ be a $\mathbf{D}$-decorated rooted tree. A cut c is a subset of E. Then, c defines a forest $F^c(\tilde{T})\in\mathcal{F}_{\mathbf{D}}$ by eliminating c from E. The remaining part $R^c(\tilde{T})\in\mathcal{T}_{\mathbf{D}}$ is the tree in $F^c(\tilde{T})$ that contains the root of $\tilde{T}$ and the pruned part $P^c(\tilde{T})\in\mathcal{F}_{\mathbf{D}}$ is the disjoint union of the other trees in $F^c(\tilde{T})$. A cut c is admissible if it appears at most once on any directed path from the root to leaves in $\tilde{T}$. The total cut is a virtual cut, which represents a cut above the root. In such a case, we set $R^c(\tilde{T})=\emptyset$ and $P^c(\tilde{T})=\tilde{T}$. The set of admissible cuts and the total cut of $\tilde{T}$ is denoted by $\mathrm{Adm}(\tilde{T})$.

By the definition, $\tilde{F}\in\mathcal{F}_{\mathbf{D}}$ is given by the disjoint union of $\tilde{T}_1,\cdots,\tilde{T}_\ell\in\mathcal{T}_{\mathbf{D}}$ and it is simply denoted by $\tilde{T}_1\cdots\tilde{T}_\ell$. The grafting operator $B_d^+:\mathcal{F}_{\mathbf{D}}\to\mathcal{T}_{\mathbf{D}}$ $(d\in\mathbf{D})$ is defined as follows: for $\tilde{F}=\tilde{T}_1\cdots\tilde{T}_\ell$ $(\tilde{T}_1,\cdots,\tilde{T}_\ell\in\mathcal{T}_{\mathbf{D}})$, $B_d^+(\tilde{F})=(T',\sigma')$ is defined by adding to $\tilde{F}$ a vertex ρ' (the root of T') with the decoration $\sigma'(\rho')=d$ and edges $e_j=\rho'\to\rho_j$ $(j=1,\cdots,\ell)$ from ρ' to ρ_j, where ρ_j is the root of T_j. The trimming operator $B^-:\mathcal{T}_{\mathbf{D}}\to\mathcal{F}_{\mathbf{D}}$ is defined as follows: for $\tilde{T}\in\mathcal{T}_{\mathbf{D}}$, $B^-(\tilde{T})$ is defined by removing the root ρ of $\tilde{T}$ and the edges whose tail is ρ and restricting the decoration to the forest.

In what follows, we set

$$\mathbf{D}=\{d=(d_0,d_1)\in\mathbf{Z}^n_{\geq 0}\times\mathbf{Z}^n_{\geq 0}\mid |d_0|=1\}.$$

Let $\tilde{F}=(F,\sigma=(\sigma_0,\sigma_1))$ be a $\mathbf{D}$-decorated rooted forest. The degree $\deg(\tilde{F})\in\mathbf{Z}_{\geq 0}$ of $\tilde{F}$ is defined by the cardinality of the set of vertices V of $\tilde{F}$. It is also denoted by $|\tilde{F}|$. Next, we define the weight $\mathrm{wt}(\tilde{F})\in\mathbf{Z}^n$ of $\tilde{F}$. Let σ_{in}, $\sigma_{\mathrm{ex}}: V\to\mathbf{Z}^n_{\geq 0}$ be maps defined by

$$\sigma_{\mathrm{in}}(v):=\sum_{v\to v'}\sigma_0(v'), \tag{7}$$

$$\sigma_{\mathrm{ex}}(v):=\sigma_1(v)-\sigma_{\mathrm{in}}(v),$$

where the sum in (7) is taken over all the vertices v' that is connected to v by an edge $v\to v'$. We set

$$\sigma_{\mathrm{in}}(\tilde{F})=\sum_{v\in V}\sigma_{\mathrm{in}}(v),\quad \sigma_{\mathrm{ex}}(\tilde{F})=\sum_{v\in V}\sigma_{\mathrm{ex}}(v),$$

$$r(\tilde{F})=\sum_{\rho}\sigma_0(\rho), \tag{8}$$

where the sum in (8) is taken over all the roots of $\tilde{F}$. We then define the weight $\mathrm{wt}(\tilde{F})$ of $\tilde{F}$ by

$$\mathrm{wt}(\tilde{F})=\sigma_{\mathrm{ex}}(\tilde{F})-r(\tilde{F}).$$

It is also denoted by $\|\tilde{F}\|$.

We introduce a subclass of $\mathbf{D}$-decorated rooted forests as follows:

$$\mathcal{F}^+_{\mathbf{D}}=\{\tilde{F}\in\mathcal{F}_{\mathbf{D}}\mid\sigma_{\mathrm{ex}}(v)\in\mathbf{Z}^n_{\geq 0}\ (\forall v\in V)\}.$$

A $\mathbf{D}$-decorated rooted forest $\tilde{F}$ is said to be proper if $\tilde{F}\in\mathcal{F}^+_{\mathbf{D}}$. We see that $\sigma_{\mathrm{ex}}(\tilde{F})\in\mathbf{Z}^n_{\geq 0}$ if $\tilde{F}\in\mathcal{F}^+_{\mathbf{D}}$. We put

$$\mathcal{F}_{\mathbf{D},k,\ell}:=\{\tilde{F}\in\mathcal{F}_{\mathbf{D}}\mid\deg(\tilde{F})=k,\mathrm{wt}(\tilde{F})=\ell\},$$

$$\mathcal{H}:=\bigoplus_{\tilde{F}\in\mathcal{F}_{\mathbf{D}}}\mathbf{C}\tilde{F},\quad \mathcal{H}_{k,\ell}:=\bigoplus_{\tilde{F}\in\mathcal{F}_{\mathbf{D},k,\ell}}\mathbf{C}\tilde{F}.$$

Similarly, we define $\mathcal{H}^+$ (resp. $\mathcal{H}^+_{k,\ell}$) using $\mathcal{F}^+_{\mathbf{D}}$ (resp. $\mathcal{F}^+_{\mathbf{D},k,\ell}:=\mathcal{F}^+_{\mathbf{D}}\cap\mathcal{F}_{\mathbf{D},k,\ell}$) instead of $\mathcal{F}_{\mathbf{D}}$ (resp. $\mathcal{F}_{\mathbf{D},k,\ell}$).

Remark 2.5. A proper $\mathbf{D}$-decorated rooted forest $\tilde{F}$ is visualized by a rooted forest with external lines. The j-th component of $\sigma_{\mathrm{in}}(v)$ (resp. $\sigma_{\mathrm{ex}}(v)$) represents the number of internal (resp. external) lines of color j going out of the vertex v.

Now, let $\mathrm{CK}_{\mathbf{D}}=(\mathcal{H},\mu,\eta,\Delta,\varepsilon,S)$ be the Connes-Kreimer Hopf algebra of $\mathbf{D}$-decorated rooted forests. The product μ is defined by the disjoint union of forests. The unit η is defined by $\eta(1)=\emptyset$. The coproduct Δ is given for $\tilde{T}\in\mathcal{T}_{\mathbf{D}}$ by

$$\Delta(\tilde{T})=\sum_{c\in\mathrm{Adm}(\tilde{T})}P^c(\tilde{T})\otimes R^c(\tilde{T})$$

and extended to $\tilde{F}\in\mathcal{F}_{\mathbf{D}}$ algebraically. The counit ε is defined for $\tilde{F}\in\mathcal{F}_{\mathbf{D}}$ by $\varepsilon(\tilde{F})=1$ if $\tilde{F}=\emptyset$ and $\varepsilon(\tilde{F})=0$ otherwise. The antipode S is recursively determined by the bialgebraic structure. By the definition of the degree and the weight, we see they are compatible with the Hopf algebraic structure, i.e.,

$$\mu : \mathcal{H}_{k,\ell} \otimes \mathcal{H}_{k',\ell'} \to \mathcal{H}_{k+k',\ell+\ell'}, \tag{9}$$

$$\Delta : \mathcal{H}_{k,\ell} \to \bigoplus_{\substack{k'+k''=k \\ \ell'+\ell''=\ell}} \mathcal{H}_{k',\ell'} \otimes \mathcal{H}_{k'',\ell''}, \tag{10}$$

$$S : \mathcal{H}_{k,\ell} \to \mathcal{H}_{k,\ell}. \tag{11}$$

Let $\mathrm{CK}_{\mathbf{D}}^{+}$ be the restriction of $\mathrm{CK}_{\mathbf{D}}$ to the subspace $\mathcal{H}^{+}$. Then, we see that $\mathrm{CK}_{\mathbf{D}}^{+}$ is a sub Hopf algebra of $\mathrm{CK}_{\mathbf{D}}$ (cf. [5]).

3. Arbomoulds and coarbomoulds. Let $(\mathcal{A}, \mu_{\mathcal{A}}, \eta_{\mathcal{A}})$ be a unital commutative $\mathbf{C}$-algebra. An $\mathcal{A}$-valued arbomould $\mathfrak{M}^{\bullet}$ is a map $\mathfrak{M}^{\bullet} : \mathcal{F}_{\mathbf{D}} \to \mathcal{A}$. By linearly extending $\mathfrak{M}^{\bullet}$ to $\mathcal{H}$, $\mathfrak{M}^{\bullet}$ defines a $\mathbf{C}$-linear map $\mathfrak{M} \in \mathcal{L}(\mathcal{H}, \mathcal{A})$, where $\mathcal{L}(\mathcal{H}, \mathcal{A})$ denotes the set of $\mathbf{C}$-linear maps from $\mathcal{H}$ to $\mathcal{A}$. We write the value of $\mathfrak{M}^{\bullet}$ (resp. $\mathfrak{M}$) at $\tilde{F} \in \mathcal{F}_{\mathbf{D}}$ (resp. $h \in \mathcal{H}$) by $\mathfrak{M}^{\tilde{F}}$ (resp. $\mathfrak{M}(h)$). The product $\mathfrak{P}^{\bullet} = \mathfrak{M}^{\bullet} \times \mathfrak{N}^{\bullet}$ of arbomoulds $\mathfrak{M}^{\bullet}$ and $\mathfrak{N}^{\bullet}$ is defined by the convolution product of $\mathfrak{M}$ and $\mathfrak{N}$, i.e., $\mathfrak{P}^{\tilde{F}} = \mathfrak{M} * \mathfrak{N}(\tilde{F})$ for $\tilde{F} \in \mathcal{F}_{\mathbf{D}}$, where the convolution product $\mathfrak{M} * \mathfrak{N}$ of $\mathfrak{M}$ and $\mathfrak{N}$ is defined by $\mathfrak{M} * \mathfrak{N} := \mu_{\mathcal{A}} \circ (\mathfrak{M} \otimes \mathfrak{N}) \circ \Delta$.

Definition 3.1. (1) An arbomould $\mathfrak{M}^{\bullet}$ is said to be separative if $\mathfrak{M}$ is a character, i.e., $\mathfrak{M}$ satisfies $\mathfrak{M} \circ \mu = \mu_{\mathcal{A}} \circ (\mathfrak{M} \otimes \mathfrak{M})$. (2) An arbomould $\mathfrak{M}^{\bullet}$ is said to be antiseparative if $\mathfrak{M}$ is an infinitesimal character, i.e., $\mathfrak{M}$ satisfies $\mathfrak{M} \circ \mu = \mathfrak{M} \otimes e_{\mathcal{A}} + e_{\mathcal{A}} \otimes \mathfrak{M}$, where $e_{\mathcal{A}} := \eta_{\mathcal{A}} \circ \varepsilon$.

Let $(\mathcal{B}, \mu_{\mathcal{B}}, \eta_{\mathcal{B}})$ be a unital commutative $\mathcal{A}$-algebra. A coarbomould $\mathbf{B}_{\bullet}$ is a map $\mathbf{B}_{\bullet} : \mathcal{F}_{\mathbf{D}} \to \mathrm{End}_{\mathcal{A}}(\mathcal{B})$. We write the image of $\varphi \in \mathcal{B}$ by the morphism $\theta \in \mathrm{End}_{\mathcal{A}}(\mathcal{B})$ by $\theta \cdot \varphi$.

In what follows, we take $\mathcal{A} = \mathbf{C}[[x]]$ and $\mathcal{B} = \mathbf{C}[[x, y]]$ and

$$\mathbf{B}_{\bullet} : \mathcal{F}_{\mathbf{D}} \to \mathbf{C}[y, \partial_y] \subset \mathrm{End}_{\mathcal{A}}(\mathcal{B})$$

is fixed by the one constructed by the following rules (cf. [5] and [6]).

i) $\mathbf{B}_{\emptyset} = 1$.

ii) For the forest $\bullet_d \in \mathcal{F}_{\mathbf{D}}$ $(d = (d_0, d_1) \in \mathbf{D})$ consisting of a single vertex v with the decoration $\sigma(v) = d$, we put

$$\mathbf{B}_{\bullet_d} = y^{d_1} \partial_y^{d_0}.$$

iii) When $\tilde{T} = B_d^{+}(\tilde{F})$ with $d = (d_0, d_1) \in \mathbf{D}$ and $\tilde{F} \in \mathcal{F}_{\mathbf{D}}$, we put

$$\mathbf{B}_{\tilde{T}} = (\mathbf{B}_{\tilde{F}} \cdot y^{d_1}) \partial_y^{d_0}.$$

iv) When $\tilde{F} = \tilde{T}_1^{k_1} \cdots \tilde{T}_\ell^{k_\ell}$ for $\tilde{T}_1, \cdots, \tilde{T}_\ell \in \mathcal{F}_{\mathbf{D}}$ with $\tilde{T}_i \neq \tilde{T}_j$ $(i \neq j)$ and $k_1, \cdots, k_\ell \in \mathbf{Z}_{\geq 0}$, we put

$$\mathbf{B}_{\tilde{F}} = \left(\prod_{j=1}^{\ell} \frac{(\mathbf{B}_{\tilde{T}_j} \cdot y^{r(\tilde{T}_j)})^{k_j}}{k_j!} \right) \partial_y^{r(\tilde{F})}.$$

By the construction of $\mathbf{B}_{\bullet}$, we see that $\mathbf{B}_{\bullet}$ satisfies the following properties.

(12) $\mathbf{B}_{\tilde{F}}$ is written in the form $\beta_{\tilde{F}} y^{\sigma_{\mathrm{ex}}(\tilde{F})} \partial_y^{r(\tilde{F})}$ with a constant $\beta_{\tilde{F}} \in \mathbf{Q}$.

(13) $\mathbf{B}_{\tilde{F}} = 0$ if $\tilde{F} \in \mathcal{F}_{\mathbf{D}} \setminus \mathcal{F}_{\mathbf{D}}^{+}$.

(14) For any $\tilde{F} \in \mathcal{F}_{\mathbf{D}}$ and $\varphi, \psi \in \mathbf{C}[[x, y]]$,

$$\mathbf{B}_{\tilde{F}}(\varphi\psi) = \sum_{\tilde{F}', \tilde{F}''} \mathbf{B}_{\tilde{F}'}(\varphi) \mathbf{B}_{\tilde{F}''}(\psi),$$

where the sum is taken over all the pairs $(\tilde{F}', \tilde{F}'') \in \mathcal{F}_{\mathbf{D}} \times \mathcal{F}_{\mathbf{D}}$ satisfying $\tilde{F} = \tilde{F}'\tilde{F}''$.

4. The arbomould expansion of Θ. In this section, we give an arbomould(-coarbomould) expansion of Θ of the following form (see [17] for the convergence of the expansion):

$$\Theta = \sum \mathfrak{M}^{\bullet} \mathbf{B}_{\bullet} := \sum_{\tilde{F} \in \mathcal{F}_{\mathbf{D}}^{+}} \mathfrak{M}^{\tilde{F}} \mathbf{B}_{\tilde{F}}. \tag{15}$$

In what follows, we consider arbomoulds defined only on $\mathcal{F}_{\mathbf{D}}^{+}$ since their values at $\mathcal{F}_{\mathbf{D}} \setminus \mathcal{F}_{\mathbf{D}}^{+}$ are not used by (13). Let $A_j(x, y)$ be written as follows:

$$A_j(x, y) = \lambda_j y_j + \sum_{\alpha \in \mathbf{Z}_{\geq 0}^n} a_{j,\alpha}(x) y^{\alpha},$$

where $a_{j,\alpha}(x) \in x\mathbf{C}\{x\}$. We define an arbomould $\mathfrak{a}^{\bullet} : \mathcal{F}_{\mathbf{D}}^{+} \to \mathbf{C}[[x]]$ by

$$\mathfrak{a}^{\tilde{F}} = \begin{cases} a_{j,d_1}(x) & \text{if} \quad \tilde{F} = \bullet_{(e_j, d_1)}, \\ 0 & \text{otherwise}, \end{cases} \tag{16}$$

where e_j is the j-th unit vector. We set

$$\mathcal{Z} = \sum \mathfrak{a}^{\bullet} \mathbf{B}_{\bullet}. \tag{17}$$

Then, $\mathcal{X} = \mathcal{X}_0 + \mathcal{Z}$ and the equation $\Theta\mathcal{X} = \mathcal{X}_0\Theta$ is equivalent to $[\mathcal{X}_0, \Theta] = \Theta\mathcal{Z}$. Therefore, using

$$[\mathcal{X}_0, \mathfrak{M}^{\tilde{F}} \mathbf{B}_{\tilde{F}}] = \left[\left(x^2 \frac{\partial}{\partial x} + \nabla_\lambda \right) \mathfrak{M}^{\tilde{F}} \right] \mathbf{B}_{\tilde{F}}, \tag{18}$$

$$\Theta\mathcal{Z} = \sum (\mathfrak{M}^{\bullet} \times \mathfrak{a}^{\bullet}) \mathbf{B}_{\bullet}, \tag{19}$$

the problem is reduced to constructing an arbomould $\mathfrak{M}^{\bullet}$ satisfying $\mathfrak{M}^{\emptyset} = 1$ and

$$\left(x^2 \frac{\partial}{\partial x} + \nabla_\lambda \right) \mathfrak{M}^{\bullet} = \mathfrak{M}^{\bullet} \times \mathfrak{a}^{\bullet}, \tag{20}$$

where $\nabla_\lambda \mathfrak{M}^{\bullet}$ is an arbomould defined by $\nabla_\lambda \mathfrak{M}^{\tilde{F}} =$

$(\lambda \cdot \|\tilde{F}\|)\mathfrak{M}^{\tilde{F}}$ for $\tilde{F} \in \mathcal{F}^+_{\mathbf{D}}$. Then, $\mathfrak{M}^\bullet$ is recursively and uniquely determined by solving

$$(21)\qquad \left(x^2\frac{\partial}{\partial x} + \lambda \cdot \|\tilde{T}\|\right)\mathfrak{M}^{\tilde{T}} = a_{j,d_1}(x)\mathfrak{M}^{B^-\tilde{T}}$$

for $\tilde{T} \in \mathcal{T}^+_{\mathbf{D}}$ with the root ρ having the decoration $\sigma(\rho) = (e_j, d_1)$, and by setting

$$(22)\qquad \mathfrak{M}^{\tilde{F}} := \mathfrak{M}^{\tilde{T}_1}\cdots\mathfrak{M}^{\tilde{T}_\ell}$$

for $\tilde{F} = \tilde{T}_1\cdots\tilde{T}_\ell$ ($\ell \geq 2$) with $\tilde{T}_j \in \mathcal{T}^+_{\mathbf{D}}$ ($j = 1, \cdots, \ell$). Then, we have the following

Theorem 4.1 ([10]). *There exists uniquely a separative arbomould $\mathfrak{M}^\bullet : \mathcal{F}^+_{\mathbf{D}} \to \mathbf{C}[[x]]$ such that $\Theta = \sum \mathfrak{M}^\bullet \mathbf{B}_\bullet$ defines a tangent-to-identity formal diffeomorphism satisfying $\Theta\mathcal{X} = \mathcal{X}_0\Theta$.*

5. Resurgent formal series. In this section, we review the results in [9] and [11] related to the resurgence of formal series. We first recall the definition of the formal Borel transform. The formal Borel transform $\mathfrak{B} : x\mathbf{C}[[x]] \to \mathbf{C}[[\xi]]$ is a $\mathbf{C}$-linear morphism defined by

$$\mathfrak{B} : \tilde{\varphi}(x) = \sum_{k=1}^{\infty} \varphi_k x^k \mapsto \hat{\varphi}(\xi) = \sum_{k=1}^{\infty} \varphi_k \frac{\xi^{k-1}}{(k-1)!}.$$

It is linearly extended to $\mathfrak{B} : \mathbf{C}[[x]] \to \mathbf{C}\delta \oplus \mathbf{C}[[\xi]]$ by

$$\mathfrak{B} : \varphi_0 + \tilde{\varphi}(x) \in \mathbf{C}1 \oplus x\mathbf{C}[[x]] \mapsto \varphi_0\delta + \hat{\varphi}(\xi),$$

where the symbol δ denotes the image of 1 by $\mathfrak{B}$. We put $\mathbf{C}[[x]]_1 := \mathfrak{B}^{-1}(\mathbf{C}\delta \oplus \mathbf{C}\{\xi\})$. The convolution product $\hat{\varphi}_1 * \hat{\varphi}_2$ of $\hat{\varphi}_1$, $\hat{\varphi}_2 \in \mathbf{C}\{\xi\}$ is defined by

$$\hat{\varphi}_1 * \hat{\varphi}_2 := \int_0^{\xi} \hat{\varphi}_1(\xi - \xi')\hat{\varphi}_2(\xi')d\xi'.$$

We see that $\hat{\varphi}_1 * \hat{\varphi}_2 \in \mathbf{C}\{\xi\}$ and satisfies $\hat{\varphi}_1 * \hat{\varphi}_2 = \mathfrak{B}(\tilde{\varphi}_1\tilde{\varphi}_2)$ if $\tilde{\varphi}_1$, $\tilde{\varphi}_2 \in x\mathbf{C}[[x]]_1$. The multiplication $\cdot * \cdot$ is naturally extended to $\mathbf{C}\delta \oplus \mathbf{C}\{\xi\}$ so that δ is the unit in the algebra.

For a detailed description of singular points of holomorphic germs, we use discrete filtered sets introduced in [1].

Definition 5.1. A discrete filtered set is a family $\Omega = (\Omega_L)_{L\in\mathbf{R}_{\geq 0}}$ of subsets in $\mathbf{C}$ such that

i) Ω_L is a finite set,
ii) $\Omega_{L_1} \subset \Omega_{L_2}$ if $L_1 \leq L_2$,
iii) there exists $R > 0$ such that $\Omega_R = \emptyset$.

Notation 5.2. (1) For a discrete filtered set Ω, we put

$$R(\Omega) := \sup\{L \in \mathbf{R}_{\geq 0} \mid \Omega_L = \emptyset\},$$

$$\mathcal{S}_\Omega := \{(L, \omega) \in \mathbf{R} \times \mathbf{C} \mid L \geq 0 \text{ and } \omega \in \Omega_L\}.$$

(2) Let E be a discrete closed subset of $\mathbf{C}$ that does not contain 0. Then, it can be regarded as a discrete filtered set by identifying it with a discrete filtered set Ω defined by $\Omega_L := \{\omega \in E \mid |\omega| \leq L\}$ for each $L \in \mathbf{R}_{\geq 0}$. We use the same notation E to denote it.

Let Ω and Ω' be discrete filtered sets. When $\Omega_L \subset \Omega'_L$ holds for any $L \in \mathbf{R}_{\geq 0}$, it is denoted by $\Omega \subset \Omega'$. Next, we define $\Omega * \Omega'$ (resp. $\Omega \cup \Omega'$) by $(\Omega * \Omega')_L := \{\omega + \omega' \mid \omega \in \Omega_{L_1}, \omega' \in \Omega_{L_2}, L_1 + L_2 = L\} \cup \Omega_L \cup \Omega'_L$ (resp. $(\Omega \cup \Omega')_L := \Omega_L \cup \Omega'_L$). In particular, $\underbrace{\Omega * \cdots * \Omega}_{n \text{ times}}$ is denoted by Ω^{*n} ($n \geq 1$). Since it consists an inductive system, a discrete filtered set $\Omega^{*\infty}$ is defined by

$$\Omega^{*\infty} := \varinjlim_{n} \Omega^{*n}.$$

Definition 5.3. (1) A Lipschitz continuous path $\gamma : [0, a] \to \mathbf{C}$ ($a > 0$) satisfying $\gamma(0) = 0$ is said to be Ω-allowed if it satisfies $(L_\gamma(t), \gamma(t)) \notin \overline{\mathcal{S}_\Omega}$ for any $t \in [0, a]$, where $L_\gamma(t)$ is the length of the path $\gamma|_{[0,t]}$ and $\overline{\mathcal{S}_\Omega}$ is the closure of $\mathcal{S}_\Omega$ in $\mathbf{R} \times \mathbf{C}$. The set of Ω-allowed paths is denoted by Π_Ω. (2) A Lipschitz continuous path $\gamma : [0, a] \to \mathbf{C}$ is said to be Ω-adherent if it satisfies $\gamma|_{[0,t]} \in \Pi_\Omega$ for any $t \in [0, a)$ and $(L_\gamma(a), \gamma(a)) \in \overline{\mathcal{S}_\Omega}$. The set of Ω-adherent paths is denoted by Π^{ad}_Ω. (3) A holomorphic germ $\hat{\varphi}$ at $\xi = 0$ is said to be Ω-continuable if it is analytically continuable along any Ω-allowed path. The set of Ω-continuable germs is denoted by $\widehat{\mathscr{R}}_\Omega$. (4) An Ω-continuable germ $\hat{\varphi}$ is said to be simple if, for any path $\gamma \in \Pi^{\mathrm{ad}}_\Omega$, the analytic continuation $\mathrm{cont}_\gamma\hat{\varphi}$ of $\hat{\varphi}$ along γ has a simple singularity at the end point ω of γ, i.e., it has the following form on the universal covering space of $\{\xi \mid 0 < |\xi - \omega| < r\}$ for sufficiently small $r > 0$:

$$\frac{C_\gamma}{2\pi i(\xi - \omega)} + \frac{1}{2\pi i}\hat{\chi}_\gamma(\xi - \omega)\log(\xi - \omega) + \mathrm{reg}_\gamma(\xi - \omega),$$

where $C_\gamma \in \mathbf{C}$ and $\hat{\chi}_\gamma$, $\mathrm{reg}_\gamma \in \mathbf{C}\{\xi\}$. We associate such a singularity with $C_\gamma\delta + \hat{\chi}_\gamma(\xi) \in \mathbf{C}\delta \oplus \mathbf{C}\{\xi\}$, and hence, a formal series in $\mathbf{C}[[x]]_1$ by $\mathfrak{B}^{-1}$. The set of simple Ω-continuable germs is denoted by $\widehat{\mathscr{R}}^{\mathrm{simp}}_\Omega$.

We put

$$\mathscr{R}_\Omega := \mathbf{C}1 \oplus \widetilde{\mathscr{R}}_\Omega, \quad \widetilde{\mathscr{R}}_\Omega := \mathfrak{B}^{-1}(\widehat{\mathscr{R}}_\Omega),$$
$$\mathscr{R}^{\mathrm{simp}}_\Omega := \mathbf{C}1 \oplus \widetilde{\mathscr{R}}^{\mathrm{simp}}_\Omega, \quad \widetilde{\mathscr{R}}^{\mathrm{simp}}_\Omega := \mathfrak{B}^{-1}(\widehat{\mathscr{R}}^{\mathrm{simp}}_\Omega)$$

An element of $\mathscr{R}_\Omega$ (resp. $\mathscr{R}_\Omega^{\mathrm{simp}}$) is called Ω-resurgent formal series (resp. simple Ω-resurgent formal series). By extending the results in [14] and [15], we have the following

Theorem 5.4 ([10]). *Let Ω and Ω' be discrete filtered sets.* (1) *For any $\hat{\varphi} \in \widehat{\mathscr{R}}_\Omega^{\mathrm{simp}}$ and $\hat{\psi} \in \widehat{\mathscr{R}}_{\Omega'}^{\mathrm{simp}}$, we have $\hat{\varphi} * \hat{\psi} \in \widehat{\mathscr{R}}_{\Omega * \Omega'}^{\mathrm{simp}}$.* (2) *For any $\varphi \in \mathscr{R}_\Omega^{\mathrm{simp}}$ and $\psi \in \mathscr{R}_{\Omega'}^{\mathrm{simp}}$, we have $\varphi\psi \in \mathscr{R}_{\Omega*\Omega'}^{\mathrm{simp}}$.*

6. The resurgence structure of $\mathfrak{M}^\bullet$. In what follows, we study the resurgence structure of the arbomould $\mathfrak{M}^\bullet$ constructed in Section 4.

We inductively define a family of discrete filtered sets $\Omega_{\tilde{F}}$ ($\tilde{F} \in \mathcal{F}_{\mathbf{D}}^+$) by the following rules.

i) $\Omega_\emptyset = \emptyset$.

ii) When $\tilde{T} \in \mathcal{T}_{\mathbf{D}}^+$, we put

$$\Omega_{\tilde{T}} = \begin{cases} \tilde{\Omega}_{B^-\tilde{T}} & \text{if } \|\tilde{T}\| = 0, \\ \{-\lambda \cdot \|\tilde{T}\|\} \cup \Omega_{B^-\tilde{T}} & \text{if } \|\tilde{T}\| \neq 0, \end{cases}$$

where $\tilde{\Omega}$ is a discrete filtered set defined by

$$\tilde{\Omega}_L = \begin{cases} \Omega_L & (L < 2R(\Omega)), \\ \{0\} \cup \Omega_L & (L \geq 2R(\Omega)) \end{cases}$$

for a discrete filtered set Ω.

iii) When $\tilde{F} = \tilde{T}_1 \cdots \tilde{T}_\ell$ ($\ell \geq 2$) with $\tilde{T}_j \in \mathcal{T}_{\mathbf{D}}^+$ ($j = 1, \cdots, \ell$), we put $\Omega_{\tilde{F}} = \Omega_{\tilde{T}_1} * \cdots * \Omega_{\tilde{T}_\ell}$.

Then, we have the following

Theorem 6.1 ([10]). *For any $\tilde{F} \in \mathcal{F}_{\mathbf{D}}^+$, we have $\mathfrak{M}^{\tilde{F}} \in \mathscr{R}_{\Omega_{\tilde{F}}}^{\mathrm{simp}}$.*

Remark 6.2. The assumption (3) is used to make the singularities of $\mathfrak{M}^{\tilde{F}}$ in the Borel plane simple.

Let Δ_ω be the alien derivation at $\omega \in \mathbf{C} \setminus \{0\}$ (see [14] and [17] for the definition). Then, by Theorem 6.1, we see $\Delta_\omega \mathfrak{M}^{\tilde{F}} \in \mathbf{C}[[x]]_1$ for any $\tilde{F} \in \mathcal{F}_{\mathbf{D}}^+$. We define an arbomould $\mathfrak{c}_\omega^\bullet : \mathcal{F}_{\mathbf{D}}^+ \to \mathbf{C}[[x]]_1$ by

$$\mathfrak{c}_\omega^\bullet = (\Delta_\omega \mathfrak{M})^\bullet \times (\mathfrak{M}^{\times(-1)})^\bullet, \tag{23}$$

where $(\mathfrak{M}^{\times(-1)})^\bullet$ is the multiplicative inverse of the separative arbomould $\mathfrak{M}^\bullet$ defined by $\mathfrak{M} \circ S$. Then, we have

Theorem 6.3 ([10]). *The arbomould $\mathfrak{c}_\omega^\bullet$ ($\omega \in \mathbf{C} \setminus \{0\}$) satisfies the following:*

i) $\mathfrak{c}_\omega^\bullet$ is antiseparative, and hence, $\mathfrak{c}_\omega^{\tilde{F}} = 0$ for any $\tilde{F} \in \mathcal{F}_{\mathbf{D}}^+ \setminus \mathcal{T}_{\mathbf{D}}^+$.

ii) $\mathfrak{c}_\omega^{\tilde{T}} \in \mathbf{C}$ for any $\tilde{T} \in \mathcal{T}_{\mathbf{D}}^+$.

iii) $\mathfrak{c}_\omega^{\tilde{T}} = 0$ if $\omega \neq -\lambda \cdot \|\tilde{T}\|$.

By (12) and Theorem 6.3, we see

$$\mathcal{C}_\omega := \sum \mathfrak{c}_\omega^\bullet \mathbf{B}_\bullet$$

defines a first order differential operator of the form

$$\sum_{j=1}^n \sum_{\alpha \in \mathbf{Z}_{\geq 0}^n} \mathfrak{C}_\omega^{j,\alpha} y^\alpha \partial_{y_j}, \quad \mathfrak{C}_\omega^{j,\alpha} = \sum_{\tilde{T} \in \mathcal{U}^{j,\alpha}} \beta_{\tilde{T}} \mathfrak{c}_\omega^{\tilde{T}},$$

where

$$\mathcal{U}^{j,\alpha} := \{\tilde{T} \in \mathcal{T}_{\mathbf{D}}^+ \mid r(\tilde{T}) = e_j, \sigma_{\mathrm{ex}}(\tilde{T}) = \alpha\}.$$

For later convenience, we extend $\mathfrak{C}_\omega^{j,\alpha}$ to $\alpha \in \mathbf{Z}^n$ by setting $\mathfrak{C}_\omega^{j,\alpha} = 0$ for $\alpha \notin \mathbf{Z}_{\geq 0}^n$. We obtain from (23) the following relation as operators in $\mathrm{End}_{\mathbf{C}}(\mathscr{R}[[y]])$:

$$[\Delta_\omega, \Theta] = \mathcal{C}_\omega \Theta, \tag{24}$$

where $\mathscr{R}$ is the space of resurgent formal series given by the union of $\mathscr{R}_\Omega$ over all the discrete filtered sets. It describes explicitly the resurgence structure of Θ: the singularity structure of Θ at any point $\xi = \omega \in \mathbf{C} \setminus \{0\}$ in the Borel plane is written by Θ itself though it is constructed at $\xi = 0$. We can also derive similar descriptions for the alien operator Δ_ω^+ and the symbolic Stokes automorphism $\mathfrak{S}_\theta$ in a direction θ (see [14], [17] for the definitions of these operators). See [10] for the details.

7. The resurgence structure of transseries solutions. Let $\varphi = (\varphi_1, \cdots, \varphi_n)$ be a formal series defined by

$$\varphi_j(x, y) = \Theta \cdot y_j = \sum_{\alpha \in \mathbf{Z}_{\geq 0}^n} \varphi_{j,\alpha}(x) y^\alpha \in \mathbf{C}[[x, y]],$$

where

$$\varphi_{j,\alpha}(x) = \sum_{\tilde{T} \in \mathcal{U}^{j,\alpha}} \beta_{\tilde{T}} \mathfrak{M}^{\tilde{T}}.$$

We see $\varphi(x, Ce^{-\lambda/x})$ formally gives a transseries solution of (6), where

$$Ce^{-\lambda/x} = (C_1 e^{-\lambda_1/x}, \cdots, C_n e^{-\lambda_n/x})$$

with $(C_1, \cdots, C_n) \in \mathbf{C}^n$. In particular, $\varphi(x, 0) \in \mathbf{C}^n[[x]]$ is a formal power series solution of (6).

We define discrete filtered sets Λ_α ($\alpha \in \mathbf{Z}_{\geq 0}^n$) by $\Lambda_\alpha := \bigcup_j \{(e_j - \alpha) \cdot \lambda\}$, where the sum is taken over j satisfying $e_j - \alpha \neq 0$. We also define discrete filtered sets Ω_α ($\alpha \in \mathbf{Z}_{\geq 0}^n$) by $(\Lambda_0)^{*\infty}$ when $\alpha = 0$ and by

$$\left(\bigcup_{1 \leq \ell \leq |\alpha|} \bigcup_{\substack{\alpha_1 + \cdots + \alpha_\ell = \alpha \\ \alpha_j \in \mathbf{Z}_{\geq 0}^n \setminus \{0\}}} \Lambda_{\alpha_1} * \cdots * \Lambda_{\alpha_\ell} \right) * (\Lambda_0)^{*\infty}$$

when $\alpha \neq 0$. Then, we have the following

Lemma 7.1 ([10]). *For any $\tilde{T} \in \mathcal{T}^{+}_{\mathbf{D}}$, we have $\Omega_{\tilde{T}} \subset \tilde{\Omega}_{\sigma_{\mathrm{ex}}(\tilde{T})}$. More precisely, we have $\Omega_{\tilde{T}} \subset \Omega_0$ when $\sigma_{\mathrm{ex}}(\tilde{T}) = 0$.*

The Ω_0-resurgence of $\varphi_{j,0}(x)$ was proved in [9]. Further, using the estimates in [9] for resurgent formal series associated with rooted trees, we have the following

Theorem 7.2 ([10]). *For each j and α, $\varphi_{j,\alpha}$ is simple $\tilde{\Omega}_\alpha$-resurgent. More precisely, $\varphi_{j,0}$ is simple Ω_0-resurgent.*

Remark 7.3. We can also obtain the Borel summability of $\varphi_{j,\alpha}$ except for the direction derived from Ω_α. See [10] for the details.

By the relation (24), we have the following bridge equation:

$$\Delta_\omega \varphi_j(x,y) = \mathcal{C}_\omega \cdot \varphi_j(x,y). \tag{25}$$

To be precise, we have the following

Theorem 7.4 ([10]). *For each j, α and $\omega \neq 0$, $\Delta_\omega \varphi_{j,\alpha}$ is given by*

$$\Delta_\omega \varphi_{j,\alpha} = \left(\sum_{k=1}^{n} \nu_k \mathfrak{C}^{k,\mu+e_k}_{\omega} \right) \varphi_{j,\nu}$$

if there exists $\mu \in \bigcup_{k=1}^{n} (-e_k + \mathbf{Z}^n_{\geq 0})$ and $\nu \in \mathbf{Z}^n_{\geq 0}$ satisfying $\omega = -\lambda \cdot \mu$ and $\alpha = \mu + \nu$. Otherwise, $\Delta_\omega \varphi_{j,\alpha} = 0$.

For example, $\Delta_\omega \varphi_{j,0}$ survives only at $\omega = \lambda_k$ $(k = 1, \cdots, n)$ and we see

$$\Delta_{\lambda_k} \varphi_{j,0} = \mathfrak{C}^{k,0}_{\lambda_k} \varphi_{j,e_k}.$$

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