A note on the Peterson hit problem for the Steenrod algebra

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Abstract: Let $P_k = \oplus_{n\geqslant 0}(P_k)_n \cong \mathbf{F}_2[x_1, x_2, \dots, x_k]$ be the graded polynomial algebra over the prime field of two elements $\mathbf{F}_2$, in k generators $x_1, x_2, \dots, x_k$, each of degree 1. Being the mod-2 cohomology of the classifying space $B(\mathbf{Z}/2)^k$, the algebra P_k is a module over the mod-2 Steenrod algebra $\mathcal{A}$.

In this Note, we explicitly compute the hit problem of some generic degrees $r(2^s-1)+2^s m$ in P_k, where $r = k-1 = 4, m \in \{8; 10; 11\}$ and s an arbitrary non-negative integer. Moreover, as a consequence, we get the dimension results for polynomial algebra in some generic degrees and in the cases $k = 5$ and 6.

Key words: Polynomial algebra; Steenrod algebra; hit problem.

1. Introduction. Denote by $P_k = H^*((\mathbf{R}P^\infty)^k)$ the modulo-2 cohomology algebra of the direct product of k copies of infinite dimensional real projective spaces $\mathbf{R}P^\infty$. Then, P_k is isomorphic to the graded polynomial algebra $\mathbf{F}_2[x_1, x_2, \dots, x_k]$ of k variables, in which each x_j is of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbf{F}_2$ of two elements.

The $\mathcal{A}$-module structure of P_k is explicitly determined by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & i > 1, \end{cases}$$

and the Cartan formula $Sq^n(xy) = \sum_{i=0}^{n} Sq^i(x)Sq^{n-i}(y)$, where $x, y \in P_k$ (see Steenrod and Epstein [18]).

A polynomial f in P_k is called *hit* if it can be written as a finite sum $f = \sum_{u\geqslant 0} Sq^{2^u}(h_u)$ for suitable polynomials h_u. That means f belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$.

The *Peterson hit problem* is to find a minimal generating set for P_k regarded as a module over the mod-2 Steenrod algebra. Equivalently, this problem is to find a basis for the vector space

$$\mathbf{F}_2 \otimes_{\mathcal{A}} P_k \cong P_k/\mathcal{A}^+P_k$$

in each degree n, where $\mathcal{A}^+$ is an ideal of $\mathcal{A}$ generated by all Steenrod squares of positive degrees. Such a basis may be represented by a list of monomials of degree n.

This problem has first been studied by Peterson [10], Wood [28], Singer [16], Priddy [14], who pointed out its relationship with some classical problems in homotopy theory such as the cobordism theory of manifolds, the modular representation theory of linear groups, Adams spectral sequences of stable homotopy of spheres, and stable homotopy type of the classifying space of finite groups. Then, this problem was investigated by Wood [28], Carlisle and Wood [1], Silverman [17], Nam [8,9], Mothebe [7], Sum [19,21], Cho'n and Hà [3], Kameko [5,6] and others. Recently, the hit problem and its applications to representations of general linear groups have been presented in the books of Walker and Wood [26,27].

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{u_i}-1)$, where $u_i > 0$. This result implies a result of Wood, which originally is a conjecture of Peterson [10].

Theorem 1.1 (See Wood [28]). *If $\mu(n) > k$, then $(\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)_n = 0$.*

Let GL_k be the general linear group over the field $\mathbf{F}_2$. This group acts naturally on P_k by matrix substitution. Since the two actions of $\mathcal{A}$ and GL_k upon P_k commute with each other, there is an action of GL_k on $\mathbf{F}_2 \otimes_{\mathcal{A}} P_k$. One of our main tools is

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Kameko's homomorphism $\widetilde{Sq}^0_* : \mathbf{F}_2 \otimes_{\mathcal{A}} P_k \to \mathbf{F}_2 \otimes_{\mathcal{A}} P_k$, which is induced by an $\mathbf{F}_2$-linear map $\phi_k : P_k \to P_k$, given by

$$\phi_k(x) = \begin{cases} y, & \text{if } x = x_1x_2\dots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. The map ϕ_k is not an $\mathcal{A}$-homomorphism. However, $\phi_k Sq^{2i} = Sq^i \phi_k$ and $\phi_k Sq^{2i+1} = 0$ for any non-negative integer i.

Theorem 1.2 (Kameko [4]). *Let d be a non-negative integer. If $\mu(2d+k) = k$, then*

$$\widetilde{Sq}^0_* : (\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)_{2d+k} \longrightarrow (\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)_d$$

is an isomorphism of GL_k-modules.

From the results of Wood [28] and Kameko [4], the hit problem is reduced to the case of degree n of the form

$$n = r(2^s - 1) + 2^s m \tag{1.1}$$

where r, s, m are non-negative intergers such that $1 \leqslant r < k$ and $\mu(m) < r$.

For $r = k-1$ and $m > 0$, the problem was studied by Crabb and Hubbuck [2], Nam [8], Repka and Selick [15], Walker and Wood [25], Sum [21].

Now, the $\mathbf{F}_2$-vector space $\mathbf{F}_2 \otimes_{\mathcal{A}} P_k$ was explicitly calculated by Peterson [10] for $k = 1, 2$, by Kameko [4] for $k = 3$ and by Sum [20,21] for $k = 4$. However, for $k > 4$, it is still unsolved, even in the case of $k = 5$ with the help of computers.

For $r = k-1 = 4$ and $m = 0$, the vector space $(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_n$ is explicitly computed by Phuc and Sum [11,12], and in the case $r = k-2 = 3$, $m = 1$ by Phuc [13]. In the present paper, we study the hit problem of some generic degrees $r(2^s-1) + 2^s m$ in P_k, where $r = k-1 = 4, m \in \{8; 10; 11\}$ and s an arbitrary non-negative integer. Moreover, as a consequence, we get the dimension results for polynomial algebra in some generic degrees and in the cases $k = 5$ and 6.

The proofs of the results of this Note will be published in detail elsewhere.

2. The Main Results. From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k. The following is one of our main results.

Theorem 2.1. *Let $n = (k-1).(2^s - 1) + 11.2^s$ with s an arbitrary non-negative integer. For $k = 5$, then*

$$\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{4(2^s-1)+11.2^s} = \begin{cases} 315, & \text{if } s = 0, \\ 1024, & \text{if } s = 1, \\ 1984, & \text{if } s \geqslant 2. \end{cases}$$

Note that this theorem has been proved by Mothebe [7] for $s = 0$, and by Walker and Wood [26] for $s = 1$, and by another method.

The proof of the above theorem is too long and very technical. It is proved by explicitly determining all admissible monomials of degree $n = 4(2^s - 1) + 11.2^s$ in P_5, for any $s \geqslant 0$.

Sketch proof of Theorem 2.1. We first recall some notations and definitions in [21]. For any $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

Then, f_i is a homomorphism of $\mathcal{A}$-modules.

Denote $I = \{i_1, i_2, \dots, i_r\}$ and

$$\mathcal{U}_k = \{(i; I) : 1 \leqslant i < i_1 < \dots < i_r \leqslant k, 0 \leqslant r < k\}.$$

Let $(i; I) \in \mathcal{U}_k$, $x_{(I,u)} = x_{i_u}^{2^{r-1}+\dots+2^{r-u}} \Pi_{u<t\leqslant r} x_{i_t}^{2^{r-t}}$ for $r = \ell(I)$ be the length of I, $x_{(\varnothing,1)} = 1$. For a monomial $x \in P_{k-1}$, we define the monomial $\varphi_{(i;I)}(x)$ in P_k by setting

$$\varphi_{(i;I)}(x) = \begin{cases} (x_i^{2^r-1} f_i(x))/x_{(I,u)} & \text{if there exists } u \text{ such that } x \text{ is } u\text{-compatible with } (i, I), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have an $\mathbf{F}_2$-linear map $\varphi_{(i;I)} : P_{k-1} \to P_k$.

For $k = 5$, we have $n = (k-1)(2^s-1) + 2^s m = \sum_{i=0}^{3}(2^{s+i} - 1)$ and set $d_1 = s+3, d_2 = -s+2, d_3 = s+1, d_4 = s$. Thus, using the results in Sum [21] we see that $B_5(4.(2^s-1) + 11.2^s) = \Phi(B_4(4.(2^s-1) + 11.2^s))$ is a minimal set of generators for $\mathcal{A}$-module P_5 in degree $4.(2^s-1) + 11.2^s$ for any $s \geqslant 4$. Here,

$$\Phi(B_4(n)) = \Phi^0(B_4(n)) \cup \Phi^+(B_4(n))$$

where

$\Phi^0(B_4(n)) = \cup_{1\leqslant i\leqslant 5} \varphi_{(i;\varnothing)}(B_4(n)) = \cup_{1\leqslant i\leqslant 5} f_i(B_4(n))$,
$\Phi^+(B_4(n)) = \cup_{(i;I)\in\mathcal{U}_5, 0<\ell(I)\leqslant 4} \varphi_{(i;I)}(B_4(n)) \setminus P_5^0$,
$P_5^0 = \langle\{x = x_1^{a_1} \dots x_5^{a_5} : a_1a_2\dots a_5 = 0\}\rangle$.

Moreover, we set $m = 2^{d_1-d_4} + 2^{d_2-d_4} + 2^{d_3-d_4} - 3 = 11$, using the result in Sum [21] one get

$$|B_5(n)| = (2^5-1).|B_4(11)| \text{ for any } s \geqslant 4.$$

Based on the result in [21], we obtain $(\mathbf{F}_2 \otimes_{\mathcal{A}} P_4)_{11} = 64$. And therefore, $\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{4.(2^s-1)+11.2^s} = 1984$, for any $s \geqslant 4$. Hence, we need only to prove the theorem for $s = 0,1,2,3$ by the direct computations. The proof of the theorem in these cases is too long and very technical. It is proved by explicitly determining all admissible monomials of degree $n = 4(2^s-1)+11.2^s$ in P_5, for $s \leqslant 3$. The ideas of proofs are from Kameko's squaring operation [4] combining with the results in Sum [19,21]. Note that the results dimension of $(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{4.(2^s-1)+11.2^s}, 2 \leqslant s \leqslant 3$ has been verified by using a computer calculation program of V. H. Vu in SAGE. I would like to say thank you for his support. □

Moreover, from the results of this theorem and in Sum [21], we get the dimension results for polynomial algebra of six variables as follows:

Corollary 2.2 (Tin [23]). *For any integer $d > 5$, there exist exactly* 19845 *admissible monomials of degree $m_1 = 5(2^d-1)+3.2^d$ in P_6. Consequently,*

$$\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_6)_{m_1} = 19845.$$

Corollary 2.3 (Tin [24]). *For any $\ell > 4$, there exist exactly* 64512 *admissible monomials of degree $m_2 = 5(2^\ell-1)+13.2^{\ell+1}$ in P_6. Consequently, $\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_6)_{m_2} = 64512$.*

It is easy to check that $\mu(56) = 4 = \alpha(56+\mu(56)) = \alpha(60)$ and we have $m_2 = 5(2^t-1)+7.2^{t+3} = 5(2^t-1)+56.2^t$ hence, using the results in [21] one get

$$|B_6(5(2^t-1)+7.2^{t+3})| = (2^6-1)|B_5(56)| = 124992,$$

for any integer $t \geqslant k-1 = 5$. We get the corollary following

Corollary 2.4. *There exist exactly* 124992 *admissible monomials of degree $m_3 = 5(2^t-1)+7.2^{t+3}$ in P_6, for any $t \geqslant 5$. Consequently, $\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_6)_{m_3} = 124992$.*

By a simple calculation, we have $\mu(116) = 4$ and $\alpha(116+\mu(116)) = 4$. Using the results in [21] one get the corollary following

Corollary 2.5. *For any integer $r > 4$, there exist exactly* 124992 *admissible monomials of degree $m_4 = 5(2^r-1)+29.2^{r+2}$ in P_6. Consequently, $\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_6)_{m_4} = 124992$.*

Set $t(k,d) = \max\{0, k-\alpha(d+k)-\zeta(d+k)\}$ where $\zeta(n)$ the greatest integer u such that n is divisible by 2^u, that means $n = 2^{\zeta(n)}m$, with m an odd integer. We recall a result in [22] the following

Theorem 2.6 (Tin and Sum [22]). *Let d be an arbitrary non-negative integer. Then*

$$(\widetilde{Sq}^0_*)^{s-t} : (\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^s-1)+2^s d} \longrightarrow (\mathbf{F}_2 \otimes_{\mathcal{A}} P_k)_{k(2^t-1)+2^t d}$$

is an isomorphism of GL_k-modules for every $s \geqslant t$ if and only if $t \geqslant t(k,d)$.

It is easy to check that for $k = 5$ and $d = 56$ then

$$t(k,d) = \max\{0, k-\alpha(d+k)-\zeta(d+k)\} = 0.$$

Then, from the results of Theorems 2.1 and 2.6 we get the corollary following

Corollary 2.7. *For any integer $u \geqslant 0$, there exist exactly* 1984 *admissible monomials of degree $5(2^u-1)+7.2^{u+3}$ in P_5. Consequently, $\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^u-1)+7.2^{u+3}} = 1984$.*

Similarly, for $k = 5$, $d = 116$ then $t(k,d) = \max\{0, k-\alpha(d+k)-\zeta(d+k)\} = 0$. We have the corollary following

Corollary 2.8. *For any integer $v \geqslant 0$, there exist exactly* 1984 *admissible monomials of degree $5(2^v-1)+29.2^{v+2}$ in P_5. Consequently, $\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{5(2^v-1)+29.2^{v+2}} = 1984$.*

Next, we determine the hit problem in degree n of the form (1.1) with $r = k-1 = 4$, $m = 8$ and s an arbitrary non-negative integer.

Theorem 2.9. *Let $n = 4.(2^s-1)+8.2^s$ with s an arbitrary non-negative integer. Then*

$$\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{4.(2^s-1)+8.2^s} = \begin{cases} 174, & \text{if } s = 0,\\ 641, & \text{if } s = 1,\\ 1426, & \text{if } s = 2,\\ 1706, & \text{if } s = 3,\\ 1705, & \text{if } s \geqslant 4. \end{cases}$$

Similarly, the following is our third main result.

Theorem 2.10. *Let $n = (k-1).(2^s-1)+10.2^s$ with s an arbitrary non-negative integer. For $k = 5$, then*

$$\dim(\mathbf{F}_2 \otimes_{\mathcal{A}} P_5)_{4(2^s-1)+10.2^s} = \begin{cases} 280, & \text{if } s = 0,\\ 961, & \text{if } s = 1,\\ 1905, & \text{if } s = 2,\\ 2171, & \text{if } s = 3,\\ 2170, & \text{if } s \geqslant 4. \end{cases}$$

The ideas of the proofs of Theorems 2.9 and 2.10 are the same as in proof of Theorem 2.1.

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