Some probabilistic value distributions of the Riemann zeta function and its derivatives

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Abstract: In this paper, we give an announcement of our results on uniform distribution and ergodic value distribution of the Riemann zeta function and its derivatives.

Key words: Uniform distribution; Birkhoff's ergodic theorem; Riemann zeta function; derivative.

In this paper, two topics related to the Riemann zeta function will be discussed: uniform distribution on the derivatives of the Riemann zeta function and an application of Birkhoff's ergodic theorem to the Riemann zeta function and its derivatives in connection with affine Boolean transformations. This is an announcement of our results, and the proofs will be given in some other papers.

The first approach to uniform distribution properties for the Riemann zeta function was done by H. A. Rademacher [Rad75] in 1956. He proved that a sequence consisting of the constant multiple of the imaginary part of the zero points of the Riemann zeta function is uniformly distributed modulo one under the Riemann hypothesis. After then, P. D. T. A. Elliott [Ell72] remarked that the same result holds unconditionally in 1972 and E. Hlawka [Hla75] finally proved it unconditionally in 1975. Moreover, a number of mathematicians began to consider generalizations and extensions of their results in [Fuj78], [AM08], [FSZ09], and [Ste12a].

Let $a \in \mathbf{C}$. We say an element z in $\mathbf{C}$ is an *a-point* of a function f if $f(z)$ is equal to a. J. Steuding [Ste12a] proved that a sequence consisting of the constant multiple of the imaginary part of the a-points of the Riemann zeta function is uniformly distributed modulo one. This generalizes Hlawka's result for all a-points of the Riemann zeta function. We further extend this result of Steuding to the derivatives of the Riemann zeta function. Below we denote by $\zeta^{(k)}(s)$ the k-th derivative of the Riemann zeta function $\zeta(s)$ for non-negative integer k, where, $\zeta^{(0)}(s)$ denotes the Riemann zeta function $\zeta(s)$ itself.

Theorem 1. *Let a be a complex number and k a positive integer. For any non-zero real number α, the sequence $\{\alpha\gamma_a^{(k)}\}_{\gamma_a^{(k)}>1}$ running over the imaginary parts of the a-points of $\zeta^{(k)}(s)$, is uniformly distributed modulo one.*

In relation to Theorem 1, we remark that Hlawka's result is the case when a and k are equal to 0, and Steuding's result is the case when k is equal to 0.

Our second result is related to the value distribution of the Riemann zeta function and its derivatives which is obtained by using ergodic transformations. In [LW09], M. Lifshitz and M. Weber investigated the value distribution of the Riemann zeta function by using the Cauchy random walk. This result implies that most of the values of the Riemann zeta function on the critical line are quite small. More precisely, they proved that almost surely

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\zeta\left(\frac{1}{2}+iS_n\right)=1+o\left(\frac{(\log N)^b}{N^{1/2}}\right)$$

holds for any $b>2$ where $\{S_n\}_{n=1}^{\infty}$ is the Cauchy random walk.

The first approach to investigate the value distribution of the Riemann zeta function by using an ergodic transformation was done by Steuding. Steuding in [Ste12b] studied the value distribution of the Riemann zeta function on the vertical lines $\sigma+i\mathbf{R}$ with respect to the *Boolean transformation* $T:\mathbf{R}\to\mathbf{R}$ given by

$$T(x) := \frac{1}{2}\cdot\left(x-\frac{1}{x}\right)\quad(x\neq 0)$$

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and $T(0) := 0$.

We are interested in studying the value distribution of not only the Riemann zeta function itself, but also its derivatives, on the vertical lines $\sigma + i\mathbf{R}$ with respect to a more general ergodic transformation, which we shall call *affine Boolean transformation* $T_{\alpha,\beta} : \mathbf{R} \to \mathbf{R}$ given by

$$T_{\alpha,\beta}(x) := \frac{\alpha}{2}\left(\frac{x+\beta}{\alpha} - \frac{\alpha}{x-\beta}\right) \quad (x \neq \beta)$$

and $T(\beta) := \beta$ for a positive real number α and an arbitrary real number β. We have the following result on the value distribution for the Riemann zeta function and its derivatives.

Theorem 2. *Let $T_{\alpha,\beta}$ be an affine Boolean transformation. Then for any non-negative integer k and any s satisfying $\mathrm{Re}(s) > -1/2$ and $\mathrm{Re}(s) \neq 1$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \zeta^{(k)}\left(s + iT^n_{\alpha,\beta}x\right) = \frac{\alpha}{\pi} \int_{\mathbf{R}} \frac{\zeta^{(k)}(s+i\tau)d\tau}{\alpha^2 + (\tau-\beta)^2}$$

for almost all x in $\mathbf{R}$.

Denoting the right-hand side of the above formula by $l^{(k)}_{\alpha,\beta}(s)$, we have

$$l^{(k)}_{\alpha,\beta}(s) = \begin{cases} \zeta^{(k)}(s+\alpha+i\beta) + A_k(s), \\ \qquad -1/2 < \mathrm{Re}(s) < 1, s \neq 1-\alpha-i\beta; \\ (-1)^k\gamma_k - \dfrac{k!}{(2\alpha)^{k+1}}, \\ \qquad -1/2 < \mathrm{Re}(s) < 1, s = 1-\alpha-i\beta; \\ \zeta^{(k)}(s+\alpha+i\beta), \\ \qquad \mathrm{Re}(s) > 1; \end{cases}$$

where

$$A_k(s) := \frac{(-1)^k k!}{i^{k+1}}\left(\frac{1}{(\beta+i\alpha-i(s-1))^{k+1}} - \frac{1}{(\beta-i\alpha-i(s-1))^{k+1}}\right)$$

and

$$\gamma_k := \lim_{N\to\infty}\left(\sum_{n=1}^{N} \frac{\log^k n}{n} - \frac{\log^{k+1} N}{k+1}\right).$$

If $k = 0$, we can extend the result to the line $\mathrm{Re}(s) = 1$ by setting

$$l^{(0)}_{\alpha,\beta}(1+it) = \zeta^{(0)}(1+\alpha+i(t+\beta)) - \frac{\alpha}{\alpha^2 + (t+\beta)^2}.$$

We remark that Steuding showed Theorem 2 when $k = 0$, $\alpha = 1$, and $\beta = 0$ so Theorem 2 is a generalization of [Ste12b, Theorem 1.1].

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