Note on a general complex Monge-Ampère equation on pseudoconvex domains of infinite type

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Abstract: Let Ω be a smoothly bounded domain in $\mathbf{C}^n$, for $n \geq 2$. For a given continuous function ϕ on $b\Omega$, and a non-negative continuous function Ψ on $\mathbf{R} \times \overline{\Omega}$, the main purpose of this note is to seek a plurisubharmonic function u on Ω, continuous on $\overline{\Omega}$, which solves the following Dirichlet problem of the complex Monge-Ampère equation

$$\begin{cases} \det\left[\dfrac{\partial^2(u)}{\partial z_i \partial \bar{z}_j}\right](z) = \Psi(u(z), z) \geqslant 0 & \text{in} \quad \Omega, \\ u = \phi & \text{on} \quad b\Omega. \end{cases}$$

In particular, the boundary regularity for the solution of this complex, fully nonlinear equation is studied when Ω belongs to a large class of weakly pseudoconvex domains of finite and infinite type in $\mathbf{C}^n$.

Key words: Pseudoconvexity; D'Angelo type; complex Monge-Ampère operator; Perron-Bremermann family.

1. Introduction. Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$, for $n \geq 2$, with the smooth boundary $b\Omega$. In the coordinates $(z_1, z_2, \dots, z_n) \in \mathbf{C}^n$, we define complex linear operators as

$$\partial = \sum_{j=1}^{n} \frac{\partial}{\partial z_j} dz_j \quad \text{and} \quad \bar{\partial} = \sum_{j=1}^{n} \frac{\partial}{\partial \bar{z}_j} d\bar{z}_j.$$

Let $u = u(z_1, \dots, z_n)$ be a function of class $C^2(\Omega)$. The complex Monge-Ampère operator is a fully nonlinear partial differential operator denoted by

$$(dd^c u)^n = \underbrace{dd^c u \wedge \dots \wedge dd^c u}_{n\text{-times}} = 4^n n! \det\left[\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right]_{j,k=1,\dots,n} dV,$$

where $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$, then $dd^c u = 2i\partial\bar{\partial}u$, and dV is the volume form

$$dV = \left(\frac{i}{2}\right)^n \prod_{j=1}^{n} dz_j \wedge d\bar{z}_k.$$

Let $\mathcal{P}(\Omega)$ be the cone of plurisubharmonic functions on Ω, $\mathcal{M}(\Omega)$ be the set of non-negative Borel measure on Ω with the weak topology. Then, following the generalized definition of E. Bedford and B. A. Taylor [1], the operator $(dd^c)^n : C^2(\Omega) \cap \mathcal{P}(\Omega) \to \mathcal{M}(\Omega)$ is extended continuously to $C(\Omega) \cap \mathcal{P}(\Omega)$ via the language of positive currents. In this sense, we study the following Dirichlet problem of a complex Monge-Ampère type

$$(1.1) \qquad \begin{cases} u \in \mathcal{P}(\Omega) \cap C(\overline{\Omega}) \\ (dd^c u)^n(z) = \Psi(u(z), z) dV, & \text{for } z \in \Omega \\ u = \phi & \text{on } b\Omega, \end{cases}$$

where Ψ has the following properties:

(a) non-negative, continuous in all variables,

(b) $u \mapsto \Psi^{\frac{1}{n}}(u, z)$ is a nondecreasing, convex function.

One of the most classical examples, $\Psi(u(z), z) = e^{u(z)}$, is studied by several mathematicians such as E. Calabi, L. Nirenberg, S. T. Yau, C. Fefferman and N. Krylov. The boundary regularity of the solution to (1.1) in the current sense is well-known by the work of E. Bedford and B. A. Taylor [1,2] on the strongly pseudoconvex domain. However, the consideration for this problem on weakly pseudoconvex domains of finite and also infinite type, in the sense of D'Angelo, still remains unsolved with

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the general right-hand side $\Psi(u,z)$. In this note, we will provide an admissible answer to this problem.

The note is organized as follows: in Section 2, we will recall some essential notions to state the main result which is proven in Section 3.

2. Geometry on the boundaries of weakly pseudoconvex domains. In this section, we are interested in some geometric materials on the boundaries of pseudoconvex domains. By these, the existence of strictly plurisubharmonic defining functions on such domains is established. For more details, the reader should be referred to the previous work [3]. Here, the most important geometric condition on boundaries is named f-property firstly introduced by T. V. Khanh and G. Zampieri [4]. This condition generalizes the classical finite geometric type and many cases of infinite type. We have three certain examples to illustrate this notion. First of all, we say that Ω admits the $t^{\frac{1}{2}}$-property if it is strongly pseudoconvex. Second, Ω is said to have the $t^{m^{-n^2 m^{n^2}}}$-property when it is pseudoconvex of finite type m in $\mathbf{C}^n$ in the sense of D'Angelo. Finally, let us define

$$\Omega = \left\{(z_1,\dots,z_n)\in\mathbf{C}^n : \sum_{j=1}^{n}\exp\left(-\frac{1}{|z_j|^{s_j}}\right) < \frac{1}{e}\right\}.$$

Obviously, Ω is smooth, pseudoconvex domain of infinite type in $\mathbf{C}^n$. Moreover, we also say that Ω has the $\ln^{\frac{1}{s}} t$-property, for $s := \max\{s_1,\dots,s_n\}$.

More generally, we have

Definition 2.1. For a smooth, monotonic, increasing function $f:[1,+\infty)\to[1,+\infty)$ with $\dfrac{f(t)}{t^{1/2}}$ decreasing, we say that Ω has an f-property if there exist a neighborhood U of $b\Omega$ and a family of functions $\{\phi_\delta\}$ such that

(i) the functions ϕ_δ are plurisubharmonic, C^2 on U, and satisfy $-1\leqslant\phi_\delta\leqslant 0$, and

(ii) $i\partial\bar\partial\phi_\delta \gtrsim f(\delta^{-1})^2 Id$ and $|D\phi_\delta|\lesssim\delta^{-1}$ for any $z\in U\cap\{z\in\Omega : -\delta<r(z)<0\}$, where r is a C^1-defining function of Ω.

Here and in what follows, $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant. Moreover, we will use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

In the case $\Psi=\Psi(z)$, on general weakly pseudoconvex domains of finite type, the solution to the equation (1.1) only belongs to Hölder classes, and we do not have any smoothness result. This is proved in [5]. The solution u can not belong to $C^2(\bar\Omega)$ even if Ω is strongly pseudoconvex. In the best case, when Ω is the unit ball, the second partial derivatives of u are locally bounded, see [1, Theorem C]. Therefore, on only f-property domains, we expect to prove "weaker" Hölder estimates for the solution of the Dirichlet problem of the complex Monge-Ampère equation. This is the main result in the previous work [3]. For this purpose, we recall the definition of the f-Hölder spaces in [3].

Definition 2.2. Let f be an increasing function such that $\lim\limits_{t\to+\infty} f(t)=+\infty$ and $f(t)\lesssim t$. For a subset A of $\mathbf{C}^m$, define the f-Hölder space on A as

$$\Lambda^f(A) = \left\{u : \|u\|_{L^\infty(A)} + \sup_{z,w\in A, z\neq w} f(|z-w|^{-1})\cdot|u(z)-u(w)|<\infty\right\}$$

and set

$$\|u\|_{\Lambda^f(A)} = \|u\|_{L^\infty(A)} + \sup_{z,w\in A, z\neq w} f(|z-w|^{-1})\cdot|u(z)-u(w)|.$$

Note that the notion of the f-Hölder space includes the standard Hölder space Λ_α by taking $f(t)=t^\alpha$ (so that $f(|h|^{-1})=|h|^{-\alpha}$) with $0<\alpha\leq 1$. When $1<\alpha\leq 2$, we also define $\Lambda^{t^\alpha}(A):=\Lambda_\alpha(A)$ where

$$\Lambda_\alpha(A)=\{u : \|u\|_{\Lambda^{t^\alpha}(A)} = \|Du\|_{\Lambda^{t^{\alpha-1}}(A)}<\infty\}.$$

Since the right hand side of (1.1) is nonlinear, we also define the corresponding Hölder classes on $\mathbf{R}\times A$ as follows:

$$\Lambda^f(\mathbf{R}\times A) = \left\{\Psi\in C(\mathbf{R}\times A) : \|\Psi\|_{\Lambda^f(\mathbf{R}\times A)} = \|\Psi\|_{L^\infty(\mathbf{R}\times A)} + \sup_{u,v\in\mathbf{R}}\sup_{\substack{z,w\in A,\\ z\neq w}}[f(|z-w|^{-1}) \times|\Psi(u,z)-\Psi(v,w)|] \text{ is finite}\right\},$$

where

$$\|\Psi\|_{L^\infty(\mathbf{R}\times A)} = \sup_{u\in\mathbf{R}, z\in A}|\Psi(u,z)|.$$

As was mentioned in the previous work [3], the f-property provides the existence of strictly plurisubharmonic (global) defining functions on weakly pseudoconvex domains of finite type and also many cases of infinite type. Actually, such defining functions are smooth if Ω is strongly pseudoconvex.

Theorem 2.3. *Let f be as in Definition 2.1 with $[g(t)]^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty$. Assume that Ω is a bounded, pseudoconvex domain admitting the f-property. Then there exists a strictly plurisubharmonic defining function of Ω which belongs to the g^2-Hölder space of $\overline{\Omega}$, that means, there is a plurisubharmonic function ρ such that*

(a) *$z \in \Omega$ if and only if $\rho(z) < 0$ and $b\Omega = \{z \in \mathbf{C}^n : \rho(z) = 0\}$;*
(b) *$i\partial\bar{\partial}\rho(X, \bar{X}) \geq |X|^2$ on Ω in the distribution sense, for any $X \in T^{1,0}(\mathbf{C}^n)$; and*
(c) *ρ is in the g^2-Hölder space of $\overline{\Omega}$, that is, $|\rho(z) - \rho(z')| \lesssim g(|z - z'|^{-1})^{-2}$ for any $z, z' \in \overline{\Omega}$.*

The main result in this note consists in the following

Theorem 2.4. *Let f be as in Definition 2.1 and $\Omega \subset \mathbf{C}^n$ be a bounded, pseudoconvex domain admitting the f-property. Suppose that a function $g : [1, \infty) \to [1, \infty)$ is defined by*

$$[g(t)]^{-1} := \int_t^\infty \frac{da}{af(a)} < \infty.$$

If $0 < \alpha \leq 2$, $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and $\Psi \geq 0$ on $\mathbf{R} \times \Omega$ such that $\Psi^{\frac{1}{n}}(u, z) \in \Lambda^{g^\alpha}(\mathbf{R} \times \overline{\Omega})$, nondecreasing, convex in u, then the following Dirichlet problem for the complex Monge-Ampère equation

$$\begin{cases} \det(u_{ij}(z)) = \Psi(u(z), z) & \text{in} \quad \Omega, \\ u = \phi & \text{on} \quad b\Omega, \end{cases} \tag{2.1}$$

has a unique plurisubharmonic solution $u \in \Lambda^{g^\alpha}(\overline{\Omega})$.

3. Proof of the main result. Based on the classical approach, the solution in the main theorem is constructed by the Perron-Bremermann family of subsolutions of second order complex fully nonlinear equations. Let $\phi \in C(b\Omega)$ and $\Psi(u, z) \in C(\mathbf{R} \times \bar{\Omega})$, that is

$$\mathcal{B}(\phi, \Psi) = \Big\{ v \in \mathcal{P}(\Omega) \cap C(\bar{\Omega}) : \det[(v)_{ij}](z) \geq \Psi(v(z), z), \text{ and } \limsup_{z \to z_0} v(z) \leq v(z_0), \text{ for all } z_0 \in b\Omega \Big\}.$$

The existence and uniqueness theorem for the problem (1.1) follows from the proof by E. Bedford and B. A. Taylor in [1, Theorem 8.3, p. 42] with modifications in [2, Theorem A, p. 40].

Theorem 3.1 (Bedford-Taylor [2]). *Let Ω be a smoothly bounded open set in $\mathbf{C}^n$. Let $\phi \in C(b\Omega)$ and $0 \leq \Psi(u, z) \in C(\mathbf{R} \times \bar{\Omega})$ be increasing, convex in u-variable. If the Perron-Bremermann family is non-empty, and its upper envelope*

$$u = \sup\{v : v \in \mathcal{B}(\phi, \Psi)\} \tag{3.1}$$

is continuous on $\bar{\Omega}$ with $u = \phi$ on $b\Omega$, then u is the unique solution to the Dirichlet problem (1.1).

The fact that the Perron-Bremermann family is non-empty is a direct consequence of the following result.

Proposition 3.2. *Let Ω be a smoothly bounded, pseudoconvex domain. Assume that there is a strictly plurisubharmonic defining function ρ of Ω such that $\rho \in \Lambda^{g^2}(\bar{\Omega})$. Let $0 < \alpha \leq 2$, and $\phi \in \Lambda^{t^\alpha}(b\Omega)$, and let $\Psi \geq 0$ on $\mathbf{R} \times \bar{\Omega}$ with $\Psi^{1/n} \in \Lambda^{g^\alpha}(\mathbf{R} \times \bar{\Omega})$. Then, for each $\zeta \in b\Omega$, there exists $v_\zeta \in \Lambda^{g^\alpha}(\overline{\Omega}) \cap \mathcal{P}(\Omega)$ such that*

(i) *$v_\zeta(z) \leq \phi(z)$ for all $z \in b\Omega$, and $v_\zeta(\zeta) = \phi(\zeta)$,*
(ii) *$\|v_\zeta\|_{\Lambda^{g^\alpha}(\overline{\Omega})} \leq C_0$ and*
(iii) *$\det(H(v_\zeta)(z)) \geq \Psi(v_\zeta(z), z)$,*

where C_0 is a positive constant depending only on Ω and $\|\phi\|_{\Lambda^{t^\alpha}(b\Omega)}$.

Proof of Proposition 3.2. For each $\zeta \in b\Omega$, the family $\{v_\zeta\}$ is defined by:

Case 1: if $0 < \alpha \leq 1$ then we choose

$$v_\zeta(z) = \phi(\zeta) - K[-2\rho(z) + |z - \zeta|^2]^{\frac{\alpha}{2}}, \quad z \in \overline{\Omega};$$

Case 2: if $1 < \alpha \leq 2$ then we choose

$$v_\zeta(z) = \phi(\zeta) - \sum_{j=1}^n 2\,\mathrm{Re}\,\frac{\partial\phi(\zeta)}{\partial\zeta_j}(z_j - \zeta_j) - K[-2\rho(z) + |z - \zeta|^2]^{\frac{\alpha}{2}}, \quad z \in \overline{\Omega};$$

where ρ is defined by Theorem 2.3, and K will be chosen step by step later.

The proof of the assertions (i) and (ii) is exactly contained in [3, Proposition 3.2], with $K \geq \|\phi\|_{\Lambda^{t^\alpha}}$. To establish (iii), we make a simple modification:

$$(v_\zeta(z))_{ij} = K\frac{\alpha}{2}(-2\rho(z) + |z - \zeta|^2)^{\frac{\alpha}{2}-2}\Big[(-2\rho(z) + |z - \zeta|^2)(2\rho(z)_{ij} - \delta_{ij}) + \Big(1 - \frac{\alpha}{2}\Big)(-2\rho_i + \bar{z}_i - \bar{\zeta}_i)\overline{(-2\rho_j + \bar{z}_j - \bar{\zeta}_j)}\Big].$$

Hence

$$i\partial\bar{\partial}v_\zeta(X, X) \geq K\frac{\alpha}{2}(-2\rho(z) + |z - \zeta|^2)^{\frac{\alpha}{2}-1}(2i\partial\bar{\partial}\rho(X, X) - |X|^2)$$

$$\geq K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\frac{\alpha}{2}-1}|X|^2,$$

for any $X \in T^{1,0}(\mathbf{C}^n)$. Here the last inequality follows from Theorem 2.3 (b). Thus v_ζ is plurisubharmonic and furthermore, we obtain

$$\det[(v_\zeta)_{ij}](z) \geq \left[K\frac{\alpha}{2}(-2\rho(z)+|z-\zeta|^2)^{\left(\frac{\alpha}{2}-1\right)}\right]^n. \tag{3.2}$$

Now, since $0<\alpha\leq 2$, we might choose

$$K \geq \max\left\{\frac{2}{\alpha}\max_{z\in\overline{\Omega},\zeta\in b\Omega}(-2\rho(z) + |z-\zeta|^2)^{1-\frac{\alpha}{2}}\|\Psi^{1/n}\|_{L^\infty(\mathbf{R}\times\Omega)}, \|\phi\|_{\Lambda^{t^\alpha}}\right\}.$$

Then

$$\begin{aligned}\det[(v_\zeta)_{ij}](z) &\geq \|\Psi^{1/n}\|^n_{L^\infty(\mathbf{R}\times\Omega)}\\ &\geq (\Psi^{1/n}(v_\zeta(z),z))^n\\ &= \Psi(v_\zeta(z),z),\end{aligned} \tag{3.3}$$

for all $z\in\Omega$, and $\zeta\in b\Omega$. This completes the proof of Proposition 3.2. □

Proof of Theorem 2.4. First, immediately, the set $\mathcal{B}(\phi,\Psi)$ is non-empty. In particular, it contains the family of $\{v_\zeta\}_{\zeta\in b\Omega}$ in Proposition 3.2. The proof of this theorem will be completed if the upper envelope defined in (3.1) has the properties
(a) $u(\zeta)=\phi(\zeta)$ for all $\zeta\in b\Omega$; and
(b) $u\in\Lambda^{g^\alpha}(\bar{\Omega})$.
We note that the uniqueness of the solution follows from the Minimum Principle (cf. [2, Proposition 3, p. 45]).

Next, we define another upper envelope, for each $z\in\overline{\Omega}$, as

$$v(z) := \sup_{\zeta\in b\Omega}\{v_\zeta(z)\}.$$

By the first property of $\{v_\zeta\}$ in Proposition 3.2, we have

$$\begin{aligned}&v(\zeta)\geq v_\zeta(\zeta)=\phi(\zeta) \quad \text{for all } \zeta\in b\Omega,\\ &v(z)\leq\phi(z) \quad \text{for all } z\in b\Omega,\end{aligned} \tag{3.4}$$

and so $v=\phi$ on $b\Omega$.

From the second property in Proposition 3.2, we have

$$|v_\zeta(z)-v_\zeta(z')|\leq C_0(g^\alpha(|z-z'|^{-1}))^{-1} \quad \text{for all } z,z'\in\bar{\Omega}.$$

Notice that C_0 is independent of ζ so taking the supremum in ζ, the theory of the modulus of continuity again implies that

$$|v(z)-v(z')|\leq C_0(g^\alpha(|z-z'|^{-1}))^{-1} \quad \text{for all } z,z'\in\bar{\Omega}.$$

The first inequality in (3.3) also shows that

$$\begin{aligned}\det[(v_\zeta)_{ij}](z) &\geq \|\Psi^{1/n}\|^n_{L^\infty(\mathbf{R}\times\Omega)}\\ &\geq (\Psi^{1/n}(v(z),z))^n\geq\Psi(v(z),z).\end{aligned}$$

By Proposition 2.8 in [1], the following inequality holds

$$\det[(v)_{ij}](z)\geq\inf_{\zeta\in b\Omega}\{\det[(v_\zeta)_{ij}](z)\}\geq\Psi(v(z),z),$$

for all $z\in\Omega$. So, we conclude that $v\in\mathcal{B}(\phi,\Psi)\cap\Lambda^{g^\alpha}(\overline{\Omega})$ and $v(\zeta)=\phi(\zeta)$ for any $\zeta\in b\Omega$.

By a similar construction there exists a plurisuperharmonic function $w\in\Lambda^{g^\alpha}(\bar{\Omega})$ such that $w(\zeta)=\phi(\zeta)$ for any $\zeta\in b\Omega$. Thus, $v(z)\leq u(z)\leq w(z)$ for any $z\in\bar{\Omega}$, and hence $u(\zeta)=\phi(\zeta)$ for any $\zeta\in b\Omega$. We also obtain

$$\begin{aligned}&|u(z)-u(\zeta)|\\ &\quad\leq\max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})},\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}(g^\alpha(|z-\zeta|^{-1})^{-1}\end{aligned} \tag{3.5}$$

for any $z\in\bar{\Omega},\zeta\in b\Omega$. Here, the inequality follows from the facts that $w,v\in\Lambda^{g^\alpha}(\bar{\Omega})$ and $v(\zeta)=u(\zeta)=w(\zeta)=\phi(\zeta)$ for any $\zeta\in\partial\Omega$.

Finally, using the method by J. B. Walsh in [6], we will show that (3.5) also holds for all $\zeta\in\Omega$. For any small vector $\tau\in\mathbf{C}^n$, we define

$$V(z,\tau)=\begin{cases}u(z) & \text{if } z+\tau\notin\Omega, z\in\overline{\Omega},\\ \max\{u(z),V_\tau(z)\}, & \text{if } z,z+\tau\in\Omega,\end{cases}$$

where

$$V_\tau(z)=u(z+\tau)+(K_1|z|^2-K_2-K_3)g^{-\alpha}(|\tau|^{-1})$$

and here

$$K_1\geq\max_{k\in\{1,\dots,n\}}\binom{n}{k}^{1/k}\|\Psi^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}(\mathbf{R}\times\overline{\Omega})},\ K_2\geq K_1|z|^2,$$

and

$$K_3\geq\max\{\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})},\|w\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}.$$

We will show that $V(z,\tau)\in\mathcal{B}(\phi,\Psi)$. Observe that $V(z,\tau)\in\mathcal{P}(\Omega)$ for all z,τ. Moreover, for $z\in\partial\Omega$ and $z+\tau\in\Omega$, we have

$$\begin{aligned}V_\tau(z)-u(z)&\\ =u(z+\tau)&-u(z)\\ &+(K_1|z|^2-K_2-K_3)g^{-\alpha}(|\tau|^{-1})\\ \leq\max\{&\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})},\|v\|_{\Lambda^{g^\alpha}(\bar{\Omega})}\}g^{-\alpha}(|\tau|^{-1})\end{aligned} \tag{3.6}$$

$$+ (K_1|z|^2 - K_2 - K_3)g^{-\alpha}(|\tau|^{-1})$$
$$\leq 0.$$

Here the first inequality follows from (3.5) and the second follows from the choices of K_2 and K_3. This implies that $\limsup_{z\to\zeta} V(z,\tau) \leq \phi(\zeta)$ for all $\zeta \in b\Omega$. For the proof of $\det[V(z,\tau)_{ij}] \geq \Psi(V(z,\tau), z)$, we need the following lemma.

Lemma 3.3. *Let* $(\alpha_{ij}) \geq 0$ *and* $\beta \in (0, +\infty)$. *Then*

$$\det[\alpha_{ij} + \beta I] \geq \sum_{k=0}^{n} \beta^k \det(\alpha_{ij})^{(n-k)/n}.$$

Proof of Lemma 3.3. Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$ be the eigenvalues of (α_{ij}). We have

$$\text{(3.7)} \quad \det[\alpha_{ij} + \beta] = \prod_{j=1}^{n} (\lambda_j + \beta) \geq \sum_{k=0}^{n} \left(\beta^k \prod_{j=k+1}^{n} \lambda_j\right) \geq \sum_{k=0}^{n} (\beta^k \det[\alpha_{ij}]^{(n-k)/n}).$$

Here the last inequality follows from

$$\det[\alpha_{ij}] = \prod_{j=1}^{n} \lambda_j \leq \left(\prod_{j=k+1}^{n} \lambda_j\right)^{n/(n-k)}.$$

□

Coming back to the proof of the main theorem, for any $z, z+\tau \in \Omega$ we have

$$\begin{aligned} \text{(3.8)} \quad \det[(V_\tau(z))_{ij}] &= \det[u_{ij}(z+\tau) + K_1 g^{-\alpha}(|\tau|^{-1}) I] \\ &\geq \det[u_{ij}(z+\tau)] \\ &\quad + \sum_{k=1}^{n} K_1^k [g^\alpha(|\tau|^{-1})]^{-k} \\ &\quad \times \det[u_{ij}(z+\tau)]^{\frac{n-k}{n}} \\ &\geq \Psi(u(z+\tau), z+\tau) \\ &\quad + \sum_{k=1}^{n} K_1^k [g^\alpha(|\tau|^{-1})]^{-k} \\ &\quad \times (\Psi(u(z+\tau), z+\tau))^{\frac{n-k}{n}}, \end{aligned}$$

where the first inequality is derived by Lemma 3.3. Since $\Psi^{\frac{1}{n}} \in \Lambda^{g^\alpha}(\mathbf{R} \times \Omega)$, we obtain

$$\Psi^{\frac{1}{n}}(u(z), z) - \Psi^{\frac{1}{n}}(u(z+\tau), z+\tau) \leq g^{-\alpha}(|\tau|^{-1}) \|\Psi^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}(\mathbf{R}\times\Omega)},$$

for any $z, z+\tau \in \Omega$. Hence

$$\begin{aligned} \text{(3.9)} \quad \Psi(u(z), z) &\leq \Psi(u(z+\tau), z+\tau) \\ &\quad + \sum_{k=1}^{n} \binom{n}{k} \Psi(u(z+\tau), z+\tau)^{(n-k)/n} \\ &\quad \times (g^{-\alpha}(|\tau|^{-1}) \|\Psi^{\frac{1}{n}}\|_{\Lambda^{g^\alpha}})^k. \end{aligned}$$

Combining (3.8) and (3.9) with the choice of K_1, we get

$$\det[(V_\tau)_{ij}](z) \geq \Psi(u(z), z), \quad \text{for any} \quad z, z+\tau \in \Omega.$$

We conclude that $V(z,\tau) \in \mathcal{B}(\phi, \Psi)$. It follows that for all $z \in \Omega$, $V(z,\tau) \leq u(z)$. If $z, z+\tau \in \Omega$, this yields

$$\begin{aligned} u(z+\tau) - u(z) &\leq V(\tau, z) - u(z) \\ &\quad - (K_1|z|^2 - K_2 - K_3) g^{-\alpha}(|\tau|^{-1}) \\ &\leq (-K_1|z|^2 + K_2 + K_3) g^{-\alpha}(|\tau|^{-1}) \\ &\leq (K_2 + K_3) g^{-\alpha}(|\tau|^{-1}). \end{aligned}$$

By reversing the role of z and $z+\tau$, we assert that $u \in \Lambda^{g^\alpha}(\overline{\Omega})$. This completes the proof. □

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