A note on balancing binomial coefficients

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(Communicated by Masaki KASHIWARA, M.J.A., Sept. 14, 2015)

Abstract: In 2014, T. Komatsu and L. Szalay studied the balancing binomial coefficients. In this paper, we focus on the following Diophantine equation

$$\binom{1}{5}+\binom{2}{5}+\cdots+\binom{x-1}{5}=\binom{x+1}{5}+\cdots+\binom{y}{5}$$

where $y>x>5$ are integer unknowns. We prove that the only integral solution is $(x,y)=(14,15)$. Our method is mainly based on the linear form in elliptic logarithms.

Key words: Balancing problem; binomial coefficient; linear form in elliptic logarithms.

1. Introduction. In a recent paper [5], T. Komatsu and L. Szalay studied the balancing binomial coefficients, namely, the Diophantine equation

$$\binom{1}{k}+\binom{2}{k}+\cdots+\binom{x-1}{k}=\binom{x+1}{l}+\cdots+\binom{y}{l}$$

in the positive integer unknowns $x>k$ and $y>x$. In particular, the equation has infinitely many solutions when $k=l=1$ (see also [3]), and no solutions when $k=l=2$ or 3 (see also [1]). Moreover, when $k=l>3$, it has finitely many solutions. However, for the case $k=l=5$, the authors only got one solution $(x,y)=(14,15)$ through a computer search with $x\le 300$. In this paper, we will completely solve the equation

$$\binom{1}{5}+\binom{2}{5}+\cdots+\binom{x-1}{5}=\binom{x+1}{5}+\cdots+\binom{y}{5}, \tag{1}$$

where $y>x>5$, and our result is

Theorem 1.1. *Equation (1) has only one integral solution* $(x,y)=(14,15)$.

Our method of proof, which is mainly based on the linear form in elliptic logarithms, is motivated by [4], and further by [6]. Throughout this paper, we use the notations of [6].

2. Proof of Theorem 1.1. Note that

$$\binom{1}{k}+\binom{2}{k}+\cdots+\binom{x}{k}=\binom{x+1}{k+1},$$

we can therefore rewrite (1) as

$$\binom{x}{6}+\binom{x+1}{6}=\binom{y+1}{6}. \tag{2}$$

Set $u=(x-2)^2$ and $v=(y-1)(y-2)$, (2) becomes

$$2u^3-10u^2+8u=v^3-8v^2+12v. \tag{3}$$

The transformation

$$X=\frac{-4u-17v+58}{2u-3v},$$
$$Y=\frac{-146u^2-5v^2+686u-188v}{(2u-3v)^2}$$

yields a minimal Weierstrass model for (3), specifically,

$$E: Y^2=X^3-X^2-30X+81. \tag{4}$$

With *Magma*, we can find the Mordell-Weil group $E(\mathbf{Q})\cong\mathbf{Z}\oplus\mathbf{Z}\oplus\mathbf{Z}$ is generated by the points $P_1=(3,-3)$, $P_2=(-6,3)$ and $P_3=(11,31)$.

Let $Q_0=(X_0,Y_0)$ be the image on (4) of the point at infinity on (3), then we have $X_0=7+2\alpha+3\alpha^2$ and $Y_0=-17-15\alpha-8\alpha^2$ where $\alpha=\sqrt[3]{2}$. Note that $Q_0\in E(\mathbf{Q}(\alpha))$. Also note that $v=\alpha u+\beta$ where $\beta=(8-5\alpha)/3$ is the asymptote of the curve.

It is easy to verify that

$$\frac{dv}{6u^2-20u+8}=-\frac{1}{4}\frac{dX}{Y}. \tag{5}$$

Note that for $v\ge 30$, $u(v)$ given by (3) can be

2010 Mathematics Subject Classification. Primary 11D25, 11G05, 11Y50.

doi: 10.3792/pjaa.91.110

viewed as a strictly increasing function of v, we therefore have

$$\int_v^\infty \frac{dv}{6u^2 - 20u + 8} = \frac{1}{4}\int_{X_0}^{X} \frac{dX}{Y}. \tag{6}$$

It is also easy to verify that $6u^2 - 20u + 8 > 3v^2$ for $v \geq 30$. Thus, we have

$$\int_v^\infty \frac{dv}{6u^2 - 20u + 8} < \frac{1}{3}\int_v^\infty \frac{dv}{v^2} = \frac{1}{3v}. \tag{7}$$

Let $P = m_1P_1 + m_2P_2 + m_3P_3$ be an arbitrary point on (4) with integral coordinates u, v on (3). We have

$$\int_{X_0}^{X} \frac{dX}{Y} = \int_{X_0}^{\infty} \frac{dX}{Y} - \int_{X}^{\infty} \frac{dX}{Y} = \omega(\phi(Q_0) - \phi(P)), \tag{8}$$

where $\omega = 5.832948\ldots$ is the fundamental real period of E, and

$$\begin{aligned}\phi(P) &= \phi(m_1P_1 + m_2P_2 + m_3P_3)\\ &= m_1\phi(P_1) + m_2\phi(P_2) + m_3\phi(P_3) + m_0\end{aligned}$$

with $m_0 \in \mathbf{Z}$ and all ϕ-function are in $[0, 1)$. Put $M = \max_{1\leq i\leq 3} |m_i|$, it follows $|m_0| \leq 3M$. By Zagier's algorithm (see [8]), we have $u_1 = \omega\phi(P_1) = 4.158074\ldots$, $u_2 = \omega\phi(P_2) = 2.851605\ldots$, $u_3 = \omega\phi(P_3) = 0.627538\ldots$, and $u_0 = \omega\phi(Q_0) = 5.289657\ldots$. Let

$$\begin{aligned}L(P) &= \omega(\phi(Q_0) - \phi(P))\\ &= u_0 - m_0\omega - m_1u_1 - m_2u_2 - m_3u_3,\end{aligned}$$

we then obtain the lower bound

$$|L(P)| > \exp(-c_4(\log(3M) + c_5)(\log\log(3M) + c_6)^6), \tag{9}$$

where $c_4 = 7 \times 10^{160}$, $c_5 = 2.1$, and $c_6 = 21.2$, by applying David's result [2] (see also [7]).

By (6) and (7), we also have

$$|L(P)| = 4\int_v^\infty \frac{dv}{6u^2 - 20u + 8} \leq \frac{4}{3v}. \tag{10}$$

For $v \geq 30$, it is easy to verify that

$$h(P) \leq \log(4u + 17v - 58) < 3.044523 + \log v. \tag{11}$$

Here u and v are required to be integral. Moreover we have

$$\hat{h}(P) \geq c_1M^2 \tag{12}$$

where $c_1 = 0.125612\ldots$ is the least eigenvalue of the Néron-Tate height pairing matrix. Note that Silverman's bound for the difference of heights on elliptic curves gives that

$$2\hat{h}(P) - h(P) < 7.846685. \tag{13}$$

By (10), (11), (12), and (13), we obtain the upper bound

$$|L(P)| < \exp(11.1789 - 0.251224M^2). \tag{14}$$

Together with (9) and (14), we therefore find an absolute upper bound $M_0 = 1.4 \times 10^{86}$ for M. Applying the LLL algorithm (cf. [6]), we may reduce this bound to $M = 11$. Through a computer search, we therefore prove that $(x, y) = (14, 15)$ is the only integral solution of (1).

Remark 2.1. It is of interest to mention that all integral solutions (u, v) of (3) are given in Table I, by slightly modifying our proof and then through a similar computer search.

Table I. Integral solutions (u, v) of (3)

$(0,0)$	$(0,2)$	$(0,6)$
$(1,0)$	$(1,2)$	$(1,6)$
$(4,0)$	$(4,2)$	$(4,6)$
$(9,12)$	$(144,182)$	$(-56,-70)$

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