Conformally invariant systems of differential operators associated to maximal parabolics of quasi-Heisenberg type

By Toshihisa KUBO

Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

(Communicated by Kenji FUKAYA, M.J.A., Feb. 12, 2013)

Abstract: Let G_0 be a simple Lie group and Q_0 a maximal parabolic subgroup of quasi-Heisenberg type. In this paper we construct conformally invariant systems of differential operators associated to a homogeneous line bundle $\mathcal{L}_s \to G_0/Q_0$. The systems that we construct yield explicit homomorphisms between appropriate generalized Verma modules. We also determine whether or not these homomorphisms are standard.

Key words: Intertwining differential operator; generalized Verma module; real flag manifold.

1. Introduction. The main work of this paper concerns systems of differential operators that are equivariant under an action of a Lie algebra. We call such systems *conformally invariant*. To explain the meaning of the equivariance condition, suppose that $\mathcal{V} \to M$ is a vector bundle over a smooth manifold M and $\mathfrak{g}$ is a Lie algebra of first order differential operators that act on sections of $\mathcal{V}$. A linearly independent list $D_1, \ldots, D_n$ of linear differential operators on sections of $\mathcal{V}$ is called a *conformally invariant system* if, for each $X \in \mathfrak{g}$, there are smooth functions $C^X_{ij}(m)$ on M so that, for all $1 \leq i \leq n$, and sections f of $\mathcal{V}$, we have

$$([X, D_i]f)(m) = \sum_{j=1}^{n} C^X_{ji}(m)(D_j f)(m), \tag{1.1}$$

where $[X, D_i] = XD_i - D_iX$. (For the precise definition see for example Definition 2.3 of [14].)

A typical example for a conformally invariant system of one differential operator is the wave operator $\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2}$ on the Minkowski space $\mathbf{R}^{3,1}$. If X is an element of $\mathfrak{g} = \mathfrak{so}(4,2)$ acting as a first-order differential operators on sections of an appropriate line bundle over $\mathbf{R}^{3,1}$ then there is a smooth function C^X on $\mathbf{R}^{3,1}$ so that

$$[X, \Box] = C^X \Box.$$

(See for example the introduction of [11].)

The notion of conformally invariant systems generalizes that of quasi-invariant differential operators introduced by Kostant in [12] and is related to a work of Huang ([9]). It is also compatible with the definition given by Ehrenpreis in [7]. Conformally invariant systems are explicitly or implicitly presented in the work of Barbasch-Sahi-Speh ([1]), Davidson-Enright-Stanke ([5]), Enright-Wallach ([8]), Kable ([10]), Kobayashi-Ørsted ([11]), among others. Much of the published work is for the case that $M = G_0/Q_0$ with $Q_0 = L_0N_0$, N_0 abelian. Conformally invariant systems are also related to works of Dobrev (see for instance [6]) in mathematical physics. The systematic study of conformally invariant systems started with the work of Barchini-Kable-Zierau in [2] and [3].

An important consequence of the identity (1.1) is that the common kernel of the operators in the conformally invariant system $D_1, \ldots, D_n$ is invariant under a Lie algebra action. The representation theoretic question of understanding the common kernel as a $\mathfrak{g}$-module is an open question (except for a small number of very special examples).

Although the theory of conformally invariant systems can be viewed as a geometric-analytic theory, it is also closely related to algebraic objects such as generalized Verma modules. It has been shown in [3] that a conformally invariant system yields a homomorphism between certain generalized Verma modules. The classification of non-

2010 Mathematics Subject Classification. Primary 22E46; Secondary 17B10, 22E47.

doi: 10.3792/pjaa.89.41

standard homomorphisms between generalized Verma modules is an open problem.

The main goal of this paper is to build systems of differential operators that satisfy the condition (1.1), when M is a homogeneous manifold G_0/Q_0 with Q_0 a maximal parabolic subgroup of quasi-Heisenberg type. (We shall describe the class of maximal parabolics in Section 2.) This is to construct systems $D_1, \ldots, D_n$ acting on sections of bundles $\mathcal{V}_s \to G_0/Q_0$ over G_0/Q_0 in a systematic manner and to determine the bundles $\mathcal{V}_s$ on which the systems are conformally invariant. The systems that we build yield explicit homomorphisms between appropriate generalized Verma modules. We also classify whether or not these homomorphisms are standard.

The full detail of this paper can be found in the preprints [14] and [13], and will appear elsewhere.

2. Construction of Ω_k systems. The aim of this section is to construct systems of operators that satisfy the condition (1.1). Let G be a complex, simple, connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Fix a maximal connected solvable subgroup B. Write $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{u}$ for its Lie algebra with $\mathfrak{h}$ a Cartan subalgebra and $\mathfrak{u}$ the nilpotent radical. Let $\mathfrak{q} \supset \mathfrak{b}$ be a standard parabolic subalgebra of $\mathfrak{g}$. We define $Q = N_G(\mathfrak{q})$, a parabolic subgroup of G. Write $Q = LN$ for the Levi decomposition of Q.

Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$ in which the complex parabolic subalgebra $\mathfrak{q}$ has a real form $\mathfrak{q}_0$, and let G_0 be the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. Define $Q_0 = N_{G_0}(\mathfrak{q}) \subset Q$, and write $Q_0 = L_0 N_0$. We will work with G_0/Q_0 for a class of maximal parabolic subgroups Q_0 whose unipotent radicals N_0 are two-step nilpotent.

Next, let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the set of roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Let Δ^+ be the positive system attached to $\mathfrak{b}$ and denote by Π the set of simple roots. We write $\mathfrak{g}_\alpha$ for the root space for $\alpha \in \Delta$. For each subset $S \subset \Pi$, let $\mathfrak{q}_S$ be the corresponding standard parabolic subalgebra. Write $\mathfrak{q}_S = \mathfrak{l}_S \oplus \mathfrak{n}_S$ with Levi factor $\mathfrak{l}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_S} \mathfrak{g}_\alpha$ and nilpotent radical $\mathfrak{n}_S = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_S} \mathfrak{g}_\alpha$, where $\Delta_S = \{\alpha \in \Delta \mid \alpha \in \operatorname{span}(\Pi \setminus S)\}$. If Q_0 is a maximal parabolic subgroup then there exists a unique simple root $\alpha_{\mathfrak{q}} \in \Pi$ so that $\mathfrak{q} = \mathfrak{q}_{\{\alpha_{\mathfrak{q}}\}}$. Let $\lambda_{\mathfrak{q}}$ be the fundamental weight for $\alpha_{\mathfrak{q}}$. The weight $\lambda_{\mathfrak{q}}$ is orthogonal to any roots α with $\mathfrak{g}_\alpha \subset [\mathfrak{l}, \mathfrak{l}]$. Hence it exponentiates to a character $\chi_{\mathfrak{q}}$ of L. As $\chi_{\mathfrak{q}}$ takes real values on L_0, for $s \in \mathbf{C}$, character $\chi^s = |\chi_{\mathfrak{q}}|^s$ is well-defined on L_0. Let $\mathbf{C}_{\chi^s}$ be the one-dimensional representation of L_0 with character χ^s. The representation χ^s is extended to a representation of Q_0 by making it trivial on N_0. Then it deduces a line bundle $\mathcal{L}_s$ on G_0/Q_0 with fiber $\mathbf{C}_{\chi^s}$.

By the Bruhat theory, G_0/Q_0 admits an open dense submanifold $\bar{N}_0 Q_0/Q_0$. We restrict our bundle to this submanifold. The systems that we study act on smooth sections of the restricted bundle. By slight abuse of notation we refer to the restricted bundle as $\mathcal{L}_s$.

Now observe that $\mathfrak{g}$ has a $\mathbf{Z}$-grading $\mathfrak{g} = \bigoplus_{i=-r}^{r} \mathfrak{g}(i)$ so that $\mathfrak{q} = \mathfrak{g}(0) \oplus \bigoplus_{i>0} \mathfrak{g}(i) = \mathfrak{l} \oplus \mathfrak{n}$. If $\Delta(\mathfrak{g}(r)) = \{\gamma_j \in \Delta \mid \mathfrak{g}_{\gamma_j} \subset \mathfrak{g}(r)\}$ then, for $1 \leq k \leq 2r$, we define polynomial maps τ_k by

$$\tau_k : \mathfrak{g}(1) \to \mathfrak{g}(-r+k) \otimes \mathfrak{g}(r) \qquad (2.1)$$
$$X \mapsto \frac{1}{k!} (\operatorname{ad}(X)^k \otimes \operatorname{Id})\omega,$$

with $\omega = \sum_{\gamma_j \in \Delta(\mathfrak{g}(r))} X_{-\gamma_j} \otimes X_{\gamma_j}$, where X_{γ_j} are root vectors for γ_j so that $\{X_{\gamma_j}, X_{-\gamma_j}, [X_{\gamma_j}, X_{-\gamma_j}]\}$ is an $\mathfrak{sl}(2)$-triple. Each τ_k can be thought of as giving the symbols of the differential operators that we study.

Let W be an irreducible constituent of $\mathfrak{g}(-r+k) \otimes \mathfrak{g}(r)$ as an L-module. If W^* is its dual space (with respect to the Killing form) then there exists an L-intertwining operator $\tilde{\tau}_k|_{W^*} \in \operatorname{Hom}_L(W^*, \mathcal{P}^k(\mathfrak{g}(1)))$ so that $\tilde{\tau}_k|_{W^*}(Y^*)(X) = Y^*(\tau_k(X))$ for $Y^* \in W^*$, where $\mathcal{P}^k(\mathfrak{g}(1))$ is the space of polynomials on $\mathfrak{g}(1)$ of homogeneous degree k. If $\tilde{\tau}_k|_{W^*} \not\equiv 0$ then we call the irreducible constituent W *special* for τ_k.[*1)] Observe that, as $\mathcal{P}^k(\mathfrak{g}(1)) \cong \operatorname{Sym}^k(\mathfrak{g}(-1))$ via the Killing form, the linear map $\tilde{\tau}_k|_{W^*}$ can be thought of as a map from W^* to $\operatorname{Sym}^k(\mathfrak{g}(-1))$. Then we consider the following composition of linear maps:

$$W^* \overset{\tilde{\tau}_k|_{W^*}}{\to} \operatorname{Sym}^k(\mathfrak{g}(-1)) \overset{\sigma}{\hookrightarrow} \mathcal{U}(\bar{\mathfrak{n}}) \overset{R}{\to} \mathbf{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}. \qquad (2.2)$$

Here, $\sigma : \operatorname{Sym}^k(\mathfrak{g}(-1)) \to \mathcal{U}(\bar{\mathfrak{n}})$ is the symmetrization operator, R is the infinitesimal right translation, and $\mathbf{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ is the space of $\bar{\mathfrak{n}}$-invariant differential operators for $\mathcal{L}_s$ with $\bar{\mathfrak{n}}$ the opposite nilradical of $\mathfrak{n}$. Let $\Omega_k|_{W^*} : W^* \to \mathbf{D}(\mathcal{L}_s)^{\bar{\mathfrak{n}}}$ be the composition of linear maps, namely, $\Omega_k|_{W^*} = R \circ \sigma \circ \tilde{\tau}_k|_{W^*}$. For simplicity we write $\Omega_k(Y^*) =$

[*1)] There is a certain discrepancy of the definition for special constituents between this paper and [14]. See the introduction of [13] for some remark on this matter.

$\Omega_k|_{W^*}(Y^*)$ for the k-th order differential operator arising from $Y^* \in W^*$.

Now, given basis $\{Y_1^*, \ldots, Y_m^*\}$ for W^*, we have a system of differential operators

$$\Omega_k(Y_1^*), \ldots, \Omega_k(Y_m^*) \tag{2.3}$$

for $\mathcal{L}_s$. We call the system of operators the $\Omega_k|_{W^*}$ *system.* When the irreducible constituent W is not important, we simply refer to each $\Omega_k|_{W^*}$ system as an Ω_k system.

It is not necessary for the $\Omega_k|_{W^*}$ system to be conformally invariant; the conformal invariance of the operators (2.3) strongly depends on the complex parameter s of the line bundle $\mathcal{L}_s$. Then we say that the $\Omega_k|_{W^*}$ system has *special value* s_0 if the system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$.

The special values for the case that $\dim([\mathfrak{n}, \mathfrak{n}]) = 1$ for $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$, a parabolic subalgebra of Heisenberg type, are studied by Barchini-Kable-Zierau in [2] and [3], and myself in [15]. We then consider a more general case; namely, $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{n}$ is a maximal parabolic subalgebra with $\mathfrak{n}$ satisfying the conditions that $[\mathfrak{n}, [\mathfrak{n}, \mathfrak{n}]] = 0$ and $\dim([\mathfrak{n}, \mathfrak{n}]) > 1$. We call such parabolic subalgebras *quasi-Heisenberg type.* In this case we have $r = 2$ in (2.1). Therefore the Ω_k systems for $k \geq 5$ are zero. In the next section we find the special values of the Ω_1 system and Ω_2 systems for $\mathfrak{q}$ under consideration.

3. Special values of *s*. The aim of this section is to find the special values of the Ω_1 system and Ω_2 systems for maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heisenberg type.

We start with describing the results for the Ω_1 system. When $k = 1$, since $\tau_1 : \mathfrak{g}(1) \to \mathfrak{g}(-1) \otimes \mathfrak{g}(2)$ is linear, if V is a special constituent in $\mathfrak{g}(-1) \otimes \mathfrak{g}(2)$ then $V \cong \mathfrak{g}(1)$. By identifying $\mathfrak{g}(1)^* \cong \mathfrak{g}(-1)$ via the Killing form, for $\Delta(\mathfrak{g}(1)) = \{\alpha_1, \ldots, \alpha_m\}$ the set of roots contributing to $\mathfrak{g}(1)$, the Ω_1 system is given by $R(X_{-\alpha_1}), \ldots, R(X_{-\alpha_m})$ via the composition of maps in (2.2). (Here $X_{-\alpha_i}$ are root vectors for $-\alpha_i$.)

Theorem 1 ([14, Theorem 5.7])**.** *Let $\mathfrak{q}$ be a maximal parabolic subalgebra of quasi-Heisenberg type. The Ω_1 system is conformally invariant on $\mathcal{L}_s$ if and only if $s = 0$.*

Outline of the proof. A direct computation shows that, for $X \in \mathfrak{g}$ acting as a first order differential operator on $\mathcal{L}_s$, the bracket $[X, R(X_{-\alpha_i})]$ is of the form

$$[X, R(X_{-\alpha_i})] = \sum_{\alpha_j \in \Delta(\mathfrak{g}(1))} C_{ji}^X R(X_{-\alpha_j}) + sF_{ji}^X,$$

where C_{ji}^X and F_{ji}^X are some smooth functions on $\bar{N}_0$. Now the identity (1.1) holds if and only if $s = 0$. □

Next, we describe the results for the Ω_2 systems. To determine special constituents, we decompose $\mathfrak{g}(0) \otimes \mathfrak{g}(2)$ into irreducible constituents by using a standard character formula. The special constituents are classified as type 1a, type 1b, type 2, and type 3. To describe the types of the special constituents, we now briefly observe the structure of $[\mathfrak{g}(0), \mathfrak{g}(0)]$. Note that the subspace $\mathfrak{g}(0)$ is $\mathfrak{g}(0) = \mathfrak{l}$, the Levi subalgebra of $\mathfrak{q}$.

Let α_γ be a simple root that is not orthogonal to the highest root for $\mathfrak{g}$. Note that if maximal parabolic subalgebra $\mathfrak{q}$ is of quasi-Heisenberg type then $[\mathfrak{l}, \mathfrak{l}]$ is either simple or the direct sum of two or three simple ideals with only one simple ideal containing the root space $\mathfrak{g}_{\alpha_\gamma}$. (For the details see Section 4.3 of [14].) Given Dynkin type $\mathcal{T}$ of $\mathfrak{g}$, if we write $\mathcal{T}(i)$ for the Lie algebra together with the choice of maximal parabolic subalgebra $\mathfrak{q} = \mathfrak{q}_{\{\alpha_i\}}$ determined by α_i then the three simple factors occur only when $\mathfrak{q}$ is of type $D_n(n-2)$. So, if $\mathfrak{q}$ is not of type $D_n(n-2)$ then there are at most two simple factors. In this case we denote by $\mathfrak{l}_\gamma$ (resp. $\mathfrak{l}_{n\gamma}$) the simple ideal of $\mathfrak{l}$ that contains (reps. does not contain) $\mathfrak{g}_{\alpha_\gamma}$. Thus $\mathfrak{l}$ may decompose into $\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}_\gamma \oplus \mathfrak{l}_{n\gamma}$, where $\mathfrak{z}(\mathfrak{l})$ is the center of $\mathfrak{l}$. Note that when $[\mathfrak{l}, \mathfrak{l}]$ is a simple ideal, we have $\mathfrak{l}_{n\gamma} = \{0\}$. The maximal parabolic subalgebras $\mathfrak{q}$ of quasi-Heisenberg type with the above decomposition of $\mathfrak{l}$ are parametrized as:

$$\begin{aligned} &B_n(i)\ (3 \leq i \leq n), \\ &C_n(i)\ (2 \leq i \leq n-1), \\ &D_n(i)\ (3 \leq i \leq n-3), \end{aligned} \tag{3.1}$$

and

$$E_6(3),\ E_6(5),\ E_7(2),\ E_7(6),\ E_8(1),\ F_4(4). \tag{3.2}$$

Here, the Bourbaki conventions [4] are used for the labels of the simple roots. Note that when $\mathfrak{g}$ is of type A_n or type G_2, there is no maximal parabolic subalgebra of quasi-Heisenberg type. As $\mathfrak{n}$ is two-step nilpotent, we have $\mathfrak{g}(2) = \mathfrak{z}(\mathfrak{n})$, the center of $\mathfrak{n}$. Thus $\mathfrak{g}(0) \otimes \mathfrak{g}(2) = \mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ may be written as $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n}) = (\mathfrak{z}(\mathfrak{l}) \otimes \mathfrak{z}(\mathfrak{n})) \oplus (\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})) \oplus (\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n}))$.

Now observe that if $V(\nu)$ is a special constituent of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$ with highest weight ν then, as $V(\nu)^*$

Table I. Types of special constituent

Type	$V(\mu+\epsilon_\gamma)$	$V(\mu+\epsilon_{n\gamma})$
$B_n(i)$, $3 \le i \le n-2$	Type 1a	Type 1a
$B_n(n-1)$	Type 1a	Type 1b
$B_n(n)$	Type 2	—
$C_n(i)$, $2 \le i \le n-1$	Type 3	Type 2
$D_n(i)$, $3 \le i \le n-3$	Type 1a	Type 1a
$E_6(3)$	Type 1a	Type 1a
$E_6(5)$	Type 1a	Type 1a
$E_7(2)$	Type 1a	—
$E_7(6)$	Type 1a	Type 1a
$E_8(1)$	Type 1a	—
$F_4(4)$	Type 2	—

Table II. Line bundles with special values for Ω_2 systems

Parabolic $\mathfrak{q}$	$\Omega_2\vert_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\vert_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i), 3 \le i \le n-2$	$n-i-\frac{1}{2}$	1
$B_n(n-1)$	$\frac{1}{2}$	?
$B_n(n)$	-1	—
$C_n(i)$, $2 \le i \le n-1$	?	-1
$D_n(i)$, $3 \le i \le n-3$	$n-i-1$	1
$E_6(3)$	1	2
$E_6(5)$	1	2
$E_7(2)$	2	—
$E_7(6)$	1	3
$E_8(1)$	3	—
$F_4(4)$	-1	—

is embedded into $\mathrm{Sym}^2(\mathfrak{g}(-1)) \cong \mathrm{Sym}^2(\mathfrak{g}(1))^* \subset \mathfrak{g}(1)^* \otimes \mathfrak{g}(1)^*$ (see (2.2)), we have $V(\nu) \hookrightarrow \mathfrak{g}(1) \otimes \mathfrak{g}(1)$. (Here recall that the duality is with respect to the Killing form.) Thus the highest weight ν is of the form $\mu+\epsilon$, where μ is the highest weight for $\mathfrak{g}(1)$ and ϵ is some weight for $\mathfrak{g}(1)$. It is shown in Section 6 of [14] that, for $\mathfrak{q}$ under consideration, there are exactly one or two special constituents of $\mathfrak{l} \otimes \mathfrak{z}(\mathfrak{n})$; one is an irreducible constituent of $\mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and the other is equal to $\mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$. We denote by $V(\mu+\epsilon_\gamma)$ and $V(\mu+\epsilon_{n\gamma})$ the special constituents so that $V(\mu+\epsilon_\gamma) \subset \mathfrak{l}_\gamma \otimes \mathfrak{z}(\mathfrak{n})$ and $V(\mu+\epsilon_{n\gamma}) = \mathfrak{l}_{n\gamma} \otimes \mathfrak{z}(\mathfrak{n})$.

Definition 1 ([14, Definition 6.20])**.** Let μ be the highest weight for $\mathfrak{g}(1)$, and let $\epsilon = \epsilon_\gamma$ or $\epsilon_{n\gamma}$. We say that a special constituent $V(\mu+\epsilon)$ is of

(1) **type 1a** if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and both μ and ϵ are long roots,
(2) **type 1b** if $\mu+\epsilon$ is not a root with $\epsilon \neq \mu$ and either μ or ϵ is a short root,
(3) **type 2** if $\mu+\epsilon = 2\mu$ is not a root, or
(4) **type 3** if $\mu+\epsilon$ is a root.

Table I shows the types of the special constituents $V(\mu+\epsilon)$ for each maximal parabolic subalgebra $\mathfrak{q}$. A dash in the column for $V(\mu+\epsilon_{n\gamma})$ indicates that $\mathfrak{l}_{n\gamma} = \{0\}$ for the case. (So there is no special constituent $V(\mu+\epsilon_{n\gamma})$.) One may observe from Table I that in most of the cases special constituents $V(\mu+\epsilon)$ are either type 1a or type 2. Then, in this paper, we find the special values for these cases.

Write

$$\Delta_{\mu+\epsilon}(\mathfrak{g}(1)) = \{\alpha \in \Delta(\mathfrak{g}(1)) \mid \mu+\epsilon-\alpha \in \Delta(\mathfrak{g}(1))\},$$

and we denote by $|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|$ the number of elements in $\Delta_{\mu+\epsilon}(\mathfrak{g}(1))$.

Theorem 2 ([14, Theorem 7.16, Corollary 7.23])**.** *Suppose that $V(\mu+\epsilon)$ is a special constituent of* type 1a *or* type 2*.*

(a) *If $V(\mu+\epsilon)$ is of* type 1a *then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if*

$$s = \frac{|\Delta_{\mu+\epsilon}(\mathfrak{g}(1))|}{2} - 1.$$

(b) *If $V(\mu+\epsilon)$ is of* type 2 *then the $\Omega_2|_{V(\mu+\epsilon)^*}$ system is conformally invariant on $\mathcal{L}_s$ if and only if*

$$s = -1.$$

Outline of the proof. It is not easy to find the special values of s for the Ω_2 systems by a direct computation. (See Section 5 of [2].) We then use two reduction techniques to compute the special values. First, in order to show the equivariance condition (1.1) for $D_j = \Omega_2(Y_j^*)$, it is enough to compute $[X, \Omega_2(Y_j^*)]$ at the identity e. (Here we regard $\mathfrak{g}$ as the Lie algebra of first order differential operators.) Furthermore, we can show that it is even sufficient to compute only $[X_h, \Omega_2(Y_l^*)]$ at e, where X_h and Y_l^* are a highest weight vector for $\mathfrak{g}(1) \subset \mathfrak{g}$ and a lowest weight vector for V^*, respectively. Now, by computing $[X_h, \Omega_2(Y_l^*)]$ at e, we can obtain the special values. Note that the conditions that $\mu+\epsilon$ is not a root and that both μ and ϵ are long roots play a role in the computation. (For the details, see Section 7.3 of [14].) □

Let λ_i be the fundamental weight for the simple root α_i that determines the maximal parabolic subalgebra $\mathfrak{q}$. Table II summarizes the special values of s of the line bundles $\mathcal{L}_s = \mathcal{L}(s\lambda_i)$ on which the Ω_2 systems are conformally invariant. When $\mathfrak{q}$ is of type $B_n(n-1)$, the constituent $V(\mu+\epsilon_{n\gamma})$ is of type 1b, and when $\mathfrak{q}$ is of type $C_n(i)$, the constituent $V(\mu+\epsilon_\gamma)$ is of type 3. Therefore, a question mark is put for these cases in Table II.

4. Homomorphism φ_{Ω_k}. In this section we briefly discuss about homomorphisms φ_{Ω_k} between generalized Verma modules that are induced by conformally invariant Ω_k systems. (For the full description see [13].)

If an Ω_k system is conformally invariant on the line bundle $\mathcal{L}_{s_0}$ then the system of operators yields a finite dimensional simple $\mathfrak{l}$-submodule F of generalized Verma module $M_\mathfrak{q}[\mathbf{C}_{-s_0\lambda_\mathfrak{q}}] = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} \mathbf{C}_{-s_0\lambda_\mathfrak{q}}$, on which $\mathfrak{n}$ acts trivially. Then the inclusion map $\iota \in \mathrm{Hom}_L(F, M_\mathfrak{q}[\mathbf{C}_{-s_0\lambda_\mathfrak{q}}])$ induces a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism

$$\varphi_{\Omega_k} \in \mathrm{Hom}_{\mathcal{U}(\mathfrak{g}),L}(M_\mathfrak{q}[F], M_\mathfrak{q}[\mathbf{C}_{-s_0\lambda_\mathfrak{q}}])$$

between generalized Verma modules, that is given by

$$\begin{aligned} M_\mathfrak{q}[F] &\overset{\varphi_{\Omega_k}}{\to} M_\mathfrak{q}[\mathbf{C}_{-s_0\lambda_\mathfrak{q}}] \\ u \otimes Y &\mapsto u \cdot \iota(Y), \end{aligned} \tag{4.1}$$

where $M_\mathfrak{q}[F] = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{q})} F$.

Observe that there is a quotient map from a (full) Verma module to a generalized Verma module. A homomorphism φ between generalized Verma modules is then called *standard* if φ comes from a homomorphism between the corresponding full Verma modules. The following theorem classifies whether or not the maps φ_{Ω_k} for $k = 1, 2$ induced by the Ω_1 system and Ω_2 systems are standard.

Theorem 3 ([13, Theorems 5.3, 6.5, 6.6, 6.38])**.**
(a) *The map φ_{Ω_1} is standard.*
(b) *The standardness of φ_{Ω_2} depends on $V(\mu+\epsilon)$.*

Table III summarizes the results for φ_{Ω_2}.

The proof for Theorem 3 is a bit too long for this paper. Thus, for the rest of this section, we only describe key facts to show that φ_{Ω_k} is non-standard.

First recall that if an $\Omega_k|_{W^*}$ system is conformally invariant on $\mathcal{L}_{s_0}$ then the system of operators yields a finite dimensional simple $\mathfrak{l}$-submodule F of $M_\mathfrak{q}[\mathbf{C}_{-s_0\lambda_\mathfrak{q}}]$ on which $\bar{\mathfrak{n}}$ acts trivially.

Proposition 1 ([13, Proposition 3.4])**.** *If W^* has highest weight ν then the simple $\mathfrak{l}$-module F has highest weight $\nu - s_0\lambda_\mathfrak{q}$.*

To show that φ_{Ω_k} is non-standard, it suffices to show that the standard map φ_{std} between the two generalized Verma module is zero. It is known exactly when the standard map is zero. To describe the criterion efficiently, we now parametrize generalized Verma modules by their infinitesimal characters. Thus, by using Proposition 1, we express (4.1) as

$$M_\mathfrak{q}(\nu - s_0\lambda_\mathfrak{q} + \rho) \to M_\mathfrak{q}(-s_0\lambda_\mathfrak{q} + \rho), \tag{4.2}$$

where ρ is half the sum of the positive roots.

The main idea to show the standard map φ_{std} is zero is a use of a theorem of Lepowsky. Write

$$\mathbf{P}_\mathfrak{l}^+ = \{\zeta \in \mathfrak{h}^* \mid \langle \zeta, \alpha^\vee \rangle \in 1 + \mathbf{Z}_{\geq 0} \text{ for all } \alpha \in \Pi(\mathfrak{l})\},$$

where $\Pi(\mathfrak{l})$ is the set of simple roots for $\mathfrak{l}$. As usual, if there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism from $M(\eta)$ into $M(\zeta)$ then we write $M(\eta) \subset M(\zeta)$.

Theorem 4 ([16, Proposition 3.3])**.** *Let $\eta, \zeta \in \mathbf{P}_\mathfrak{l}^+$, and assume that $M(\eta) \subset M(\zeta)$. Then the standard map φ_{std} from $M_\mathfrak{q}(\eta)$ to $M_\mathfrak{q}(\zeta)$ is zero if and only if $M(\eta) \subset M(s_\alpha\zeta)$ for some $\alpha \in \Pi(\mathfrak{l})$.*

Theorem 4 reduces the existence problem of the non-zero standard map φ_{std} between generalized Verma modules to that of the non-zero map between appropriate Verma modules. It is well known when a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism between Verma modules exists. To describe the condition, we first introduce the definition of a *link* of two weights.

Definition 2 (Bernstein-Gelfand-Gelfand)**.** Let $\lambda, \delta \in \mathfrak{h}^*$ and $\beta_1, \ldots, \beta_t \in \Delta^+$. Set $\delta_0 = \delta$ and $\delta_i = s_{\beta_i} \cdots s_{\beta_1}\delta$ for $1 \leq i \leq t$. We say that the sequence $(\beta_1, \ldots, \beta_t)$ ***links*** δ to λ if
(1) $\delta_t = \lambda$ and
(2) $\langle \delta_{i-1}, \beta_i^\vee \rangle \in \mathbf{Z}_{\geq 0}$ for $1 \leq i \leq t$.

Table III. The homomorphism φ_{Ω_2} arising from the Ω_2 systems

Parabolic $\mathfrak{q}$	$\Omega_2\|_{V(\mu+\epsilon_\gamma)^*}$	$\Omega_2\|_{V(\mu+\epsilon_{n\gamma})^*}$
$B_n(i)$, $3 \leq i \leq n-2$	standard	non-standard
$B_n(n-1)$	standard	?
$B_n(n)$	standard	—
$C_n(i)$, $2 \leq i \leq n-1$	?	standard
$D_n(i)$, $3 \leq i \leq n-3$	non-standard	non-standard
$E_6(3)$	non-standard	non-standard
$E_6(5)$	non-standard	non-standard
$E_7(2)$	non-standard	—
$E_7(6)$	non-standard	non-standard
$E_8(1)$	non-standard	—
$F_4(4)$	standard	—

Theorem 5 *(BGG-Verma)*. *Let* $\lambda, \delta \in \mathfrak{h}^*$. *The following conditions are equivalent:*

(1) $M(\lambda) \subset M(\delta)$.
(2) $L(\lambda)$ *is a composition factor of* $M(\delta)$.
(3) *There exists a sequence* $(\beta_1, \ldots, \beta_t)$ *with* $\beta_i \in \Delta^+$ *that links* δ *to* λ,

where $L(\lambda)$ *is the unique irreducible quotient of* $M(\lambda)$.

Observe that if there is a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism (not necessarily standard) from $M_{\mathfrak{q}}(\eta)$ to $M_{\mathfrak{q}}(\zeta)$ then $M(\eta) \subset M(\zeta)$. By taking into account Theorem 5 and this observation, in our setting, Theorem 4 is equivalent to the following proposition.

Proposition 2 ([13, Propostion 4.6]). *Let* $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ *and* $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ *be the generalized Verma modules in* (4.2). *Then the standard map from* $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ *to* $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ *is zero if and only if there exists* $\alpha \in \Pi(\mathfrak{l})$ *so that* $-\alpha - s_0\lambda_{\mathfrak{q}} + \rho$ *is linked to* $\nu - s_0\lambda_{\mathfrak{q}} + \rho$.

Proof. First observe that since there exists a non-zero $\mathcal{U}(\mathfrak{g})$-homomorphism φ_{Ω_k} from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$, we have $M(\nu - s_0\lambda_{\mathfrak{q}} + \rho) \subset M(-s_0\lambda_{\mathfrak{q}} + \rho)$. Therefore, by Theorem 4 and Theorem 5, the standard map from $M_{\mathfrak{q}}(\nu - s_0\lambda_{\mathfrak{q}} + \rho)$ to $M_{\mathfrak{q}}(-s_0\lambda_{\mathfrak{q}} + \rho)$ is zero if and only if there exists $\alpha \in \Pi(\mathfrak{l})$ so that $s_\alpha(-s_0\lambda_{\mathfrak{q}} + \rho)$ is linked to $\nu - s_0\lambda_{\mathfrak{q}} + \rho$. As $\langle \lambda_{\mathfrak{q}}, \alpha^\vee \rangle = 0$ and $\langle \rho, \alpha^\vee \rangle = 1$ for $\alpha \in \Pi(\mathfrak{l})$, we have $s_\alpha(-s_0\lambda_{\mathfrak{q}} + \rho) = -\alpha - s_0\lambda_{\mathfrak{q}} + \rho$. Now this proposition follows. □

Using this criterion we can show that the map φ_{Ω_k} is non-standard for each non-standard case in Table III.

Acknowledgments. This work is part of author's Ph.D. thesis at Oklahoma State University. The author would like to thank his advisor, Prof. Leticia Barchini, for her generous guidance. He would also like to thank Profs. Anthony Kable and Roger Zierau for their valuable comments on this work, and the referee for careful reading.

References

[1] D. Barbasch, S. Sahi and B. Speh, Degenerate series representations for GL$(2n, \mathbf{R})$ and Fourier analysis, in *Symposia Mathematica, Vol. XXXI* (*Rome, 1988*), 45–69, Sympos. Math., XXXI, Academic Press, London, 1990.

[2] L. Barchini, A. C. Kable and R. Zierau, Conformally invariant systems of differential equations and prehomogeneous vector spaces of Heisenberg parabolic type, Publ. Res. Inst. Math. Sci. **44** (2008), no. 3, 749–835.

[3] L. Barchini, A. C. Kable and R. Zierau, Conformally invariant systems of differential operators, Adv. Math. **221** (2009), no. 3, 788–811.

[4] N. Bourbaki, *Éléments de mathématique*, Masson, Paris, 1981.

[5] M. G. Davidson, T. J. Enright and R. J. Stanke, Differential operators and highest weight representations, Mem. Amer. Math. Soc. **94** (1991), no. 455, iv+102 pp.

[6] V. K. Dobrev, Canonical construction of differential operators intertwining representations of real semisimple Lie groups, Rep. Math. Phys. **25** (1988), no. 2, 159–181.

[7] L. Ehrenpreis, Hypergeometric functions, in *Algebraic analysis*, vol. I, Academic Press, Boston, MA, 1988, pp. 85–128.

[8] T. J. Enright and N. R. Wallach, Embeddings of unitary highest weight representations and generalized Dirac operators, Math. Ann. **307** (1997), no. 4, 627–646.

[9] J.-S. Huang, Intertwining differential operators and reducibility of generalized Verma modules, Math. Ann. **297** (1993), no. 2, 309–324.

[10] A. C. Kable, Conformally invariant systems of differential equations on flag manifolds for G_2 and their K-finite solutions, J. Lie Theory **22** (2012), no. 1, 93–136.

[11] T. Kobayashi and B. Ørsted, Analysis on the minimal representation of O(p, q). III. Ultrahyperbolic equations on $\mathbf{R}^{p-1,q-1}$, Adv. Math. **180** (2003), no. 2, 551–595.

[12] B. Kostant, Verma modules and the existence of quasi-invariant differential operators, in *Noncommutative harmonic analysis* (*Actes Colloq., Marseille-Luminy, 1974*), 101–128. Lecture Notes in Math., 466, Springer, Berlin, 1975.

[13] T. Kubo, On the homomorphisms between the gen-eralized Verma modules arising from conformally invariant systems, arXiv:1209.5516.

[14] T. Kubo, Special values for conformally invari-ant systems associated to maximal parabolics of quasi-Heisenberg type, arXiv:1209.1861.

[15] T. Kubo, A system of third-order differential operators conformally invariant under $\mathfrak{sl}(3, \mathbf{C})$ and $\mathfrak{so}(8, \mathbf{C})$, Pacific J. Math. **253** (2011), no. 2, 439–453.

[16] J. Lepowsky, A generalization of the Bernstein-Gelfand-Gelfand resolution, J. Algebra **49** (1977), no. 2, 496–511.