Generalizations of the Farkas identity for modulus 4 and 7

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Abstract: In this paper, we express the generating functions of the partion numbers of certain kinds by theta constants with rational characteristics. Our result is the generalizations of the result of Farkas [1] for modulus 4 and 7.

Key words: Theta functions; rational characteristic; partion.

Introduction. The aim of this paper is to express the generating functions of the partion numbers of certain kinds by theta constants with rational characteristics. Our concern is with the infinite product given by

$$(0.1)\qquad \frac{1}{\prod_{n=0}^{\infty}(1-x^{kn+1})(1-x^{kn+2})\cdots(1-x^{kn+(k-1)})},$$

where k is a positive integer with $k \geq 3$. The coefficient of x^n in the power series expansion is the number we can write n using summands that are congruent to $1, 2, \ldots (k-1)$ mod k.

Following Farkas and Kra [2], we first introduce the theta function with characteristic, which is defined by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau) := \sum_{n\in\mathbf{Z}} \exp\left(2\pi i\left[\frac{1}{2}\left(n+\frac{\epsilon}{2}\right)^2\tau + \left(n+\frac{\epsilon}{2}\right)\left(\zeta+\frac{\epsilon'}{2}\right)\right]\right),$$

where ϵ, $\epsilon' \in \mathbf{R}$, $\zeta \in \mathbf{C}$, and $\tau \in \mathbf{H}^2$, the upper half plane. The theta constants are given by

$$\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(0,\tau),$$

$$\theta'\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix} := \frac{\partial}{\partial\zeta}\theta\begin{bmatrix}\epsilon\\ \epsilon'\end{bmatrix}(\zeta,\tau)\Big|_{\zeta=0}.$$

For the basic properties of the theta functions, see Farkas and Kra [2, pp. 72–80].

Farkas [1] treated equation (0.1) for $k = 3$ and proved that for every $\tau \in \mathbf{H}^2$,

$$(0.2)\qquad \frac{6\theta'\begin{bmatrix}1\\ \frac{1}{3}\end{bmatrix}(0,\tau)}{-\omega^2\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{1}{3}\end{bmatrix}(0,\tau) - \omega\theta^3\begin{bmatrix}\frac{1}{3}\\ \frac{5}{3}\end{bmatrix}(0,\tau) + \theta^3\begin{bmatrix}\frac{1}{3}\\ 1\end{bmatrix}(0,\tau)} = \frac{2\pi i x^{\frac{1}{12}}}{\prod_{n=0}^{\infty}(1-x^{3n+1})(1-x^{3n+2})},$$

where $x = \exp(2\pi i\tau)$ and $\omega = \exp(\frac{2\pi i}{3})$.

In this paper, we treat the case where $k = 4, 7$. Our main theorem is as follows:

Theorem 0.1. *For every $\tau \in \mathbf{H}^2$, we have*

$$(0.3)\qquad \frac{2\theta'\begin{bmatrix}1\\ \frac{1}{2}\end{bmatrix}(0,\tau)\theta\begin{bmatrix}1\\ 0\end{bmatrix}(0,\tau)}{\left(\theta^2\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}(0,\tau) - i\theta^2\begin{bmatrix}\frac{1}{2}\\ 1\end{bmatrix}(0,\tau)\right)\left(\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(0,\tau) - i\theta^2\begin{bmatrix}\frac{1}{2}\\ \frac{3}{2}\end{bmatrix}(0,\tau)\right)} = \frac{\sqrt{2}\pi i\exp\left(\frac{\pi i}{4}\right)x^{\frac{1}{8}}}{\prod_{n=0}^{\infty}(1-x^{4n+1})(1-x^{4n+2})(1-x^{4n+3})},$$

and

$$(0.4)\qquad \left(\frac{14\sum_{j=1}^{3}(-1)^{j-1}\theta^6\begin{bmatrix}1\\ \frac{2j-1}{7}\end{bmatrix}(0,\tau)\theta'\begin{bmatrix}1\\ \frac{2j-1}{7}\end{bmatrix}(0,\tau)}{\prod_{k=1}^{3}\theta^2\begin{bmatrix}1\\ \frac{2k-1}{7}\end{bmatrix}(0,\tau)}\right)^3 \times \frac{\left\{\theta'\begin{bmatrix}1\\ 1\end{bmatrix}(0,\tau)\right\}^4}{\prod_{l=1}^{3}\left(\sum_{m=1}^{7}\zeta_{14}^{8l+3-(2l-1)m}\theta^7\begin{bmatrix}\frac{2l-1}{7}\\ \frac{2m-1}{7}\end{bmatrix}(0,\tau)\right)} = \frac{(2\pi i)^7 x^{\frac{1}{4}}}{\prod_{n=0}^{\infty}(1-x^{7n+1})(1-x^{7n+2})(1-x^{7n+3})(1-x^{7n+4})(1-x^{7n+5})(1-x^{7n+6})},$$

where $\zeta_{14} = \exp(\frac{2\pi i}{14})$ and $x = \exp(2\pi i\tau)$.

1. Theta functional formula. The results in this section can be proved in the same way as

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equation (0.2). For the detailed proof, see Farkas [1].

Lemma 1.1.

$$\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau)\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau) - i\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau)\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau)$$
$$- i\theta^2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau) - \theta^2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau)$$
$$= c(\tau)\ \theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau),$$

where $c(\tau)$ is a constant independent of ζ.

Proof. By the basic properties of the theta functions, we can show that the both sides have the same zeros. The quotient is then an elliptic function with no poles and hence constant, which proves the lemma. □

Proposition 1.2. *For every $(\zeta,\tau) \in \mathbf{C} \times \mathbf{H}^2$, we have*

$$4\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)\theta'\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau)$$
$$= i\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau)\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)\left(\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau) - i\theta^2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\right)$$
$$\times\left(\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau) - i\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau)\right).$$

Proof. From Lemma 1.1, it follows that

$$c(\tau) = \frac{\left(\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau) - i\theta^2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\right)\left(\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau) - i\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau)\right)}{\theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\zeta,\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\zeta,\tau)}.$$

If we now differentiate both numerator and denominator and set $\zeta = \tau/4$, we obtain

$$c(\tau) = \frac{\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\frac{\tau}{4},\tau)\left(2\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\frac{\tau}{4},\tau)\theta'\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\frac{\tau}{4},\tau) - 2i\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\frac{\tau}{4},\tau)\theta'\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\frac{\tau}{4},\tau)\right)}{\theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}(\frac{\tau}{4},\tau)\theta'\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}(\frac{\tau}{4},\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}(\frac{\tau}{4},\tau)\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}(\frac{\tau}{4},\tau)}.$$

Using the basic properties of the theta functions, we can show that $c(\tau)$ becomes

$$4 \times \frac{\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(0,\tau)\theta'\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)}{i\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau)\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}(0,\tau)}.$$

□

Using Jacobi's derivative formula,

$$\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix} = -\pi\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

we obtain the following corollary:

Corollary 1.3.

$$\frac{4\theta'\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{i\left(\theta^2\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} - i\theta^2\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\right)\left(\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - i\theta^2\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}\right)}$$
$$= -\pi\frac{\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\theta\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}}.$$

In the same way as Proposition 1.2, we can prove the following proposition:

Proposition 1.4. *For every $(\zeta,\tau) \in \mathbf{C} \times \mathbf{H}^2$, we have*

$$(1.1)\quad \theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau)\prod_{k=1}^{3}\theta^2\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)$$
$$\times\left(\sum_{l=1}^{7}\zeta_{14}^{11-l}\theta^7\begin{bmatrix} \frac{1}{7} \\ \frac{2l-1}{7} \end{bmatrix}(\zeta,\tau)\right)$$
$$= -14\sum_{k=1}^{3}(-1)^{k-1}\theta^6\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)\theta'\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)$$
$$\times\prod_{l=1}^{7}\theta\begin{bmatrix} \frac{1}{7} \\ \frac{2l-1}{7} \end{bmatrix}(\zeta,\tau),$$

$$(1.2)\quad \theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau)\prod_{k=1}^{3}\theta^2\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)$$
$$\times\left(\sum_{l=1}^{7}\zeta_{14}^{19-3l}\theta^7\begin{bmatrix} \frac{3}{7} \\ \frac{2l-1}{7} \end{bmatrix}(\zeta,\tau)\right)$$
$$= -14\sum_{k=1}^{3}(-1)^{k-1}\theta^6\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)\theta'\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)$$
$$\times\prod_{l=1}^{7}\theta\begin{bmatrix} \frac{3}{7} \\ \frac{2l-1}{7} \end{bmatrix}(\zeta,\tau),$$

and

$$(1.3)\quad \theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau)\prod_{k=1}^{3}\theta^2\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)$$
$$\times\left(\sum_{l=1}^{7}\zeta_{14}^{27-5l}\theta^7\begin{bmatrix} \frac{5}{7} \\ \frac{2l-1}{7} \end{bmatrix}(\zeta,\tau)\right)$$
$$= -14\sum_{k=1}^{3}(-1)^{k-1}\theta^6\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)\theta'\begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix}(0,\tau)\times$$

$$\times \prod_{l=1}^{7} \theta \begin{bmatrix} \frac{5}{7} \\ \frac{2l-1}{7} \end{bmatrix} (\zeta, \tau).$$

Remark. We note that

$$\sum_{k=1}^{3} (-1)^{k-1} \theta^6 \begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix} (0,\tau) \theta' \begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix} (0,\tau)$$

$$= \theta^7 \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,\tau) \times \left(\frac{\theta^7 \begin{bmatrix} 1 \\ \frac{1}{7} \end{bmatrix} (0,\tau)}{\theta^7 \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,\tau)} \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{7} \end{bmatrix} (0,\tau)}{\theta \begin{bmatrix} 1 \\ \frac{1}{7} \end{bmatrix} (0,\tau)} - \frac{\theta^7 \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (0,\tau)}{\theta^7 \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,\tau)} \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (0,\tau)}{\theta \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (0,\tau)} + \frac{\theta' \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,\tau)}{\theta \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,\tau)} \right),$$

and

$$\frac{\theta^7 \begin{bmatrix} 1 \\ \frac{1}{7} \end{bmatrix} (0,i\infty)}{\theta^7 \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,i\infty)} \frac{\theta' \begin{bmatrix} 1 \\ \frac{1}{7} \end{bmatrix} (0,i\infty)}{\theta \begin{bmatrix} 1 \\ \frac{1}{7} \end{bmatrix} (0,i\infty)} - \frac{\theta^7 \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (0,i\infty)}{\theta^7 \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,i\infty)}$$

$$\times \frac{\theta' \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (0,i\infty)}{\theta \begin{bmatrix} 1 \\ \frac{3}{7} \end{bmatrix} (0,i\infty)} + \frac{\theta' \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,i\infty)}{\theta \begin{bmatrix} 1 \\ \frac{5}{7} \end{bmatrix} (0,i\infty)}$$

$$= -\pi \left(\frac{\cos^7 \frac{\pi}{14}}{\cos^7 \frac{5\pi}{14}} \tan \frac{\pi}{14} - \frac{\cos^7 \frac{3\pi}{14}}{\cos^7 \frac{5\pi}{14}} \tan \frac{3\pi}{14} + \tan \frac{5\pi}{14} \right)$$

$$= -\pi \times 18.89184837 \ldots \neq 0,$$

which imply that

$$\sum_{k=1}^{3} (-1)^{k-1} \theta^6 \begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix} (0,\tau) \theta' \begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix} (0,\tau) \not\equiv 0.$$

Corollary 1.5. *For every $\tau \in \mathbf{H}^2$, we have*

$$\left(\frac{14 \sum_{j=1}^{3} (-1)^{j-1} \theta^6 \begin{bmatrix} 1 \\ \frac{2j-1}{7} \end{bmatrix} (0,\tau) \theta' \begin{bmatrix} 1 \\ \frac{2j-1}{7} \end{bmatrix} (0,\tau)}{\prod_{k=1}^{3} \theta^2 \begin{bmatrix} 1 \\ \frac{2k-1}{7} \end{bmatrix} (0,\tau)} \right)^3$$

$$\times \frac{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0,\tau) \right\}^4}{\prod_{l=1}^{3} \left(\sum_{m=1}^{7} \zeta_{14}^{8l+3-(2l-1)m} \theta^7 \begin{bmatrix} \frac{2l-1}{7} \\ \frac{2m-1}{7} \end{bmatrix} (0,\tau) \right)}$$

$$= - \frac{\left\{ \theta' \begin{bmatrix} 1 \\ 1 \end{bmatrix} (0,\tau) \right\}^7}{\prod_{l=1}^{7} \theta \begin{bmatrix} \frac{1}{7} \\ \frac{2l-1}{7} \end{bmatrix} (0,\tau) \theta \begin{bmatrix} \frac{3}{7} \\ \frac{2l-1}{7} \end{bmatrix} (0,\tau) \theta \begin{bmatrix} \frac{5}{7} \\ \frac{2l-1}{7} \end{bmatrix} (0,\tau)}.$$

2. Proof of the main theorem.

2.1. The Jacobi triple product identity. For the proof of the main theorem, we use the Jacobi triple product identity, which is given by

$$\theta \begin{bmatrix} \epsilon \\ \epsilon' \end{bmatrix} (\zeta,\tau) = \exp\left(\frac{\pi i \epsilon \epsilon'}{2} \right) x^{\frac{\epsilon^2}{4}} z^{\frac{\epsilon}{2}}$$

$$\times \prod_{n=1}^{\infty} (1 - x^{2n})(1 + e^{\pi i \epsilon'} x^{2n-1+\epsilon} z)(1 + e^{-\pi i \epsilon'} x^{2n-1-\epsilon}/z),$$

where $x = \exp(\pi i \tau)$, $z = \exp(2\pi i \zeta)$. For the proof of this identity, see Farkas and Kra [2, p. 141].

2.2. Proof of equation (0.3).

Proof. We set $y = \exp(\pi i \tau / 2)$. The Jacobi triple product identity yields

$$\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \prod_{n=1}^{\infty} (1 - y^{4n})^2 (1 - y^{8n-4})^2,$$

$$\theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \theta \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$= \exp\left(\frac{3\pi i}{4} \right) y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - y^{4n})^4 (1 - y^{16n-4})(1 - y^{16n-12}).$$

It then follows that

$$\frac{\theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\theta \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \theta \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \theta \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}}$$

$$= \frac{\prod_{n=1}^{\infty} (1 - y^{4n})^2 (1 - y^{8n-4})^2}{\exp\left(\frac{3\pi i}{4}\right) y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - y^{4n})^4 (1 - y^{16n-4})(1 - y^{16n-12})}$$

$$= \frac{\prod_{n=1}^{\infty} (1 - y^{8n-4})^2}{\exp\left(\frac{3\pi i}{4}\right) y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - y^{4n})^2 (1 - y^{16n-4})(1 - y^{16n-12})}$$

$$= \frac{1}{\exp\left(\frac{3\pi i}{4}\right) y^{\frac{1}{2}} \prod_{n=1}^{\infty} (1 - y^{8n})^2 (1 - y^{16n-4})(1 - y^{16n-12})}.$$

Furthermore, the Jacobi triple product identity yields

$$\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = 2\sqrt{2} y \prod_{n=1}^{\infty} (1 - y^{8n})(1 - y^{16n}).$$

Finally, we obtain

$$
-\pi \frac{\theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}\theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}\theta\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}{\theta\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}\theta\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}\theta\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}\theta\begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}}
= \frac{2\sqrt{2}\pi y^{\frac{1}{2}} \exp\left(\frac{\pi i}{4}\right)}{\prod_{n=1}^{\infty}(1-y^{16n-4})(1-y^{16n-8})(1-y^{16n-12})}.
$$

$\square$

2.3. Proof of equation (0.3).

Proof. We set $y = \exp(\frac{2\pi i\tau}{7})$ and $\omega = \exp(\frac{2\pi i}{7})$. The Jacobi triple product identity then yields

$$
\prod_{l=1}^{7}\theta\begin{bmatrix} \frac{1}{7} \\ \frac{2l-1}{7} \end{bmatrix}(0,\tau) = \exp\left(\frac{\pi i}{2}\right)y^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-y^{7n})^7
\times \prod_{n=1}^{\infty}\left(\prod_{m=0}^{6}(1-(w^m y)^{7n-3})(1-(w^m y)^{7n-4})\right),
$$

$$
\prod_{l=1}^{7}\theta\begin{bmatrix} \frac{3}{7} \\ \frac{2l-1}{7} \end{bmatrix}(0,\tau) = \exp\left(\frac{3\pi i}{2}\right)y^{\frac{9}{8}}\prod_{n=1}^{\infty}(1-y^{7n})^7
\times \prod_{n=1}^{\infty}\left(\prod_{m=0}^{6}(1-(w^m y)^{7n-2})(1-(w^m y)^{7n-5})\right),
$$

$$
\prod_{l=1}^{7}\theta\begin{bmatrix} \frac{5}{7} \\ \frac{2l-1}{7} \end{bmatrix}(0,\tau) = \exp\left(\frac{5\pi i}{2}\right)y^{\frac{25}{8}}\prod_{n=1}^{\infty}(1-y^{7n})^7
\times \prod_{n=1}^{\infty}\left(\prod_{m=0}^{6}(1-(w^m y)^{7n-1})(1-(w^m y)^{7n-6})\right).
$$

We recall from Farkas and Kra [2, p. 289] that

$$
\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau) = -2\pi x^{\frac{1}{8}}\prod_{n=1}^{\infty}(1-x^n)^3, \quad x = \exp(2\pi i\tau).
$$

Therefore, it follows that

$$
-\frac{\left\{\theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(0,\tau)\right\}^7}{\prod_{k=1}^{3}\prod_{l=1}^{7}\theta\begin{bmatrix} \frac{2k-1}{7} \\ \frac{2l-1}{7} \end{bmatrix}(0,\tau)}
= \frac{(2\pi i)^7 x^{\frac{1}{4}}}{\prod_{n=0}^{\infty}(1-x^{7n+1})(1-x^{7n+2})(1-x^{7n+3})(1-x^{7n+4})(1-x^{7n+5})(1-x^{7n+6})},
$$

where $x = \exp(2\pi i\tau)$. $\square$

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