A remark on tame dynamics in compact complex manifolds

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Abstract: We will investigate the dynamics of a holomorphic self-map f of a compact complex manifold M such that the sequence $\{f^n\}_{n\geq 1}$ has at least one subsequence which converges uniformly on M.

Key words: Normal family; Lefschetz fixed-point theory; compact complex manifolds.

1. Introduction. Let f denote a holomorphic self-map of a compact complex manifold M. We suppose that the sequence $\{f^n\}_{n\geq 1}$ has at least one subsequence which converges uniformly on M. Our purpose is to investigate the dynamics of f by using results in [4]. First, we will show that the number of possible periods of periodic points of f is finite. This implies that there exists an integer $p \geq 1$ such that the dynamics of f^p on the minimal image of M is an 'irrationally rotation' around a pointwise-fixed closed submanifold. Moreover, when the number of periodic points of f is finite, we will show that the number of periodic points of f equals the Euler characteristic of the minimal image of M.

2. Results. Let f be a holomorphic self-map of a (connected) compact complex manifold M. We denote the n-th iterate of f by f^n, i.e. $f^n := f \circ \cdots \circ f$ (n times). Since M is compact, the image $f^n(M)$ for any $n \geq 1$ is a compact irreducible analytic subset of M. Hence, there is an integer $m \geq 1$ such that $f^m(M) = f^{m+1}(M) = \cdots$. We call $f^m(M)$ *the minimal image* and denote it by M_f. The restriction $f|M_f$ is a surjective holomorphic self-map of M_f. When $f|M_f$ is of topological degree 1, the set M_f is a complex submanifold in M (for instance, see [4]). Particularly, when $\{f^n\}_{n\geq 1}$ is a normal family on M, it is the case.

Let us introduce a concept of tameness of f.

Definition 2.1. Let f be a holomorphic self-map of a compact complex manifold M. We say that f is tame if the sequence $\{f^n\}_{n\geq 1}$ has at least one subsequence which converges uniformly on M.

We have an equivalent condition for f to be tame.

Theorem 2.2 (Theorem 2.4 (a) in [4]). *Let f be a holomorphic self-map of a compact complex manifold M. Then, $\{f^n\}_{n\geq 1}$ is a normal family on M if and only if f is tame.*

To state our theorem, we will prepare some notions and notations.

Definition 2.3. Let f be a holomorphic self-map of a compact complex manifold M and let $p \in M$. We say that p is a fixed point of f if $f(p) = p$. We denote by $\mathrm{Fix}(f)$ the set of fixed points of f. Let k be an integer ≥ 1. We say that p is a periodic point of period k of f if $f^k(p) = p$ and $f^i(p) \neq p$ for $0 < i < k$. We denote by $\mathrm{Per}(f)$ the set of periodic points of f, in other words, $\mathrm{Per}(f) := \bigcup_{n\geq 1} \mathrm{Fix}(f^n)$.

Let us denote by $\chi(N)$ the Euler characteristic of a compact manifold N and by $\sharp A$ the cardinality of a set A.

Theorem 2.4. *Let f be a tame holomorphic self-map of a compact complex manifold M. Then, the number of possible periods of periodic points of f is finite and $\mathrm{Per}(f)$ forms a (not necessarily connected) closed complex submanifold in M. Moreover, if $\dim_{\mathbf{C}} \mathrm{Per}(f) = 0$, then $\sharp\mathrm{Per}(f) = \chi(M_f)$.*

Proof. When f is tame, the minimal image M_f is a complex submanifold in M and $f|M_f$ is an automorphism on M_f. So, without loss of generality, we may assume that $M = M_f$ and f is an automorphism on M. Let $\mathrm{Aut}(M)$ denote the space of holomorphic automorphisms on M with C^0-topology. By the Bochner-Montgomery theorem [2], the space $\mathrm{Aut}(M)$ has a structure of (complex) Lie group.

By results in [4], the closure $\overline{\{f^n\}}_{n\geq 1} (\subset \mathrm{Aut}(M))$ is a commutative Lie subgroup of $\mathrm{Aut}(M)$ and there are integers $p \geq 1, q \geq 0$ such that

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$$\overline{\{f^n\}}_{n\geq 1} \simeq (\mathbf{Z}/p\mathbf{Z}) \times \mathbf{T}^q,$$

where the symbol $\simeq$ stands for an isomorphism in the sense of Lie groups and $\mathbf{T}^q$ stands for a torus of real dimension q. Particularly, $\overline{\{f^n\}}_{n\geq 1}$ contains the identity map Id_M on M and $\overline{\{f^n\}}_{n\geq 1} = \overline{\{f^n\}}_{n\in\mathbf{Z}}$. Let V_0 denote the connected component of $\overline{\{f^n\}}_{n\geq 1}$ which contains Id_M. Then, $V_0 \simeq \mathbf{T}^q$ and $f^p(\in V_0)$ generates V_0.

Let a be any integer ≥ 1. Assume that $z \in \mathrm{Fix}(f^a)$. Then, $f^{pa}(z) = z$. Since f^p generates V_0, it follows that f^{pa} also generates V_0. Hence, there is a sequence $\{f^{n_j pa}\}_{j\geq 1}$ which converges to f^p uniformly on M as $j \to +\infty$. So,

$$z = \lim_{j\to+\infty} f^{n_j pa}(z) = f^p(z).$$

This implies that $\mathrm{Fix}(f^a) \subset \mathrm{Fix}(f^p)$. Thus, the number of possible periods of periodic points of f is finite and

$$\mathrm{Per}(f) = \mathrm{Fix}(f^p),$$

where $\mathrm{Fix}(f^p)$ is obviously a closed analytic subset in M.

In order to show that $\mathrm{Fix}(f^p)$ is non-singular, we have only to consider the linearization of f^p in a neighborhood of any point $z \in \mathrm{Fix}(f^p)$. The method of the linearization is already known (for instance, see the proof of Proposition 2.5.9 in [1]) and actually f^p is conjugate to a diagonal matrix. (Since the sequence of the iterates of f^p is normal and all the eigenvalues of any fixed point of f^p have modulus 1, the Jordan normal form should be a diagonal matrix.)

Now, we will assume that $\dim_{\mathbf{C}} \mathrm{Per}(f) = 0$, i.e. $\mathrm{Per}(f)$ is a finite set. Let us show $\sharp\mathrm{Per}(f) = \chi(M)$. It can be done like the proof of the Hopf index theorem for vector fields. First, we will note that all fixed points of f^p are non-degenerate, i.e. 1 is not an eigenvalue. (Around any fixed point z of f^p, we can linearize f^p. So, if 1 is an eigenvalue of z, it follows that $\dim_{\mathbf{C}} \mathrm{Fix}(f^p) \geq 1$. This is a contradiction to the assumption $\dim_{\mathbf{C}} \mathrm{Per}(f) = 0$.) Hence, we can use the Lefschetz fixed-point formula (see p. 421 [3]), that is,

$$\sum_{z\in\mathrm{Fix}(f^p)} \iota_{f^p}(z) = L(f^p),$$

where $\iota_{f^p}(z)$ is the index of f^p at z and $L(f^p)$ is the Lefschetz number of f^p. Here, $\iota_{f^p}(z) = 1$ for any $z \in \mathrm{Fix}(f^p)$ because f^p is holomorphic. Since f^p is an element of V_0, it follows that f^p is homotopic to Id_M. Hence $L(f^p) = \chi(M)$. So, the formula implies that $\sharp\mathrm{Fix}(f^p) = \chi(M)$, that is, $\sharp\mathrm{Per}(f) = \chi(M)$. $\square$

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