An L_p-function determines ℓ_p

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(Communicated by Heisuke HIRONAKA, M.J.A., Feb. 12, 2008)

Abstract: ℓ_p is characterized by the convergence of a series defined by an L_p-function on the real line $\mathbf{R}$.

Key words: ℓ_p; absolutely continuous; integrable function; p-integral.

1. Introduction. In this paper we shall give a new characterization of the well-known sequence space ℓ_p by using an L_p-function.

Let $f(\neq 0)$ be an L_p-function defined on the real line $\mathbf{R}$ and assume $1 \leq p < +\infty$. For a sequence of real numbers $\boldsymbol{a} = \{a_k\} \in \mathbf{R}^\infty$, define

$$\Psi_p(\boldsymbol{a}; f) := \left(\sum_k \int_{-\infty}^{+\infty} \Big|f(x-a_k) - f(x)\Big|^p dx\right)^{\frac{1}{p}}$$

and

$$\Lambda_p(f) := \{\boldsymbol{a} \in \mathbf{R}^\infty \mid \Psi_p(\boldsymbol{a} : f) < +\infty\}.$$

We say $I_p(f) < +\infty$ if $f(x)$ is absolutely continuous on $\mathbf{R}$ and the *p-integral* defined by

$$I_p(f) := \int_{-\infty}^{+\infty} |f'(x)|^p dx$$

is finite.

Let g be a probability density function on $\mathbf{R}$. Then Shepp [1] proved $\Lambda_2(\sqrt{g}) \subset \ell_2$, and also $\Lambda_2(\sqrt{g}) = \ell_2$ if and only if $I_2(\sqrt{g}) < +\infty$. This paper generalizes those results.

We shall first show $\Lambda_p(f) \subset \ell_p$ (Theorem 1). Next we shall show that $I_p(f) < +\infty$ implies $\ell_p \subset \Lambda_p(f)$ for every $1 \leq p < +\infty$ (Theorem 2), and that for $1 < p < +\infty$, $\ell_p \subset \Lambda_p(f)$ implies $I_p(f) < +\infty$ (Theorem 3). In particular, for $1 < p < +\infty$ we have $\Lambda_p(f) = \ell_p$ if and only if $I_p(f) < +\infty$ (Corollary 4).

In Theorem 3, the case $p = 1$ is excluded. In fact, define $f(x) = e^{-x}$ for $x \geq 0$, and $= 0$ for $x < 0$. Then $f(x)$ is not absolutely continuous on $\mathbf{R}$, so that $I_1(f) < +\infty$ does not hold, but we have $\ell_1 = \Lambda_1(f)$.

Finally, as an illuminating example, we shall estimate $\Lambda_p(f)$ explicitly for $f(x) := \sqrt{x}e^{-x}$ for $x \geq 0$, and $:= 0$ for $x < 0$. Then we have $\Lambda_p(f) = \ell_p$, if $1 \leq p < 2$, $= \ell_{2-}$, if $p = 2$, and $= \ell_{1+\frac{p}{2}}$, if $p > 2$, where

$$\ell_{2-} := \left\{\boldsymbol{a} \,\middle|\, \sum_k a_k^2(1 + |\log|a_k||) < +\infty\right\}.$$

2. Results. Our first result is the following theorem.

Theorem 1. *Assume* $1 \leq p < +\infty$ *and let* $f(\neq 0)$ *be an* L_p*-function on* $\mathbf{R}$*. Then* $\Lambda_p(f) \subset \ell_p$*.*

Proof. Assume that $\boldsymbol{a} = \{a_k\} \in \Lambda_p(f)$, which is equivalent to $\Psi_p(\boldsymbol{a}; f) < +\infty$. Without loss of generality, we may assume $a_k \neq 0$ for every k.

First we shall prove that $\{a_k\}$ is bounded. If there is a subsequence $\{a_{k'}\}$ such that $|a_{k'}| \to +\infty$, then $\Psi_p(\boldsymbol{a}; f) < +\infty$ implies

$$0 = \lim_{k'} \int_{-\infty}^{+\infty} \Big|f(x-a_{k'}) - f(x)\Big|^p dx = 2\|f\|_{L_p}^p > 0,$$

which is a contradiction.

Next we shall prove that $\{a_k\}$ converges to 0. Assume that there exists a subsequence $a_{k'}$ such that $a_{k'} \to a_0 \neq 0$. Then we have

$$\begin{aligned} 0 &= \lim_{k'} \int_{-\infty}^{+\infty} \Big|f(x-a_{k'}) - f(x)\Big|^p dx \\ &= \int_{-\infty}^{+\infty} \Big|f(x-a_0) - f(x)\Big|^p dx, \end{aligned}$$

which implies $f(x - a_0) = f(x)$, *a.e.*(dx). This contradicts to the integrability of $f(x)$.

Finally, we shall prove

$$\rho := \inf_k \int_{-\infty}^{+\infty} \left|\frac{f(x-a_k) - f(x)}{a_k}\right|^p dx > 0.$$

Assume that there exists a subsequence $a_{k'}$ such that

2000 Mathematics Subject Classification. Primary 26D10, 46A45; Secondary 60G30.

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$$\int_{-\infty}^{+\infty}\left|\frac{f(x-a_{k'})-f(x)}{a_{k'}}\right|^p dx \to 0.$$

Then it follows that

$$\frac{f(x-a_{k'})-f(x)}{a_{k'}} \to 0 \ \mathit{in}\ L_p(\mathbf{R}).$$

On the other hand, $\frac{f(x-a_{k'})-f(x)}{a_{k'}} \to -f'(x)$ in the distribution sence. Consequently, $f(x)$ is absolutely continuous with $f'(x)=0, a.e.(dx)$, which implies $f=0$. This is a contradiction.

Summing up the above, we have

$$\begin{aligned}+\infty &> \Psi_p(\boldsymbol{a};f)^p\\ &= \sum_k \int_{-\infty}^{+\infty}\left|\frac{f(x-a_k)-f(x)}{a_k}\right|^p dx\ |a_k|^p\\ &\geq \rho\sum_k |a_k|^p,\end{aligned}$$

which proves the theorem. □

In the following theorems, we shall discuss the converse of the above theorem.

Theorem 2. *Assume* $1 \leq p < +\infty$ *and let* $f(\neq 0)$ *be an* L_p*-function on* $\mathbf{R}$. *Then* $I_p(f) < +\infty$ *implies* $\ell_p \subset \Lambda_p(f)$.

Proof. Assume $I_p(f) < +\infty$ and hence $f(x)$ is absolutely continuous. Then by Fubini's theorem, we have

$$\begin{aligned}\Psi_p(\boldsymbol{a};f)^p &= \sum_k |a_k|^p \int_{-\infty}^{+\infty} dx\left|\int_0^1 f'(x-ta_k)dt\right|^p\\ &\leq \sum_k |a_k|^p \int_{-\infty}^{+\infty} dx \int_0^1 |f'(x-ta_k)|^p dt\\ &\leq I_p(f)\sum_k |a_k|^p,\end{aligned}$$

which proves the theorem. □

Theorem 3. *Assume* $1 < p < +\infty$ *and let* $f(\neq 0)$ *be an* L_p*-function on* $\mathbf{R}$. *Then* $\ell_p \subset \Lambda_p(f)$ *implies* $I_p(f) < +\infty$.

Proof. Assume $\ell_p \subset \Lambda_p(f)$, and define

$$\psi(a) := \int_{-\infty}^{+\infty}\left|f(x-a)-f(x)\right|^p dx, \quad a \in \mathbf{R},$$

and

$$F_N(x) := \frac{f(x-u_N)-f(x)}{u_N},\ u_N := 2^{-\frac{N}{p}},\ N \geq 1.$$

Then we have

$$\sup_N 2^N\psi(u_N) = \sup_N \int_{-\infty}^{+\infty}\left|F_N(x)\right|^p dx < +\infty.$$

In fact, on the contrary, assume that for every n there exists $N(n) > n$ satisfying

$$2^{N(n)}\psi(u_{N(n)}) > 2^n.$$

Then for the sequence

$$\boldsymbol{a}_0 := \{\overbrace{u_{N(1)},\cdots,u_{N(1)}}^{2^{N(1)-1}},\cdots,\overbrace{u_{N(n)},\cdots,u_{N(n)}}^{2^{N(n)-n}},\cdots\},$$

we have $\boldsymbol{a}_0 \in \ell_p$ and $\Psi_p(\boldsymbol{a}_0;f) = +\infty$, which is a contradiction.

Since $L_p(\mathbf{R},dx), 1 < p < +\infty$, is a separable reflexive Banach space, each bounded closed ball is compact and metrizable with respect to the weak topology. Consequently, there exists a subsequence $\{F_{N_j}(x)\}$ and $h(x) \in L_p(\mathbf{R},\ dx)$ such that $\{F_{N_j}(x)\}$ converges weakly to $h(x)$. Since $F_{N_j}(x) \to -f'(x)$ in the distribution sence, $f(x)$ is absolutely continuous, $f'(x) = -h(x),\ a.e.(dx)$, and we have

$$I_p(f) = \int_{-\infty}^{+\infty}|f'(x)|^p dx = \int_{-\infty}^{+\infty}|h(x)|^p dx < +\infty.$$

□

Theorems 1, 2 and 3 yield the corollary below.

Corollary 4. *Assume* $1 < p < +\infty$ *and let* $f(\neq 0)$ *be an* L_p*-function on* $\mathbf{R}$. *Then we have* $\ell_p = \Lambda_p(f)$ *if and only if* $I_p(f) < +\infty$.

Example 5. Define $f(x) := \sqrt{x}e^{-x},\ x > 0$, and $:= 0,\ x \leq 0$. Then we have

$$\Lambda_p(f) = \begin{cases}\ell_p, & 1 \leq p < 2,\\ \ell_{2-}, & p = 2,\\ \ell_{1+\frac{p}{2}}, & p > 2.\end{cases}$$

Proof. For $1 \leq p < 2$ we have $I_p(f) < +\infty$ so that $\Lambda_p(f) = \ell_p$ by Theorem 2.

Assume $p \geq 2$ and $\boldsymbol{a} = \{a_k\} \in \Lambda_p(f)$. Then, $\boldsymbol{a} \in \ell_p$ by Theorem 1 and, since

$$\begin{aligned}&\int_{-\infty}^{+\infty}|f(x-a)-f(x)|^p dx\\ &\quad= \int_{-\infty}^{+\infty}|f(x+a)-f(x)|^p dx\end{aligned}$$

for every $a \in \mathbf{R}$, we have $\Psi_p(\boldsymbol{a};f) = \Psi_p(|\boldsymbol{a}|;f)$, where $|\boldsymbol{a}| := \{|a_k|\}$. Therefore, without loss of generality, we may assume $a_k > 0$ for every $k \geq 1$ and $\alpha := \sup_k a_k < +\infty$.

By definition we have

$$\begin{aligned}+\infty &> \Psi_p(\boldsymbol{a};f)^p\\ &= \sum_k \int_{-\infty}^{+\infty}|f(x-a_k)-f(x)|^p dx\end{aligned}$$

$$= \int_0^{a_k} |f(x)|^p dx + \int_0^{+\infty} |f(x+a_k) - f(x)|^p dx =: J_p(\boldsymbol{a}) + K_p(\boldsymbol{a})$$

so that

$$+\infty > J_p(\boldsymbol{a}) = \int_0^{a_k} x^{\frac{p}{2}} e^{-px} dx \geq \frac{2e^{-p\alpha}}{p+2} \sum_k a_k^{1+\frac{p}{2}}.$$

Therefore we have $\boldsymbol{a} \in \ell_{1+\frac{p}{2}}$ and $\Lambda_p(f) \subset \ell_{1+\frac{p}{2}}$.

In particular, assume $p = 2$. Then we have $\boldsymbol{a} \in \ell_2$ and

$$\begin{aligned}
&+\infty > K_2^{\frac{1}{p}} + \left(\sum_k a_k^2\right)^{\frac{1}{2}} \\
&\geq \left(\sum_k \int_0^{+\infty} \left|\sqrt{x+a_k} e^{-(x+a_k)} - \sqrt{x} e^{-x}\right|^2 dx\right)^{\frac{1}{2}} \\
&\quad + \left(\int_0^{+\infty} x e^{-2x} dx \sum_k \left|1 - e^{-2a_k}\right|^2\right)^{\frac{1}{2}} \\
&\geq \left(\sum_k \int_0^{+\infty} \left|\sqrt{x+a_k} - \sqrt{x}\right|^2 e^{-2(x+a_k)} dx\right)^{\frac{1}{2}} \\
&\geq \left(\sum_k \int_0^{+\infty} \frac{e^{-2(x+a_k)}}{|\sqrt{x+a_k} + \sqrt{x}|^2} dx a_k^2\right)^{\frac{1}{2}} \\
&\geq \frac{1}{2} \left(\sum_k \int_0^{+\infty} \frac{e^{-2(x+a_k)}}{|x+a_k|} dx a_k^2\right)^{\frac{1}{2}} \\
&\geq \frac{1}{2} \left(\sum_k \int_{a_k}^{+\infty} \frac{e^{-2x}}{x} dx a_k^2\right)^{\frac{1}{2}} \\
&\geq C_1 \left(\sum_k a_k^2 \log \frac{1}{a_k}\right)^{\frac{1}{2}} - C_2 \left(\sum_k a_k^2\right)^{\frac{1}{2}},
\end{aligned}$$

where C_1 and C_2 are positive constants. Consequently we have

$$\sum_k a_k^2 (1 + |\log |a_k||) < +\infty$$

and $\boldsymbol{a} \in \ell_{2-}$.

Conversely by similar discussions it is not difficult to show that $\boldsymbol{a} \in \ell_{1+\frac{p}{2}}$ implies $\Psi_p(\boldsymbol{a}; f) < +\infty$ for $p > 2$, and that $\boldsymbol{a} \in \ell_{2-}$ implies $\Psi_2(\boldsymbol{a}; f) < +\infty$ for $p = 2$. □

Reference

[1] L. A. Shepp, Distinguishing a sequence of random variables from a translate of itself, Ann. Math. Statist. **36** (1965), 1107–1112.