A geometric construction of continuous, strictly increasing singular functions

Hisashi OKAMOTO*) and Marcus WUNSCH**)

(Communicated by Heisuke HIRONAKA, M.J.A., Sept. 12, 2007)

Abstract: A parameterized family of continuous functions which was considered by the first author is re-visited in the case when they are monotonically increasing. We prove that the functions are not only continuous and strictly increasing but also singular, i.e., their derivatives are zero almost everywhere.

Key words: Continuous, monotonically increasing function; singular functions.

1. Introduction. The first author defined in [6] a family of continuous functions F_a parameterized by a $(0 < a < 1)$ and proved that they are nowhere differentiable if $2/3 \le a < 1$ and nondifferentiable almost everywhere if $0.5592\cdots < a < 2/3$. Since the author was interested in nowhere differentiable functions in [6], he did not pay much attention to the case where $0 < a < 1/2$. In fact, the function F_a with $0 < a < 1/2$ are monotonically increasing and, obviously, differentiable almost everywhere.

The purpose of the present note is to prove that F_a are continuous, strictly increasing, and singular if $0 < a < 1/3$ or $1/3 < a < 1/2$. (A function is called singular if it is not a constant function and at the same time its derivative is zero almost everywhere.)

Examples of monotone singular functions are known, but many are, as in the case of the Cantor function, not strictly monotone. Salem's functions and Minkowski's question mark function are known to be strictly monotone singular functions. Our functions F_a provide us with new examples of such functions.

2. Approximation process. We first recall the definition of F_a in [6]. F_a is defined as the limit of the uniformly converging sequence of functions f_n, which are piecewise linear continuous functions. They are geometrically constructed in the following way. We first define $f_0(x) \equiv x$ for $0 \le x \le 1$. Then f_n's are constructed so that they are linear in $k3^{-n} \le x \le (k+1)3^{-n}$ $(k = 0, 1, \cdots, 3^n - 1)$, that f_n are continuous throughout $[0, 1]$, and that f_{n+1} is defined from f_n by applying the operation shown in Fig. 1. (Each small interval $[k3^{-n}, (k+1)3^{-n}]$ is divided into three equal parts. Accordingly, the graph of f_n consists of 3^n line segments.) Although f_n depends on a, its dependence on a is not explicitly written for the sake of notational simplicity. We then set $F_a = \lim_{n\to\infty} f_n$.

By the definition, $F_{1/2}$ is the Cantor function and $F_{1/3}(x) = x$. It is also easy to prove that F_a is strictly increasing if $0 < a < 1/2$.

Remark 1. *Prof. R. Daniel Mauldin and Dr. K. Kawamura kindly showed us that our functions with $a > 1/2$ are special cases of those considered by J. Lee* [4]. *The construction in* [4] *is analytic and more general than ours; ours is geometric and has the merit of simplicity.*

The first author proved in [6] that

Theorem 1. *If $2/3 \le a < 1$, then the function F_a is continuous and nowhere differentiable. If $0 < a < 0.5592\cdots$, then F_a is differentiable almost everywhere. If $0.5592\cdots < a < 2/3$, then F_a fails to be differentiable in a set of measure one.*
He could not answer the question of the differentiability of $F_{0.5592\cdots}$ but recently Kobayashi [2] *proved that it is non-differentiable almost everywhere.*

Definition 1 (*Normal number*)**.** *Suppose that $b > 0$ is an integer. For any $x \in [0, 1]$, let*

$$x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \frac{\xi_3}{b^3} + \cdots$$

be its b-adic expansion, where $\xi_n \in \{0, 1, 2, \cdots, b-1\}$ $(n = 1, 2, \cdots)$. Let N be a positive integer

---

2000 Mathematics Subject Classification. 26A.
*) Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 606-8502, Japan.
**) Faculty of Mathematics, University of Vienna, Nordbergstrasse 15, 1090 Wien, Austria.

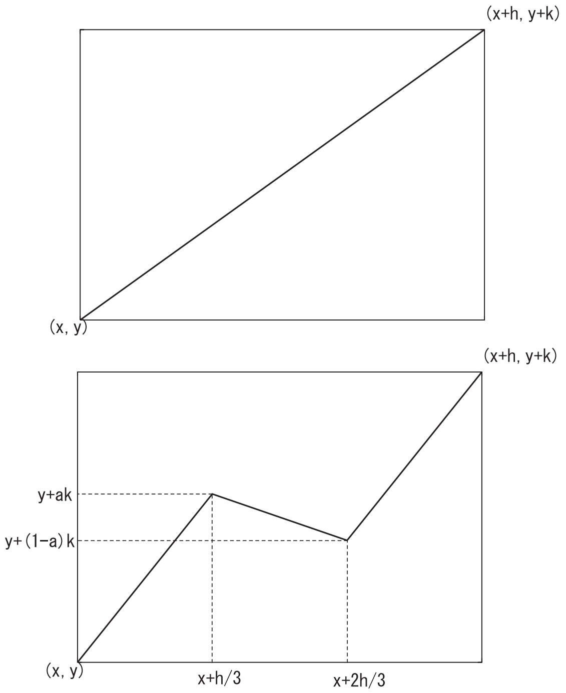

Fig. 1. The operation from f_n to f_{n+1}. Before the operation (top). After the operation (bottom). This operation is performed in each subinterval $[k/3^n, (k+1)/3^n]$.

and let a $(0 \le a < b)$ be a given integer. We then write by $A_b(a, N, x)$ the number of those $\xi_n(1 \le n \le N)$ such that $\xi_n = a$. If for any $a = 0, 1, \cdots, b-1$ we have

$$\lim_{N\to\infty} \frac{1}{N} A_b(a, N; x) = \frac{1}{b},$$

then x is called simply normal to the base b.

The following theorem is well-known (see, e.g., [3]):

Theorem 2. *The set of simply normal numbers in $[0,1]$ has full measure, i.e., its complement is a null set.*

The use of normal numbers was suggested to us by Dr. K. Kobayashi. As is shown below, and also used beautifully in [8], the use of normal numbers simplifies mathematical arguments considerably.

3. Main theorem. From now on, we assume that $0 < a < 1/2$. We prove

Theorem 3. *Suppose that $0 < a < 1/2$ and that $a \ne 1/3$. Then F_a is a continuous, strictly increasing, and singular function.*

Proof. Continuity is proved in [6]: it follows from the uniform convergence of f_n to F_a. Strict monotonicity is easily proved. Suppose now that $x \in (0,1)$ has the following ternary expansion:

$$x = \frac{\xi_1}{3} + \frac{\xi_2}{3^2} + \frac{\xi_3}{3^3} + \cdots \qquad (\xi_n = 0, 1, 2). \tag{1}$$

For a fixed n, let the number of $0, 1, 2$ among $\xi_1, \xi_2, \cdots, \xi_n$ be denoted by p_n, q_n, and r_n, respectively. We obviously have $p_n + q_n + r_n = n$. Then it follows that

$$\begin{aligned} f_n'(x) &= (3a)^{p_n + r_n}(3-6a)^{q_n} \\ &= \left[3a^{1-q_n/n}(1-2a)^{q_n/n}\right]^n. \end{aligned}$$

If x is simply normal to the base three and if

$$a^{2/3}(1-2a)^{1/3} < 1/3 \tag{2}$$

then $f_n'(x) \to 0$ as $n \to \infty$. (2) is true if

$$27(1-2a)a^2 < 1 \qquad \text{or} \qquad 54a^3 - 27a^2 + 1 > 0.$$

Hence, if we set $X = 3a$, (2) holds true if

$$2X^3 - 3X^2 + 1 > 0.$$

The validity of this is verified in elementary fashion for $0 < X < 1$ and $1 < X < 3/2$.

We have therefore proved that if $0 < a < 1/3$ or $1/3 < a < 1/2$, then $f_n'(x) \to 0$ for almost all x. We already know (see [6]) that f_n converges to F_a uniformly in $[0,1]$. But these facts do not imply by themselves that F_a' is zero almost everywhere.

To prove $F_a' = 0$ almost everywhere, we proceed as follows. Note first that F_a is monotonically increasing, whence it is differentiable almost everywhere. Accordingly, there exists a set E such that its Lebesgue measure is one, that $F_a'(x)$ exists for $x \in E$, and that all $x \in E$ are simply normal. In particular, $\lim_{n\to\infty} f_n'(x) = 0$ for $x \in E$. Also, we may assume that none of $k3^{-n}(k = 0, 1, \cdots, 3^n, n \in \mathbf{N})$ belongs to E.

Let an arbitrary $x \in E$ and an arbitrary positive integer n be fixed. Choose a k such that $k3^{-n} < x < (k+1)3^{-n}$. If $(k+1)3^{-n} - x > x - k3^{-n}$, then set $x_n = (k+1)3^{-n}$. Otherwise, set $x_n = k3^{-n}$. Accordingly, $\frac{1}{2}3^{-n} \le |x - x_n| \le 3^{-n}$. By the construction, for all $m \ge n$, the graph of f_m in $k3^{-n} \le x \le (k+1)3^{-n}$ is contained in the rectangle $k3^{-n} \le x \le (k+1)3^{-n}$, $f_n(k3^{-n}) \le y \le f_n((k+1)3^{-n})$. We therefore have

$$\left|\frac{F_a(x)-F_a(x_n)}{x-x_n}\right| \le 2\cdot 3^n \left| f_n\left(\frac{k}{3^n}\right) - f_n\left(\frac{k+1}{3^n}\right)\right|$$
$$= 2\left[3a^{1-q_n/n}(1-2a)^{q_n/n}\right]^n.$$

Since $|x-x_n| \to 0$, and since E is chosen so that the derivative exists, $F_a'(x)$ must vanish. Since E has a full measure, we are done. □

Although we have used the specific manner of construction of F_a in the proof above, a different proof is available if we use a theorem by L. Tonelli [12]. Since his theorem seems to be largely forgotten now, the proof below may be of some interest. Before we state it, we recall the concept of convergence in measure.

Definition 2. *We say that a sequence of functions $(v_n)_{n=1}^{\infty}$ converges in measure, which we shall denote by μ, to a function v if and only if*

$$\lim_{n\to\infty} \mu(\{x : |v_n(x) - v(x)| > \varepsilon\}) = 0$$

for all $\varepsilon > 0$.

Theorem 4 ([12]). *If the sequence of functions $(u_n(x))_{n=1}^{\infty}$, given on the interval (a,b), converges almost everywhere to a function $u(x)$ and if the lengths of the curves associated to u_n tend to the length of the curve of u, then*

$$u_n'(x) \to u'(x),$$

where the convergence is in (Lebesgue) measure.

With this theorem at hand, we can easily prove the required statement. As elementary calculations reveal, the lengths of the graphs of functions (f_n) indeed converge to the length of the curve F_a. In fact, the length of the graph of f_n is obviously increasing in n. It is computed as

$$\sum_{k=0}^{n} \sqrt{3^{-2n} + (1-2a)^{2n-2k} a^{2k}} \begin{pmatrix} n \\ k \end{pmatrix} 2^k,$$

hence is bounded by

$$\sum_{k=0}^{n} \left[3^{-n} + (1-2a)^{n-k} a^k\right] \begin{pmatrix} n \\ k \end{pmatrix} 2^k$$
$$= 3^{-n}(1+2)^n + (1-2a)^n \left(1 + \frac{2a}{1-2a}\right)^n = 2.$$

It therefore converges to a certain value, say, L. Let the length of the graph of F_a be denoted by L'. By definition, L' is the supremum of the length of polygonal curves inscribed to the graph of F_a. Since the graph of f_n is such a polygonal curve, we have $L' \ge L$. Suppose now that $L' > L$. Then, by the definition, there exists a polygonal curve whose vertices are on the graph of F_a and its length is greater than L. On the other hand, f_n converges uniformly to F_a. Note also that all the vertices of the graph of f_n are also on the graph of f_m for all $m \ge n$. Then it is easy to derive a contradiction from these facts. Therefore the convergence of the lengths is established.

Now Tonelli's theorem tells us that

$$\lim_{n\to\infty} \mu(\{x : |f_n'(x) - F_a'(x)| > \varepsilon\}) = 0$$

for all $\varepsilon > 0$. Let us set $G = \{x : F_a'(x) \ne 0\}$. Since convergence in measure is equivalent to

$$\int_0^1 \frac{|f_n'(x) - F_a'(x)|}{1 + |f_n'(x) - F_a'(x)|}\,\mathrm{d}x \to 0,$$

and since $f_n'(x) \to 0$ almost everywhere, we have

$$\int_G \frac{|F_a'(x)|}{1+|F_a'(x)|}\mathrm{d}x = 0.$$

But this can happen only if G is a null set. □

For the reader's convenience, we draw graphs of $F_{0.2}$ and $F_{0.4}$ in Fig. 2.

4. A generalization. The following generalization is immediate: We replace the functional operation with the one indicated in Fig. 3. Here $b \in (0,1)$ is another parameter. Let $F_{a,b}$ denote the function obtained in the limit. Again this function is a special case of what was considered by [4] if $a > b$. Since we are interested in monotone functions in this paper, we assume that $0 < a < b < 1$, which seems to be neglected in [4].

Theorem 5. *Suppose that $0 < a < b < 1$ and that $(a,b) \ne (1/3, 2/3)$. Then $F_{a,b}$ is a continuous, strictly monotone, and singular function.*

Proof. Let x have the ternary expansion (1) and p_n, q_n, and r_n be defined as in section 3. We then have

$$f_n'(x) = 3^n a^{p_n} (b-a)^{q_n} (1-b)^{r_n}.$$

Applying the same argument as in the previous section, we see that

$$a(b-a)(1-b) < \frac{1}{27} \Longrightarrow F_{a,b} \text{ is singular.}$$

On the other hand, it holds that for all $0 < a < b < 1$ except for $(a,b) = (1/3, 2/3)$, $a(b-a)(1-b) < \frac{1}{27}$ holds true. This can be verified by the arithmetic-geometric inequality

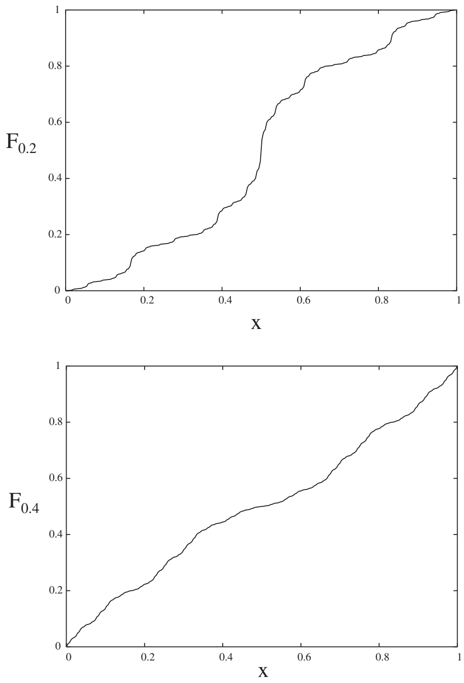

Fig. 2. The graphs of $F_{0.2}$ (top) and $F_{0.4}$ (bottom). Actually f_n with $n = 5$ are plotted.

$$\frac{1}{3} = \frac{a + (b - a) + (1 - b)}{3} \geq \sqrt[3]{a(b - a)(1 - b)}$$

with the equality if and only if $a = b - a = 1 - b$. The remaining part of the proof is the same as that of Theorem 3. □

Remark 2. *The most famous, continuous, strictly increasing, and singular function would be Salem's function* [8]*, which is sometimes called the Riesz-Nágy function. The function is generalized in* [7]. *Salem's function is perhaps the simplest, but not the oldest. The oldest known example would be Minkowski's question mark function* [5]. *His function was generalized by* [11]. *Other information can be found in* [1,9,10].

Acknowledgements. H.O. is partially supported by the Grant-in-Aid for Scientific Research from JSPS No. 14204007. He expresses his gratitude to Prof. R. Daniel Mauldin and Dr. K. Kawamura, for introducing the reference [4] to us. He also thanks Dr. Kenta Kobayashi for his helpful comments.

M.W. expresses his gratitude for the support of the Ministry of Education, Culture, Sports, Science and Technology of Japan.

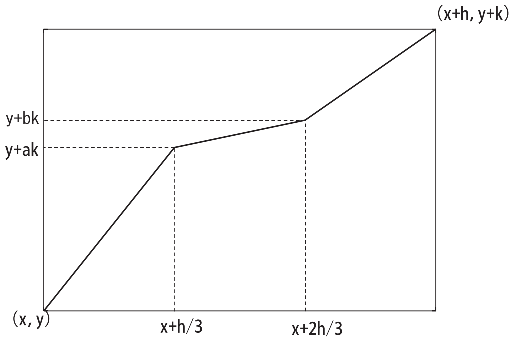

Fig. 3. The operation for $F_{a,b}$.

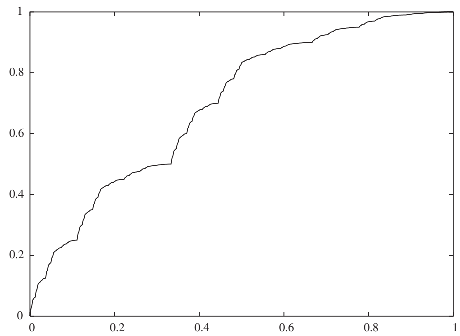

Fig. 4. The graphs of $F_{0.5,0.9}$.

References

[1] A. B. Kharazishvili, *Strange functions in real analysis*, Dekker, New York, 2000.

[2] K. Kobayashi, On the critical case of Okamoto's continuous, nowhere differentiable functions. (in preparation).

[3] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley-Intersci., New York, 1974.

[4] J. Lee, Simple constructions of nowhere differentiable continuous functions, Real Analysis Exchange, **23(1)** (1997/8), 109–112.

[5] H. Minkowski, Zur Geometrie der Zahlen, in *Verhandlungen III Inter. Math. Kongress* (*Heidelberg*, 1904), 164–173. Gesammelte Abhandlungen von H. Minkowski, Bd. II (1911), 43–52.

[6] H. Okamoto, A remark on continuous, nowhere differentiable functions, Proc. Japan Acad. Ser. A Math. Sci. **81** (2005), no. 3, 47–50.

[7] J. Paradís, P. Viader, and L. Bibiloni, Riesz-Nágy singular function revisited, J. Math. Anal. Appl. (to appear).

[8] R. Salem, On some singular monotonic functions which are strictly increasing, Trans. Amer. Math. Soc. **53** (1943), 427–439.

[9] W. Sierpińsky, Un exemple élémentaire d'une

fonction croissante qui a presque partout une dérivée nulle, Giornale di Mate. Battaglini, **(3) 6** (1916), 314–334. Oeuvres Choisies, t. II (1975), 122–140.

[10] L. Takács, An increasing continuous singular function, Amer. Math. Monthly **85** (1978), no. 1, 35–37.

[11] R. F. Tichy and J. Uitz, An extension of Minkowski's singular function, Appl. Math. Lett. **8** (1995), no. 5, 39–46.

[12] L. Tonelli, Successioni di curve e derivazione per serie, Rend. R. Accad. dei Lincei, **XXV** (1916), 22–30, 85–91. Also in Opere Scelte, **I** (1960), 305–322.