L^2-torsion invariants and homology growth of a torus bundle over S^1

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Abstract: We introduced an infinite sequence of L^2-torsion invariants for surface bundles over the circle in [4]. In this note, we investigate in detail the first two terms for a torus bundle case. In particular, we show that the first invariant can be described by the asymptotic behavior of the order of the first homology group of a cyclic covering.

Key words: L^2-torsion; hyperbolic volume; surface bundle; nilpotent quotient.

1. Introduction. L^2-analogues of the Reidemeister and the Ray-Singer torsion were initiated by Mathai [11], Carey-Mathai [1] and Lott [6]. They are defined by using the Fuglede-Kadison determinant of von Neumann algebras. It is shown in [3, 10] that the L^2-torsion for the regular representation of fundamental groups is equal to a constant multiple of Gromov's simplicial volume. Thus, for a hyperbolic manifold, it is essentially equal to its hyperbolic volume.

Recently we started to study an infinite sequence $\{\tau_k\}_{k\in\mathbf{N}}$ of L^2-torsion invariants, which should approximate the original L^2-torsion τ, of a surface bundle over the circle S^1. The purpose of this note is to show that the first invariant τ_1 can be described by the asymptotic behavior of the order of the first homology group of cyclic coverings. We give a proof only for genus one case, but we easily see that it holds for higher genera. Further we show that the second term τ_2 of our L^2-torsion invariants is trivial for all torus bundles over S^1.

First we review a definition of the Fuglede-Kadison determinant. Let π be a finitely presentable group and $\mathbf{C}\pi$ denote its group ring over $\mathbf{C}$. For an element $\sum_{g\in\pi}\lambda_g g \in \mathbf{C}\pi$, we define the $\mathbf{C}\pi$-trace $\mathrm{tr}_{\mathbf{C}\pi} : \mathbf{C}\pi \to \mathbf{C}$ by $\mathrm{tr}_{\mathbf{C}\pi}\left(\sum_{g\in\pi}\lambda_g g\right) = \lambda_e \in \mathbf{C}$, where e is the unit element in π. Next, for a matrix $B = (b_{ij}) \in M(n, \mathbf{C}\pi)$, we extend this definition of $\mathbf{C}\pi$-trace by means of

$$\mathrm{tr}_{\mathbf{C}\pi}(B) = \sum_{i=1}^{n} \mathrm{tr}_{\mathbf{C}\pi}(b_{ii}).$$

Let $l^2(\pi)$ denote the complex Hilbert space of formal sums $\sum_{g\in\pi}\lambda_g g$ which are square summable. For any matrix $B \in M(n, \mathbf{C}\pi)$, we consider the bounded π-equivariant operator

$$R_B : \bigoplus_{i=1}^{n} l^2(\pi) \to \bigoplus_{i=1}^{n} l^2(\pi)$$

defined by natural right action of B. We fix a positive real number K so that $K \geq ||R_B||_\infty$ holds, where $||R_B||_\infty$ is the operator norm of R_B.

Definition 1.1. The Fuglede-Kadison determinant of a matrix B is defined by

$$\det_{\mathbf{C}\pi}(B) = K^n \exp\left(-\frac{1}{2}\sum_{p=1}^{\infty}\frac{1}{p}\,\mathrm{tr}_{\mathbf{C}\pi}\left(I - K^{-2}BB^*\right)^p\right) \in \mathbf{R}_{>0},$$

if the infinite sum of non-negative real numbers $\sum(1/p)\,\mathrm{tr}_{\mathbf{C}\pi}\left(I-K^{-2}BB^*\right)^p$ converges to a real number. Here I is the identity matrix and B^* denotes the adjoint of B. That is, $B^* = (\overline{b_{ji}})$ and $\overline{\sum\lambda_g g} = \sum\overline{\lambda_g}g^{-1}$.

Remark 1.2. (i) The Fuglede-Kadison determinant $\det_{\mathbf{C}\pi}(B)$ is independent of the choice of the constant K.

(ii) Recently Schick [12] defined some class of groups, which includes abelian groups and amenable groups, and proved the following: If a group π belongs to this class and $\lim_{p\to\infty}(1/p)\,\mathrm{tr}_{\mathbf{C}\pi}\left(I - K^{-2}BB^*\right)^p = 0$, then the above infinite sum converges.

2. Definition of τ_k. From now on, we restrict ourselves to a surface bundle over S^1 and review the construction of its L^2-torsion invariants (see

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[4] for details). As for the definition of the original L^2-torsion τ, see [9].

Let $\Sigma_{g,1}$ be a compact oriented smooth surface of genus $g \geq 1$ with one boundary component. For an orientation preserving diffeomorphism φ of $\Sigma_{g,1}$, we form the mapping torus W_φ by taking $\Sigma_{g,1}\times[0,1]$ and gluing $\Sigma_{g,1}\times\{0\}$ and $\Sigma_{g,1}\times\{1\}$ via φ. For simplicity, we put $\pi = \pi_1(W_\varphi, *)$ and $\Gamma = \pi_1(\Sigma_{g,1}, *)$, where the base point $*$ of π and Γ is the same one on the fiber $\Sigma_{g,1}\times\{0\} \subset W_\varphi$. Then π is isomorphic to the semi-direct product of Γ and $\pi_1 S^1 \cong \mathbf{Z} = \langle t \rangle$.

Now let us consider the lower central series of Γ:

$$\Gamma_1 = \Gamma \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots,$$

where $\Gamma_k = [\Gamma_{k-1}, \Gamma_1]$ for $k \geq 2$. Let N_k be the kth nilpotent quotient $N_k = \Gamma/\Gamma_k$ and $p_k : \Gamma \to N_k$ be the natural projection. The group Γ_k is a normal subgroup of π, so that we can take the quotient group $\pi(k) = \pi/\Gamma_k$. It should be noted that $\pi(k)$ is isomorphic to the semi-direct product $N_k \rtimes \mathbf{Z}$. We denote the induced homomorphism $\pi \to \pi(k)$ by the same letter p_k. Thereby we can consider the chain complex

$$C_*\left(W_\varphi, l^2(\pi(k))\right) = l^2(\pi(k)) \otimes_{\mathbf{Z}\pi} C_*(\widetilde{W}_\varphi)$$

through the projection p_k, where $\widetilde{W}_\varphi \to W_\varphi$ is a universal covering space. By using the Laplace operator on this complex, we define the kth L^2-torsion $\tau_k(W_\varphi)$ as follows:

Definition 2.1.

$$\tau_k(W_\varphi) = \prod_{i=0}^{3} \det{}_{\mathbf{C}\pi(k)}\bigl(\Delta_i^{(k)}\bigr)^{(-1)^{i+1}i},$$

where $\Delta_i^{(k)} : C_i(\widetilde{W}_\varphi, \mathbf{C}\pi(k)) \to C_i(\widetilde{W}_\varphi, \mathbf{C}\pi(k))$ is the Laplace operator on $\mathbf{C}\pi(k)$.

Remark 2.2. For some K, a limit of $(1/p)\,\mathrm{tr}_{\mathbf{C}\pi}\bigl(I - K^{-2}\Delta_i^{(k)}(\Delta_i^{(k)})^*\bigr)^p$ on p is zero by Lück [8]. Furthermore it is easy to see $\pi(k)$ belongs to the class of groups defined by Schick. Therefore every τ_k is well-defined.

Here let us state our volume conjecture for a surface bundle over S^1.

Conjecture 2.3. *The sequence $\{\tau_k(W_\varphi)\}_{k\in\mathbf{N}}$ converges to $\tau(W_\varphi)$ when we take the limit on k.*

In our setting, Lück's formula [7] of $\tau_k(W_\varphi)$ is described as follows: Let $x_1, \dots, x_{2g}$ be a generating system of the free group $F_{2g} = \Gamma$. Then the fundamental group π is presented by

$$\pi = \bigl\langle x_1, \dots, x_{2g}, t \mid r_i = t x_i t^{-1}\left(\varphi_*(x_i)\right)^{-1}, \quad 1 \leq i \leq 2g \bigr\rangle,$$

where $\varphi_* : \Gamma \to \Gamma$ is a homomorphism induced by $\varphi : \Sigma_{g,1} \to \Sigma_{g,1}$. Applying the free differential calculus to relators $r_1, \dots, r_{2g}$, we obtain a Fox matrix

$$A = \left(\frac{\partial r_i}{\partial x_j}\right) \in M(2g, \mathbf{Z}\pi).$$

Let $p_{k*} : \mathbf{C}\pi \to \mathbf{C}\pi(k)$ be an induced homomorphism over the group rings and we put

$$A_k = \left(p_{k*}\left(\frac{\partial r_i}{\partial x_j}\right)\right) \in M(2g, \mathbf{C}\pi(k)).$$

Moreover we fix a constant K_k satisfying $K_k \geq ||R_{A_k}||_\infty$. Thereby the formula is given by

$$\begin{aligned}\log \tau_k(W_\varphi) &= -2 \log \det{}_{\mathbf{C}\pi(k)}(A_k)\\ &= -4g \log K_k\\ &\quad + \sum_{p=1}^{\infty} \frac{1}{p}\,\mathrm{tr}_{\mathbf{C}\pi(k)}\left(I - K_k^{-2} A_k A_k^*\right)^p.\end{aligned}$$

3. A formula of τ_1 and cyclic covering.

In the following, we only consider torus bundles over the circle. First we review a formula of the first invariant τ_1 (see [4, 5]).

Theorem 3.1. *The logarithm of $\tau_1(W_\varphi)$ is given by*

$$\log \tau_1(W_\varphi) = -2 \log \max\{|\alpha|, 1/|\alpha|\},$$

where α and $1/\alpha$ are the eigenvalues of the homology representation $\varphi_* \in SL(2, \mathbf{Z})$.

Remark 3.2. In other words, the first term $\log \tau_1$ is nothing but minus twice of the Mahler measure (see [2]) of the characteristic polynomial of $\varphi_* \in SL(2, \mathbf{Z})$.

From this description, we obtain the following notable corollary.

Corollary 3.3. *A mapping torus W_φ admits a hyperbolic structure if and only if W_φ has a non-trivial L^2-torsion invariant $\tau_1(W_\varphi)$.*

Therefore, in some sense, we can say that the first invariant τ_1 already approximates the simplicial volume in genus one case.

By the way, if we consider only the first term τ_1, we can define it for a manifold M with a surjection $\pi_1(M) \to T \cong \mathbf{Z}$, for example an exterior of a knot, not only for surface bundles. In this case, the above formula of τ_1 is related with the following classical result on knots (see [13]).

We fix a prime number $n \geq 2$. Let $W_{\varphi^n} \to W_\varphi$ be the n-fold cyclic covering of W_φ. Then we define $\mathrm{ord}(\varphi, n)$ to be the order of the quotient group $H_1(W_{\varphi^n}, \mathbf{Z})/\langle t \rangle$. If its order is infinity, we put $\mathrm{ord}(\varphi, n) = 0$. Here associated with

$$\pi_1(W_\varphi) \to \pi(1) = T = \langle t \rangle \ni t \mapsto \bar{t} \in \bar{T}^{(n)} := \langle \bar{t} \mid \bar{t}^n \rangle,$$

we can define the L^2-torsion invariant $\tau_1^{(n)}(W_\varphi)$. Because in this case, $\mathbf{C}\bar{T}^{(n)} = l^2(\bar{T}^{(n)}) \cong \mathbf{C}^n$ is a finite dimensional vector space. We then obtain

Theorem 3.4. *It holds that*

(i) $\log \tau_1^{(n)}(W_\varphi) = -\dfrac{2}{n} \log \mathrm{ord}(\varphi, n)$,

(ii) $\lim\limits_{n\to\infty} \log \tau_1^{(n)}(W_\varphi) = \log \tau_1(W_\varphi)$.

Proof. We consider the Fox matrix $A_1^{(n)} \in M(2, \mathbf{C}\bar{T}^{(n)})$ over $\mathbf{C}\bar{T}^{(n)}$, which is induced from A_1 by the projection $T \to \bar{T}^{(n)}$. We write $\tilde{A}_1^{(n)}$ to its induced linear endmorphism on $l^2(\bar{T}^{(n)}) \oplus l^2(\bar{T}^{(n)}) \cong \mathbf{C}^n \oplus \mathbf{C}^n = \mathbf{C}^{2n}$. Then we notice the fact that

$$\mathrm{tr}_{\mathbf{C}\bar{T}^{(n)}}(A_1^{(n)}) = \frac{1}{n}\,\mathrm{tr}(\tilde{A}_1^{(n)}),$$

where 'tr' is the ordinary trace for matrices. By using this fact and the definition of $\det_{\mathbf{C}\bar{T}^{(n)}}$, it follows that

$$\log \det{}_{\mathbf{C}\bar{T}^{(n)}}(A_1^{(n)}) = \frac{1}{n} \log |\det(\tilde{A}_1^{(n)})|,$$

where 'det' denotes the usual determinant. On the other hand, $A_1^{(n)}$ is a presentation matrix for $H_1(W_{\varphi^n}, \mathbf{Z})$ as a $\mathbf{Z}\bar{T}$-module and $\tilde{A}_1^{(n)}$ is such one as a $\mathbf{Z}$-module. Thus $|\det(\tilde{A}_1^{(n)})| = \mathrm{ord}(\varphi, n)$ holds. Therefore, by using Lück's formula mentioned in the previous section, we obtain $\log \tau_1^{(n)}(W_\varphi) = (-2/n) \log \mathrm{ord}(\varphi, n)$.

To prove the second assertion, we only have to show

$$\lim_{n\to\infty} \mathrm{tr}_{\mathbf{C}\bar{T}^{(n)}}(f(\bar{t})) = \mathrm{tr}_{\mathbf{C}T}(f(t))$$

for any $f(t) = \sum a_k t^k \in \mathbf{C}T$. Here we have written $f(\bar{t})$ to the corresponding element in $\mathbf{C}\bar{T}^{(n)}$.

By the definition, $\mathrm{tr}_{\mathbf{C}\bar{T}^{(n)}}(f(\bar{t})) = \sum_{k\equiv 0(n)} a_k$. The right hand side is equal to a finite sum $\sum_{i=1}^n (1/n) f(\zeta^i)$, where ζ is a primitive nth root of unity. Because n is prime and ζ is primitive, and then

$$\frac{1}{n}\sum_{i=1}^n a_k(\zeta^i)^k = \frac{a_k}{n}\sum_{i=1}^n \zeta^{ik} = \begin{cases} a_k & (\zeta^k = 1) \\ 0 & (\zeta^k \neq 1) \end{cases}$$

holds. Furthermore it is clear that

$$\lim_{n\to\infty} \sum_{i=1}^n \frac{1}{n} f(\zeta^i) = \int_0^1 f(e^{2\pi\sqrt{-1}\theta})\, d\theta.$$

Here we recall from [4] Theorem 5.1 that

$$\mathrm{tr}_{\mathbf{C}T}(f(t)) = \int_0^1 f(e^{2\pi\sqrt{-1}\theta})\, d\theta$$

holds. Hence we have

$$\lim_{n\to\infty} \mathrm{tr}_{\mathbf{C}\bar{T}^{(n)}}(f(\bar{t})) = \mathrm{tr}_{\mathbf{C}T}(f(t)).$$

□

Remark 3.5. The above theorem also holds for higher genera. As for a related work to Theorem 3.4, see [14].

4. Vanishing of $\log \boldsymbol{\tau_2}$. As was showing in [4], for a monodromy $\varphi : \Sigma_{1,1} \to \Sigma_{1,1}$ satisfying $|\mathrm{tr}(\varphi_*)| \leq 2$, the kth invariant $\tau_k(W_\varphi)$ is trivial for every k. In general, we can prove the vanishing of $\log \tau_2(W_\varphi)$ as follows:

Theorem 4.1. *The second term $\tau_2(W_\varphi)$ is always trivial.*

Proof. To prove this theorem, we use Lück's formula in the closed surface bundle case ([7] Theorem 2.4). For any diffeomorphism ϕ on a closed torus Σ_1, we simply denote its fundamental group $\pi_1(W_\phi)$ by $\bar{\pi}$. We then obtain the same Fox matrix

$$A = \left(\frac{\partial r_i}{\partial x_j}\right).$$

In this case, Lück's formula of the original L^2-torsion τ is described as follows:

$$\log \tau(W_\phi) = -2 \log \det{}_{\mathbf{C}\bar{\pi}}(A).$$

Now we regard as $\Sigma_{1,1} \subset \Sigma_1$ and let $\phi : \Sigma_1 \to \Sigma_1$ be a diffeomorphism induced from $\varphi : \Sigma_{1,1} \to \Sigma_{1,1}$ (namely, $\phi|_{\Sigma_{1,1}} = \varphi$ holds). Thereby we have

$$\log \tau_2(W_\varphi) = \log \tau(W_\phi) = C||W_\phi||$$

by the above formula and the definition of τ_2. Here C is a constant and $||W_\phi||$ the simplicial volume of W_ϕ (see [7]). This W_ϕ is a Seifert fibered space, or a solvable manifold. Hence $||W_\phi||$ is zero. Therefore $\log \tau_2(W_\varphi)$ is also zero. This completes the proof.

□

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