A new expression for the product of the two Dirichlet series I

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Abstract: A new expression for the product of the two Dirichlet series is given. From this new expression we derive many results. Among other things we give here another proof of Wilton's expression for the product of two Riemann's zeta functions through our new expression. Other results including mean value theorems will be treated elsewhere.

Key words: The Riemann zeta function; Dirichlet series.

Let $F(w)$ and $G(w)$ be defined by the Dirichlet series

$$F(w) := \sum_{n=1}^{\infty} a_n n^{-w} \ (a_n,\ w = x+iy \in \mathbf{C})$$
$$G(w) := \sum_{n=1}^{\infty} b_n n^{-w} \ (b_n,\ w = x+iy \in \mathbf{C})$$

which converge absolutely for $\Re w > c_a$ and $\Re w > c_b$ respectively and are extended meromorphically to the whole complex plane $\mathbf{C}$ with the following conditions:

(i)

$$A < \Re u_k \le c_a \quad \text{for} \quad \forall u_k \in S_F$$
$$A' < \Re v_k \le c_b \quad \text{for} \quad \forall v_k \in S_G$$

where S_F and S_G are the sets of all poles of the meromorphic functions $F(w)$ and $G(w)$ respectively.

(ii)

There exist $E_a, E_b \in \mathbf{R}$ and sequences $\{T_m\}_{m=1}^{\infty}$, $\{T'_m\}_{m=1}^{\infty}$ with

$$0 < T_1 < T_2 < \cdots < T_m < \cdots \to \infty \quad \text{and}$$
$$0 < T'_1 < T'_2 < \cdots < T'_m < \cdots \to \infty$$

such that

$$F(x \pm iT_m) = o(T_m) \quad \text{for} \quad \forall x > -E_a$$
$$G(x \pm iT'_m) = o(T'_m) \quad \text{for} \quad \forall x > -E_b$$

Then we have the following

Theorem 1. *If both*

$$\sum_{\substack{u_k \in S_F \\ -b<\Re(u_k-u)}} \operatorname*{Res}_{w=u_k-u} \left\{ F(u+w)G(v-w)\frac{1}{w} \right\}$$
$$:= \lim_{T\to\infty} \sum_{\substack{|\Im u_k|<T \\ u_k \in S_F \\ -b<\Re(u_k-u)}} \operatorname*{Res}_{w=u_k-u} \left\{ F(u+w)G(v-w)\frac{1}{w} \right\}$$

and

$$\sum_{\substack{v_k \in S_G \\ -b'<\Re(v_k-v)}} \operatorname*{Res}_{w=v_k-v} \left\{ G(v+w)F(u-w)\frac{1}{w} \right\}$$

converge, then we have

$$\begin{aligned} &F(u)G(v) \\ &= - \sum_{\substack{u_k \in S_F \\ -b<\Re(u_k-u)}} \operatorname*{Res}_{w=u_k-u} \left\{ F(u+w)G(v-w)\frac{1}{w} \right\} \\ &\quad - \sum_{\substack{v_k \in S_G \\ -b'<\Re(v_k-v)}} \operatorname*{Res}_{w=v_k-v} \left\{ G(v+w)F(u-w)\frac{1}{w} \right\} \\ &\quad - \frac{1}{2\pi i}\int_{(-b)} F(u+w)G(v-w)\frac{dw}{w} \\ &\quad - \frac{1}{2\pi i}\int_{(-b')} G(v+w)F(u-w)\frac{dw}{w} \end{aligned}$$

for $u+v-u_k \notin S_G$, $u+v-v_k \notin S_F$, $\Re u > c_a$, $\Re v > c_b$, $0 < b < c_a + E_a$, $0 < b' < c_b + E_b$ *where*

$$\int_{(a)} \ \textit{means} \ \int_{a-i\infty}^{a+i\infty},$$

$\operatorname{Res}_{w=a} f(w)$ *denotes the residue of* $f(w)$ *at* $w = a$ *and the above equation is extended meromorphically to the whole complex plane as far as the values in the equation are finite.*

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In particular, we have

Corollary 1.

$$F(u)G(v) = \frac{1}{2\pi i}\int_{(c)} F(u+w)G(v-w)\frac{dw}{w} + \frac{1}{2\pi i}\int_{(c')} G(v+w)F(u-w)\frac{dw}{w}$$

for $\Re u > c_a + c',\ \Re v > c_b + c,\ c > 0,\ c' > 0.$

Proof of Theorem 1. By the change of summations, we have

$$\begin{aligned}
&F(u)G(v)\\
&= \sum_{n=1}^{\infty} a_n n^{-u} \sum_{m=1}^{\infty} b_m m^{-v}\\
&= \Big\{\sum_{n<m} + \frac{1}{2}\sum_{n=m} + \frac{1}{2}\sum_{n=m} + \sum_{n>m}\Big\} a_n n^{-u} b_m m^{-v}\\
&= \sum_{m=1}^{\infty}\sum_{n\le m}{}' a_n n^{-u} b_m m^{-v} + \sum_{n=1}^{\infty}\sum_{m\le n}{}' a_n n^{-u} b_m m^{-v}
\end{aligned}$$

for $\Re u > c_a$ and $\Re v > c_b$ where $\sum'_{n\le m} f(n)$ means that we add $(1/2)f(m)$ in place of $f(m)$ when $n = m$. Applying Perron's formula [6], we have

$$\begin{aligned}
F(u)G(v) &= \sum_{m=1}^{\infty} b_m m^{-v} \frac{1}{2\pi i}\int_{(c)} F(u+w)\frac{m^w}{w}dw\\
&\quad + \sum_{n=1}^{\infty} a_n n^{-u} \frac{1}{2\pi i}\int_{(c')} G(v+w)\frac{n^w}{w}dw\\
&= \frac{1}{2\pi i}\int_{(c)} F(u+w)G(v-w)\frac{dw}{w}\\
&\quad + \frac{1}{2\pi i}\int_{(c')} G(v+w)F(u-w)\frac{dw}{w}
\end{aligned}$$

for $\Re u > c_a + c'$, $\Re v > c_b + c$, $c > 0$, $c' > 0$ which is the corollary. Next we move the path of integration from $\int_{(c)}$ to $\int_{(-b)}$ (resp. from $\int_{(c')}$ to $\int_{(-b')}$) and by the theorem of residue we lastly have the theorem. □

We show some examples. The equations in the examples are extended meromorphically to the whole complex plane as far as the values in the equations are finite.

Example 1.

$$\begin{aligned}
\zeta(u)\zeta(v) &= \zeta(u+v-1)\Big\{\frac{1}{u-1} + \frac{1}{v-1}\Big\}\\
&\quad - \frac{1}{2\pi i}\int_{(-b)} \zeta(u+w)\zeta(v-w)\frac{dw}{w}\\
&\quad - \frac{1}{2\pi i}\int_{(-b')} \zeta(v+w)\zeta(u-w)\frac{dw}{w}
\end{aligned}$$

for $u+v \neq 2$, $0 < \Re u - 1 < b < (3/2)$, $0 < \Re v - 1 < b' < (3/2)$ where $\zeta(s)$ is the Riemann zeta function.

Example 2.

$$\begin{aligned}
&\zeta(u)^2\zeta(v)^2\\
&= \{2\zeta(u+v-1)\zeta'(u+v-1)\\
&\quad - 2\gamma\zeta(u+v-1)^2\}\Big\{\frac{1}{1-u} + \frac{1}{1-v}\Big\}\\
&\quad + \zeta(u+v-1)^2\Big\{\frac{1}{(1-u)^2} + \frac{1}{(1-v)^2}\Big\}\\
&\quad - \frac{1}{2\pi i}\int_{(-b)} \zeta(u+w)^2\zeta(v-w)^2\frac{dw}{w}\\
&\quad - \frac{1}{2\pi i}\int_{(-b')} \zeta(v+w)^2\zeta(u-w)^2\frac{dw}{w}
\end{aligned}$$

for $u+v \neq 2$, $0 < \Re u - 1 < b < 1$, $0 < \Re v - 1 < b' < 1$ where $\gamma \equiv \gamma_0$ is the Euler constant.

Example 3.

$$\begin{aligned}
&\zeta(u)^3\zeta(v)^3\\
&= -\Big[\{6\zeta(u+v-1)\zeta'(u+v-1)^2\\
&\quad + 3\zeta(u+v-1)^2\zeta''(u+v-1)\\
&\quad - 9\gamma_0\zeta(u+v-1)^2\zeta'(u+v-1)\\
&\quad + 3(\gamma_0^2 - \gamma_1)\zeta(u+v-1)^3\}\Big\{\frac{1}{1-u} + \frac{1}{1-v}\Big\}\\
&\quad + \{6\zeta(u+v-1)^2\zeta'(u+v-1)\\
&\quad - 3\gamma_0\zeta(u+v-1)^3\}\Big\{\frac{1}{(1-u)^2} + \frac{1}{(1-v)^2}\Big\}\\
&\quad + 2\zeta(u+v-1)^3\Big\{\frac{1}{(1-u)^3} + \frac{1}{(1-v)^3}\Big\}\Big]\\
&\quad - \frac{1}{2\pi i}\int_{(-b)} \zeta(u+w)^3\zeta(v-w)^3\frac{dw}{w}\\
&\quad - \frac{1}{2\pi i}\int_{(-b')} \zeta(v+w)^3\zeta(u-w)^3\frac{dw}{w}
\end{aligned}$$

for $u+v \neq 2$, $0 < \Re u - 1 < b < (3/4)$, $0 < \Re v - 1 < b' < (3/4)$ where γ_n is the n-th generalized Euler constant defined by

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!}\gamma_n(s-1)^n.$$

Example 4.

$$\begin{aligned}
&\zeta(u)\zeta(v)^2\\
&= -\frac{\zeta(u+v-1)^2}{1-u}\\
&\quad + \{\zeta'(u+v-1) - 2\gamma\zeta(u+v-1)\}\frac{1}{1-v}
\end{aligned}$$

$$+\frac{\zeta(u+v-1)}{(1-v)^2}$$
$$-\frac{1}{2\pi i}\int_{(-b)}\zeta(u+w)\zeta(v-w)^2\frac{dw}{w}$$
$$-\frac{1}{2\pi i}\int_{(-b')}\zeta(v+w)^2\zeta(u-w)\frac{dw}{w}$$

for $u+v\neq 2,\ 0<\Re u-1<b<(3/2),\ 0<\Re v-1<b'<1$.

Example 5.

$$\frac{\zeta'}{\zeta}(u)\frac{\zeta'}{\zeta}(v)$$
$$=-\frac{\zeta'}{\zeta}(u+v-1)\Big\{\frac{1}{u-1}+\frac{1}{v-1}\Big\}$$
$$+\lim_{T\to\infty}\Big[\sum_{\substack{\rho;\\ |\Im\rho|<T}}\frac{\zeta'}{\zeta}(u+v-\rho)\Big\{\frac{1}{u-\rho}+\frac{1}{v-\rho}\Big\}$$
$$+\sum_{\substack{n\in\mathbf{N};\\ 1<2n<2[T^{2/3}]-1}}\frac{\zeta'}{\zeta}(u+v+2n)$$
$$\times\Big\{\frac{1}{u+2n}+\frac{1}{v+2n}\Big\}\Big]$$

where $(\zeta'/\zeta)(s):=(\zeta'(s))/(\zeta(s))$ and ρ is the non-trivial zero of $\zeta(s)$.

Lastly we restrict ourselves to the special case Example 1. We denote the first integral in the equation by $g_b(u,v)$, that is,

$$g_b(u,v)=\frac{-1}{2\pi i}\int_{(-b)}\zeta(u+w)\zeta(v-w)\frac{dw}{w}$$
$$=2(2\pi)^{u-1}\frac{1}{2\pi i}\int_{(b)}(2\pi)^{-z}\sin\frac{\pi}{2}(u-z)$$
$$\times\Gamma(z-u+1)\zeta(v+w)\zeta(z-u+1)\frac{dz}{z}$$

where we used the functional equation of $\zeta(s)$ after the change of the variables. Moreover we assume $\Re u<b$, then we have

$$g_b(u,v)$$
$$=2(2\pi)^{u-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{u-1}$$
$$\times\frac{1}{2\pi i}\int_{(b)}(2\pi n)^{-z}\sin\frac{\pi}{2}(u-z)\Gamma(z-u+1)\frac{dz}{z}$$

where $\sigma_z(n):=\sum_{d|n}d^z\ (z\in\mathbf{C})$.

We put

$$I_n(b,u)$$
$$:=\frac{1}{2\pi i}\int_{(b)}(2\pi n)^{-z}\sin\frac{\pi}{2}(u-z)\Gamma(z-u+1)\frac{dz}{z}$$

and can move the path of integration to the left as follows:

$$I_n(b_1,u)$$
$$:=\frac{1}{2\pi i}\int_{(b_1)}(2\pi n)^{-z}\sin\frac{\pi}{2}(u-z)\Gamma(z-u+1)\frac{dz}{z}$$

which is absolutely convergent for $b_1+(1/2)<\Re u$, $0<\Re u-1<b_1$.

By the formula [2, p. 325]:

$$\frac{1}{2\pi i}\int_{(c)}x^{-s}\Gamma(s+\alpha)\frac{ds}{s}=\Gamma(\alpha,x)\quad(c>0,\ \Re x>0)$$
$$\frac{1}{2\pi i}\int_{(c)}(ix)^{-s}\Gamma(s+\alpha)\frac{ds}{s}=\Gamma(\alpha,ix)$$
$$\Big(c>0,\quad \Re\alpha<\frac{1}{2}-c,\quad x\in\mathrm{R}\Big)$$

where

$$\Gamma(\alpha,x)=\int_x^{\infty}x^{\alpha-1}e^{-t}dt\quad(|\arg\alpha|<\pi)$$

is the incomplete gamma function, we have

$$I_n(b_1,u)=\frac{1}{2}e^{-\frac{\pi}{2}i(1-u)}\Gamma(1-u,2\pi in)$$
$$+\frac{1}{2}e^{\frac{\pi}{2}i(1-u)}\Gamma(1-u,-2\pi in)$$
$$=\int_{2\pi n}^{\infty}x^{-u}\cos xdx\quad(\Re u>0)$$
$$=u\int_{2\pi n}^{\infty}x^{-u-1}\sin xdx\quad(\Re u>-1)$$

where we used the formula [3, p. 149]

$$\int_u^{\infty}x^{\alpha-1}\cos xdx$$
$$=\frac{1}{2}e^{-\frac{\pi}{2}i\alpha}\Gamma(\alpha,iu)+\frac{1}{2}e^{\frac{\pi}{2}i\alpha}\Gamma(\alpha,-iu).$$

Then we have

$$g_b(u,v)$$
$$=2(2\pi)^{u-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{u-1}u\int_{2\pi n}^{\infty}x^{-u-1}\sin xdx.$$

Therefore we obtain the following theorem which is Wilton's expression for the product of two Riemann's zeta functions [7].

Theorem 2 (Wilton)**.** *For* $\Re u,\Re v>-1$, $\Re(u+v)>0$ *and* $u+v\neq 2$*, we have*

$$\zeta(u)\zeta(v)$$
$$= \zeta(u+v-1)\Big\{\frac{1}{u-1}+\frac{1}{v-1}\Big\}$$
$$+ 2(2\pi)^{u-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{u-1}u\int_{2\pi n}^{\infty}x^{-u-1}\sin x dx$$
$$+ 2(2\pi)^{v-1}\sum_{n=1}^{\infty}\sigma_{1-u-v}(n)n^{v-1}v\int_{2\pi n}^{\infty}x^{-v-1}\sin x dx.$$

Remarks. Our very simple idea of the decomposition (in the proof of Theorem 1):

$$\sum_{n<m}+\frac{1}{2}\sum_{n=m}+\frac{1}{2}\sum_{n=m}+\sum_{n>m}$$

of

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}$$

is new in place of that of Atkinson's [1]:

$$\sum_{n<m}+\sum_{n=m}+\sum_{n>m}$$

by making use of which he derived the square mean value theorem of the Riemann zeta-function now called Atkinson's formula. Motohashi generalized this Atkinson's decomposition to matrix version to develop his fourth mean value theory of the Riemann zeta-function [5]. Although Atkinson's decomposition can work only in the case of the Rimannn zeta-function and $\sum_{\chi} L(s,\chi)$, but our decomposition can work in wider class of Dirichlet series because in our decomposition we can appeal to Perron's formula to continue our procedure.

Our Theorem 1 gives a unified method to construct the mean value theorem of wider class of Dirichlet series including Wilton's formula (Theorem 2, [7]), Bellman's formula (Example 2, [4]) and so on. Lastly we should point out that Example 1 is the first example that $\zeta(\cdot)$ satisfies an integral equation whereas it is well known that $\zeta(\cdot)$ cannot satisfy any algebraic differential equation.

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