On certain hypersurfaces with non-isolated singularities in $\mathrm{P}^4(\mathbf{C})$

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Abstract: We give an example of hypersurfaces with non-isolated singularities in $\mathrm{P}^4(\mathbf{C})$, whose normalizations have isolated rational quadruple points only as singularities. From Schlessinger's criterion, it follows that these isolated rational singular points are rigid under small deformations.

Key words: Hypersurface; threefold; non-isolated singularity; ordinary singularity; normalization; rigid rational singularity.

1. An example of hypersurfaces in $\mathbf{P}^4(\mathbf{C})$ whose singularities are ordinary except at finite points. Let H_i $(1 \leq i \leq 4)$ be non-singular hypersurfaces of degrees r_i $(1 \leq i \leq 4)$, respectively, in the complex projective 4-space $\mathrm{P}^4(\mathbf{C})$ such that they are in general position at every point where they intersect. We put $D_T^{(ij)} := H_i \cap H_j$ $(1 \leq i < j \leq 4)$ and $D_T := \bigcup_{1\leq i<j\leq 4} D_T^{(ij)}$. Let f_i $(1 \leq i \leq 4)$ be the homogeneous polynomial of degree r_i which defines the hypersurface H_i. We may assume $r_1 \geq r_2 \geq r_3 \geq r_4$ because of symmetry. We choose and fix a positive integer n with $n \geq 2r_1 + 2r_2 + 2r_3$. Let T be a hypersurface in $\mathrm{P}^4(\mathbf{C})$ defined by the equation

$$
\begin{aligned}
(1.1)\quad F := Af_1f_2f_3f_4 + B(f_1f_2f_3)^2 + C(f_1f_2f_4)^2 \\
+ D(f_1f_3f_4)^2 + E(f_2f_3f_4)^2 = 0,
\end{aligned}
$$

where A, B, C, D and E are homogeneous polynomials of five variables of respective degrees $n - r_1 - r_2 - r_3 - r_4$, $n - 2r_1 - 2r_2 - 2r_3$, $n - 2r_1 - 2r_2 - 2r_4$, $n-2r_1-2r_3-2r_4$ and $n-2r_2-2r_3-2r_4$. By Bertini's theorem, T is non-singular outside D_T if we choose sufficiently generic A, B, C, D and E.

Proposition 1.1. *If the homogeneous polynomials A, B, C, D and E are chosen sufficiently generic, then T is locally isomorphic to one of the following germs of three dimensional hypersurface singularities at the origin of $\mathbf{C}^4$ at every point of T:*

(i) $w = 0$ (*simple point*),
(ii) $zw = 0$ (*ordinary double point*),
(iii) $yzw = 0$ (*ordinary triple point*),
(iv) $xyzw = 0$ (*ordinary quadruple point*),
(v) $xy^2 - z^2 = 0$ (*cuspidal point*),
(vi) $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$ (*degenerate ordinary triple point*),

where (x, y, z, w) is the coordinate on $\mathbf{C}^4$.

Proof. (i) Let $p \in D_T$ be a point satisfying $f_i(p) = 0$, $1 \leq i \leq 4$. We may assume that $A(p)B(p)C(p)D(p)E(p) \neq 0$. We make the transformations of local coordinates

$$
\begin{aligned}
&(f_1, f_2, f_3, f_4) \to \\
&\left(\sqrt[4]{\frac{A^2E}{BCD}}\frac{X}{\sqrt{1+f}}, \sqrt[4]{\frac{A^2D}{BCE}}\frac{Y}{\sqrt{1+f}},\right. \\
&\qquad\left.\sqrt[4]{\frac{A^2C}{BDE}}\frac{Z}{\sqrt{1+f}}, \sqrt[4]{\frac{A^2B}{CDE}}\frac{W}{\sqrt{1+f}}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
f := (XY)^2 + (XZ)^2 + (XW)^2 + (YZ)^2 \\
+ (YW)^2 + (ZW)^2 \\
+ XYZW(X^2 + Y^2 + Z^2 + W^2 + XYZW),
\end{aligned}
$$

and

$$
\begin{aligned}
(X + YZW, Y + XZW, Z + XYW, W + XYZ) \\
\to (X', Y', Z', W')
\end{aligned}
$$

successively in a neighborhood of p. Then the equation in (1.1) is transformed to $A'X'Y'Z'W' = 0$, where $A' := A^3/\{\sqrt{BCDE}(1 + f)^3\}$. Namely, the point p is an ordinary quadruple point.

(ii) Let $p \in D_T$ be a point where three of f_i, $1 \leq i \leq 4$, vanish, but all of f_i, $1 \leq i \leq 4$, do not. Suppose that $f_1(p) = f_2(p) = f_3(p) = 0$ and $f_4(p) \neq 0$. We write F in (1.1) as

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$$F = A'f_1f_2f_3 + C'(f_1f_2)^2 + D'(f_1f_3)^2 + E'(f_2f_3)^2 \tag{1.2}$$

where $A' := Af_4 + Bf_1f_2f_3$, $C' := Cf_4^2$, $D' := Df_4^2$ and $E' := Ef_4^2$. We may assume that both of A' and $C'D'E'$ do not vanish at p.

(ii-α) In the case of $A'(p)C'(p)D'(p)E'(p) \neq 0$: We make the transformations of local coordinates

$$(f_1, f_2, f_3) \to \left(\frac{A'}{\sqrt{C'D'}}\frac{X}{1+g}, \frac{A'}{\sqrt{C'E'}}\frac{Y}{1+g}, \frac{A'}{\sqrt{D'E'}}\frac{Z}{1+g}\right),$$

where $g := X^2 + Y^2 + Z^2 + XYZ$, and

$$(X + YZ, Y + XZ, Z + XY) \to (X', Y', Z')$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A''X'Y'Z' = 0$, where $A'' := A'^4/\{C'D'E'(1+g)^4\}$. Hence, p is an ordinary triple point.

(ii-β) In the case of $A'(p) \neq 0$, $C'(p)D'(p)E'(p) = 0$: Taking sufficiently generic C, D and E, we may assume that two of C', D' and E' do not vanish at p. Suppose that $C'(p) = 0$ and $D'(p)E'(p) \neq 0$. We put $X := f_1$, $Y := f_2$, $Z := f_3$ and $W := C'$. We may consider that (X, Y, Z, W) is a system of local coordinates at p by taking a sufficiently generic C. Using the local coordinate (X, Y, Z, W), we can write F in (1.2) as

$$F = A'XYZ + W(XY)^2 + D'(XZ)^2 + E'(YZ)^2 \tag{1.3}$$

where $A'(p)D'(p)E'(p) \neq 0$. We make the transformations of local coordinates

$$(X, Y, Z, W) \to \left(\frac{A'}{\sqrt{D'}}\frac{X'}{1+h}, \frac{A'}{\sqrt{E'}}\frac{Y'}{1+h}, \frac{A'}{\sqrt{D'E'}}\frac{Z'}{1+h}, W\right)$$

where $h := Z'^2 + (X'^2 + Y'^2 + X'Y'Z')W$, and

$$(X' + Y'Z', Y' + X'Z', Z' + X'Y'W, W) \to (X'', Y'', Z'', W)$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A''X''Y''Z'' = 0$, where $A'' := A'^4/\{D'E'(1+h)^4\}$. Hence, p is an ordinary triple point.

(ii-γ) In the case of $A'(p) = 0$, $C'(p)D'(p)E'(p) \neq 0$: We put $X := f_1$, $Y := f_2$, $Z := f_3$ and $W := A'$. We may consider that (X, Y, Z, W) is a system of local coordinates at p by taking sufficiently generic A and B. Using the local coordinate (X, Y, Z, W), we can write F in (1.2) as

$$F = XYZW + C'(XY)^2 + D'(XZ)^2 + E'(YZ)^2. \tag{1.4}$$

We make the transformation of local coordinates

$$(X, Y, Z, W) \to \left(\frac{X'}{\sqrt{C'D'}}, \frac{Y'}{\sqrt{C'E'}}, \frac{Z'}{\sqrt{D'E'}}, W\right).$$

Then the equation $F = 0$ is transformed to

$$\frac{1}{C'D'E'}\{X'Y'Z'W + (X'Y')^2 + (X'Z')^2 + (Y'Z')^2\} = 0$$

which defines the singularity (vi) in Proposition 1.1.

(iii) Let $p \in D_T$ be a point where two of $f_i, 1 \leq i \leq 4$, vanish, but more than two of $f_i, 1 \leq i \leq 4$, do not. Suppose that $f_1(p) = f_2(p) = 0$ and $f_3(p)f_4(p) \neq 0$. We write F in (1.1) as

$$F = B'f_1^2 + A'f_1f_2 + E'f_2^2, \tag{1.5}$$

where $B' := (Bf_3^2 + Cf_4^2)f_2^2 + Df_3^2f_4^2$, $A' := Af_3f_4$ and $E' := Ef_3^2f_4^2$.

(iii-α) In the case of $B'(p) \neq 0$, or $E'(p) \neq 0$: Suppose $B'(p) \neq 0$. Then F in (1.5) is written as

$$F = B'\left(f_1 + \frac{A' - \sqrt{A'^2 - 4B'E'}}{2B'}f_2\right) \times \left(f_1 + \frac{A' + \sqrt{A'^2 - 4B'E'}}{2B'}f_2\right)$$

in a neighborhood of p.

(iii-α)$_d$ If $(A'^2 - 4B'E')(p) \neq 0$, then the transfomation

$$f_1 + \frac{A' - \sqrt{A'^2 - 4B'E'}}{2B'}f_2 \longrightarrow X,$$
$$f_1 + \frac{A' + \sqrt{A'^2 - 4B'E'}}{2B'}f_2 \longrightarrow Y$$

can be regarded as that of local coordinates. By this trasformation the equation $F = 0$ is transformed to $B'XY = 0$, where B' is a non-vanishing factor. Hence p is an ordinary double point.

(iii-α)$_c$ If $(A'^2 - 4B'E')(p) = 0$, we make the transformation of local coordinates

$$\frac{A'^2 - 4B'E'}{(2B')^2} \longrightarrow X,$$
$$f_2 \longrightarrow Y,$$
$$f_1 + \frac{A'}{2B'}f_2 \longrightarrow Z$$

in a neighborhood of p. Then the equation $F = 0$ is transformed to

$$B'(Z + \sqrt{X}Y)(Z - \sqrt{X}Y) = B'(Z^2 - XY^2) = 0.$$

Hence p is a cuspidal point.

(iii-β) In the case of $B'(p) = E'(p) = 0$: We put $X := f_1$, $Y := f_2$, $Z := B'$ and $W = E'$. We may consider that (X, Y, Z, W) is a system of local coordinates at p by taking sufficiently generic B, C, D and E. Using the local coordinate (X, Y, Z, W), we can write F in (1.5) as

$$F = A'XY + ZX^2 + WY^2. \tag{1.6}$$

We may assume that $A'(p) \neq 0$. We make the transformations of local coordinates

$$\begin{aligned}
&(X, Y, Z, W) \to (X, Y, A'Z', A'W'),\\
&(X, Y, Z', W') \to\\
&\left(\frac{X'}{1+Z''W''}, \frac{Y'}{1+Z''W''}, \frac{Z''}{1+Z''W''}, \frac{W''}{1+Z''W''}\right),
\end{aligned}$$

and

$$(X' + W''Y', Y' + Z''X', Z'', W'') \longrightarrow (X'', Y'', Z'', W'')$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A''X''Y'' = 0$, where $A'' := A'/(1 + Z''W'')^3$. Hence p is an ordinary double point. □

Note: The singularities from (ii) through (v) in Proposition 1.1 are *ordinary* in the sense of Roth ([2]). Besides these four types of singularities, the *stationary point*, i.e., the singular point defined by the equation $w(xy^2 - z^2) = 0$ in $\mathbf{C}^4$, is also *ordinary*. These *ordinary* singularities arise if we project a non-singular threefold embedded in a sufficiently high dimensional complex projective space to its four dimensional linear subspace by a generic linear projection.

2. The singularity $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$.

Proposition 2.1. *In the expression $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$, we consider w as parameter. Then, if $w \neq 0$, the singularity defined by this equation is an ordinary triple point.*

Proof. The equation $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$ is a special one of the equation $F = 0$ in the case (ii-α) in the proof of Proposition 1.1 if $w \neq 0$. Hence it defines an ordinary triple point around $(0, 0, 0, w)$ with $w \neq 0$. □

Because of Proposition 2.1, the singularity $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$ might be considerd as a *degenerate* ordinary triple point.

Proposition 2.2. *Let $v : \mathrm{P}^2(\mathbf{C}) \to \mathrm{P}^5(\mathbf{C})$ be the Veronese embedding of degree 2, namely, the map defined by*

$$\begin{aligned}
&(\xi_0 : \xi_1 : \xi_2) \in \mathrm{P}^2(\mathbf{C})\\
&\quad \to (\xi_0^2 : \xi_1^2 : \xi_2^2 : \xi_0\xi_1 : \xi_0\xi_2 : \xi_1\xi_2)\\
&\quad = (x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathrm{P}^5(\mathbf{C}),
\end{aligned}$$

and let $p : \mathrm{P}^5(\mathbf{C}) \to \mathrm{P}^3(\mathbf{C})$ be the linear projection defined by

$$\begin{aligned}
&(x_0 : x_1 : x_2 : y_0 : y_1 : y_2) \in \mathrm{P}^5(\mathbf{C})\\
&\quad \to (y_0 : y_1 : y_2 : -(x_0 + x_1 + x_2))\\
&\quad = (x : y : z : w) \in \mathrm{P}^3(\mathbf{C}).
\end{aligned}$$

Then the hypersurface in $\mathrm{P}^3(\mathbf{C})$ defined by the equation $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$ coincides with $(p \circ v)(\mathrm{P}^2(\mathbf{C}))$, which is an algebraic surface with ordinary singularities, known as the Steiner surface.

The proof of this proposition is a direct calculation.

Theorem 2.3. *The normalization of the singularity defined by the equation* (vi) *in Proposition* 1.1 *at the origin of $\mathbf{C}^4$ is an isolated rational quadruple point, which is rigid under small deformations.*

Proof. We denote by S the Steiner surface, i.e., the projective variety in $\mathrm{P}^3(\mathbf{C})$ defined by the equation $(xy)^2 + (yz)^2 + (zx)^2 + xyzw = 0$, and by C_S the affine variety in $\mathbf{C}^4$ defined by the same equation, i.e., the cone over S. We denote by X the image of $\mathrm{P}^2(\mathbf{C})$ in $\mathrm{P}^5(\mathbf{C})$ by the Veronese embedding of degree 2, and by C_X the affine variety in $\mathbf{C}^6$ corresponding to X, i.e., the cone over X. Note that C_X is non-singular outside the origin of $\mathbf{C}^6$, since X is non-singular. We denote by $\overline{p} : \mathbf{C}^6 \to \mathbf{C}^4$ the linear projection induced by $p : \mathrm{P}^5(\mathbf{C}) \to \mathrm{P}^3(\mathbf{C})$ in Proposition 2.2. Since $S = p(X)$, we have $\overline{p}(C_X) = C_S$. We denote by $n : C_X \to C_S$ the restriction $\overline{p}$ to C_X. Since

$$\mathcal{O}_X(\nu) := \mathcal{O}_X([H_{\mathrm{P}^5(\mathbf{C})}]^{\otimes\nu}) \simeq \mathcal{O}_{\mathrm{P}^2(\mathbf{C})}([H_{\mathrm{P}^2(\mathbf{C})}]^{\otimes 2\nu}),$$

the map $H^0(\mathrm{P}^5(\mathbf{C}), \mathcal{O}_{\mathrm{P}^5(\mathbf{C})}(\nu)) \to H^0(X, \mathcal{O}_X(\nu))$ is surjective for every integer ν, where $[H_{\mathrm{P}^5(\mathbf{C})}]$ and $[H_{\mathrm{P}^2(\mathbf{C})}]$ denote the hyperplane line bundles on $\mathrm{P}^5(\mathbf{C})$ and $\mathrm{P}^2(\mathbf{C})$, respectively. Therefore X is *projectively normal*, and equivalently C_X is normal (cf. [3]). Hence $n : C_X \to C_S$ gives the normalization.

To see that C_X has a rational isolated singularity, we take the blowing-up $\hat{b} : \widehat{\mathbf{C}^6} \to \mathbf{C}^6$ at the origin of $\mathbf{C}^6$. We put $\widehat{C_X} := \hat{b}^{-1}(C_X)$, the proper inverse image of C_X by $\hat{b}$, and denote by $b : \widehat{C_X} \to C_X$ the restriction of $\hat{b}$ to $\widehat{C_X}$. Here we should remember that $\widehat{\mathbf{C}^6}$ can be identified with $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}$, $\widehat{C_X}$ with $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}_{|X}$, the restriction of $[H_{\mathrm{P}^5(\mathbf{C})}]^{-1}$ to X, and $b^{-1}(0)$ with the zero cross-section of the line bundle $L := [H_{\mathrm{P}^5(\mathbf{C})}]^{-1}_{|X} \to X$. By these identifications, for any open neighborhood U of $b^{-1}(0)$ in $\widehat{C_X}$, we have

$$\begin{aligned} H^q(U, \mathcal{O}_U) &\simeq \bigoplus_{\nu \geq o} H^q(X, L^{-\nu}) \\ &\simeq \bigoplus_{\nu \geq o} H^q(\mathrm{P}^2(\mathbf{C}), \mathcal{O}_{\mathrm{P}^2(\mathbf{C})}(2\nu)) = 0 \end{aligned}$$

for any $q \geq 1$. Hence $(R^q b_* \mathcal{O}_{\widehat{C_X}})_0 = 0$ for any $q \geq 1$, that is, $(C_X, 0)$ is a rational isolated singularity. The multiplicity of the affine cone C_X at the vertex 0 is four, because it is equal to the degree of X in $\mathrm{P}^4(\mathbf{C})$ ([1], p. 394, Exercise 3.4, (e)). We now refer to the following theorem due to M. Schlessinger:

Theorem ([3]). *The cone over a strongly rigid projective manifold is rigid under small deformations.*

Here, a projective manifold $Y \subset \mathrm{P}^n(\mathbf{C})$, $\dim_{\mathbf{C}} Y > 0$, is defined to be *strongly rigid* if
(i) Y is projectively normal,
(ii) $H^1(Y, \Theta_Y(\nu)) = 0$, $-\infty < \nu < \infty$,
(iii) $H^1(Y, \mathcal{O}_Y(\nu)) = 0$, $-\infty < \nu < \infty$,
where Θ_Y and $\mathcal{O}_Y$ denote the sheaves of holomorphic vector fields and holomorphic functions on Y, respectively, and $F(\nu)$ a sheaf F tensored with ν-th power of hyperplane line bundle. The fact that C_X is rigid under small deformations follows from the theorem above and Bott's theorem concerning the cohomology $H^p(\mathrm{P}^n(\mathbf{C}), \Omega^q_{\mathrm{P}^n(\mathbf{C})}(\nu))$ where $\Omega^q_{\mathrm{P}^n(\mathbf{C})}$ is the sheaf of holomorphic q-forms on $\mathrm{P}^n(\mathbf{C})$, since

$$\begin{aligned} H^1(X, \Theta_X(\nu)) &\simeq H^1(\mathrm{P}^2(\mathbf{C}), \Theta_{\mathrm{P}^2(\mathbf{C})}(2\nu)) \\ &\simeq H^1(\mathrm{P}^2(\mathbf{C}), \Omega^1_{\mathrm{P}^2(\mathbf{C})}(-2\nu-3)), \quad \text{and} \\ H^1(X, \mathcal{O}_X(\nu)) &\simeq H^1(\mathrm{P}^2(\mathbf{C}), \mathcal{O}_{\mathrm{P}^2(\mathbf{C})}(2\nu)). \end{aligned}$$

□

Corollary 2.4. *The normalization of the hypersurface in* $\mathrm{P}^4(\mathbf{C})$ *defined by the equation* (1.1) *has isolated rational quadruple points only as singularities. These isolated rational singular points are rigid under small deformations.*

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