# Global existence of solutions to the generalized Proudman-Johnson equation

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**Abstract:** We consider the equation $f_{xxt} + ff_{xxx} - af_xf_{xx} = \nu f_{xxxx}$, $x \in (0,1)$, $t > 0$, where $a \in \mathbf{R}$ is a constant, with the periodic boundary condition. We show that any solution exists globally in time if $-3 \leq a \leq 1$.

**Key words:** Proudman-Johnson equation; global existence.

**1. Introduction.** We consider the generalized Proudman-Johnson equation proposed by [7]. It is an equation for $f = f(x,t)$ and is written as

$$f_{xxt} + ff_{xxx} - af_xf_{xx} = \nu f_{xxxx}, \quad 0 < t, \quad x \in (0,1). \tag{1.1}$$

Here $\nu > 0$ is a constant called viscosity, $t$ the time variable, $x$ the space, and subscripts stand for differentiation. For the sake of simplicity we only consider it with the periodic boundary condition. Also, $\int_0^1 f(x,t)dx \equiv 0$ is assumed. It is possible to make $\nu$ be unity by a suitable change of scales. But we do not employ this and leave $\nu$ as it is.

Equation (1.1) with $a = -(m-3)/(m-1)$ is derived from the Navier-Stokes equations for incompressible viscous fluid in $\mathbf{R}^m$ by assuming a special similarity form on the velocity field; see [7] and the references therein. The case of $m = 2$ was considered by Proudman and Johnson [6], whence (1.1) with $a = 1$ is now called the Proudman-Johnson equation.

Based on their numerical experiments, Okamoto and Zhu [7] suggested that the solution of (1.1) exists globally in time if $a_0 \leq a \leq 1$ and that some solutions may blow up in finite time if $a < a_0$ or $1 < a$. (They were actually unable to determine $a_0$, the lower limit of the global existence.) As for the global existence, they could prove it mathematically only in the case where $a = 0, -2$, and $a = -1/(2k)$ $(k = 1, 2, 3, \cdots)$. In our previous paper [1] we proved that the conjecture was true for $a = 1$. The purpose of the present paper is to prove the conjecture in the case where $-3 \leq a < 1$:

**Theorem 1.** *Suppose that $-3 \leq a \leq 1$. Then any solution exists for all $t \in [0, \infty)$.*

This theorem is not rigorously stated in that it does not specify the class of solutions but this will be clear after we have explained a local-existence theorem in the next section.

**Remark 1.1.** [7] suggested $a_0 \sim -3$ but Fig. 6 in [7] is misleading because it suggests $a_0 > -3$, though the figure shows the case of a different boundary condition.

**Remark 1.2.** Based on the result of [7], we believe that, if $a < -3$ or $1 < a$, solutions with large initial data blow up in finite time, while solutions with small initial date exist globally in time. We are, however, unable to prove this.

**2. Proof of the Theorem.** Proof is carried out separately in the cases of $-3 \leq a < -1$, $-1 \leq a < 0$, and $0 < a < 1$. The global existence in the case of $a = 0$ is known in [7]. The global existence in this case is a consequence of the fact that the maximum principle holds for $f_{xx}$. Accordingly $\|f_{xx}(t)\|_\infty \leq \|f_{xx}(0)\|_\infty$ holds true. (Hereafter $\|\ \|_p$ denote the norm of $L^p(0,1)$ $(1 \leq p \leq \infty)$ and $g(\cdot,t)$, which is regarded as a function of $x$ only and $t$ is regarded as a parameter, is denoted by $g(t)$.) In the case where $a \neq 0$, we will derive similar but different a priori estimates to prove the global existence.

To begin with, we remark the following local existence theorem:

**Theorem 2.** *Let $a$ be any real number. For all $g \in L^2(0,1)$ satisfying $\int_0^1 g(x)dx = 0$, there exists a $T > 0$ such that (1.1) has a unique solution in $0 \leq t \leq T$ satisfying the periodic boundary condition and $f_x(x,0) = g(x)$.*

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Accordingly, the global existence holds true if we have shown that $\|f_x(t)\|_2$ is bounded in $0 \le t \le T$ for any $T > 0$. The proof of Theorem 2 is given in section 3 and we are now going to derive a priori estimates.

**2.1. The case of $0 < a < 1$.** We differentiate (1.1) to obtain

$$f_{xxxt} + ff^{(4)} + (1-a)f_x f_{xxx} - a f_{xx}^2 = \nu f^{(5)}. \tag{2.1}$$

We define $\Phi(u)$ by

$$\Phi(u) = \begin{cases} |u|^{1/(1-a)} & (u < 0) \\ 0 & (0 \le u). \end{cases}$$

By (2.1), we obtain

$$\frac{d}{dt}\int_0^1 \Phi(f_{xxx})dx = \int_0^1 \Phi'(f_{xxx})\big(af_{xx}^2 + \nu f^{(5)}\big)dx - \int_0^1 \Phi'(f_{xxx})\big(ff^{(4)} + (1-a)f_x f_{xxx}\big)dx.$$

It holds that $\int_0^1 f_{xx}^2 \Phi'(f_{xxx})dx \le 0$, since $\Phi$ is a monotone decreasing function. Further

$$\int_0^1 f^{(5)}\Phi'(f_{xxx})dx = -\int_0^1 \big(f^{(4)}\big)^2 \Phi''(f_{xxx})dx \le 0,$$

since $\Phi$ is a convex function. Finally we have

$$\int_0^1 ff^{(4)}\Phi'(f_{xxx})dx = -\int_0^1 f_x \Phi(f_{xxx})dx.$$

Since $\Phi(u) = (1-a)u\Phi'(u)$, we obtain

$$\frac{d}{dt}\int_0^1 \Phi(f_{xxx})dx \le 0.$$

This implies that

$$\int_{\{f_{xxx}<0\}} |f_{xxx}(x,t)|^{1/(1-a)}\, dx \le c,$$

where $c$ is a constant independent of $t$. Hereafter $c$ denotes a positive constant which is independent of $t$ but may be different in different contexts. By Hölder's inequality, we obtain

$$\int_{\{f_{xxx}<0\}} |f_{xxx}(x,t)|\, dx \le c.$$

Since

$$0 = \int_0^1 f_{xxx}dx = \int_{\{f_{xxx}>0\}} f_{xxx}dx + \int_{\{f_{xxx}<0\}} f_{xxx}dx,$$

we conclude that

$$\int_0^1 |f_{xxx}|\, dx \le c,$$

whence

$$\max_{0\le x\le 1} |f_{xx}(x,t)| \le c.$$

This a priori estimate and the local existence theorem guarantee the global existence.

**2.2. The case of $-1 \le a < 0$.** We define $\Phi(u)$ by $\Phi(u) = |u|^{-1/a}$. Suppose for the moment that $-1 < a < 0$ and compute

$$\frac{d}{dt}\int_0^1 \Phi(f_{xx})dx = \nu\int_0^1 \Phi'(f_{xx})f_{xxxx}dx + \int_0^1 \Phi'(f_{xx})(af_x f_{xx} - ff_{xxx})dx.$$

Integrating by parts, we have

$$\int_0^1 \Phi'(f_{xx})ff_{xxx}dx = -\int_0^1 f_x\Phi(f_{xx})dx.$$

Since $\Phi(u) = -au\Phi'(u)$, it follows that

$$\frac{d}{dt}\int_0^1 \Phi(f_{xx})dx = \nu\int_0^1 \Phi'(f_{xx})f_{xxxx}dx = -\int_0^1 \Phi''(f_{xx})f_{xxx}^2 dx \le 0.$$

(Here $\Phi''$ appears and we have to assume that $-1 < a$.) We therefore obtain the following bound:

$$\int_0^1 |f_{xx}(x,t)|^{-1/a}\, dx \le \int_0^1 |f_{xx}(x,0)|^{-1/a}\, dx. \tag{2.2}$$

Since the solutions depend continuously on $a$, this inequality holds for $a = -1$, too. (2.2) implies that

$$\max_{0\le x\le 1} |f_x(x,t)| \le c,$$

and the global existence follows.

**2.3. The case of $-3 \le a < -1$.** We first consider the case of $a = -3$. If we differentiate the Burgers equation $f_t + ff_x = \nu f_{xx}$ twice, we then obtain (1.1) with $a = -3$. The global existence follows from that of the Burgers equation.

We next consider the case where $-3 < a < -1$. In this case we integrate (1.1) to obtain

$$f_{xt} + ff_{xx} - \frac{1+a}{2}f_x^2 = \nu f_{xxx} + \gamma(t), \tag{2.3}$$

where $\gamma(t)$ depends only on $t$. Integrating this equation in $0 < x < 1$, we see that

$$\gamma(t) = -\frac{3+a}{2}\int_0^1 f_x^2 dx \le 0.$$

Define $\Phi(u)$ by

$$\Phi(u) = \begin{cases} 0 & (u \leq 0) \\ u^{-2/(1+a)} & (0 < u). \end{cases}$$

We now have

$$\frac{d}{dt}\int_0^1 \Phi(f_x)dx = \int_0^1 \Phi'(f_x)\Big(-ff_{xx} + \frac{1+a}{2}f_x^2\Big)dx + \int_0^1 \Phi'(f_x)\big(\nu f_{xxx} + \gamma(t)\big)dx.$$

Note that

$$\gamma(t)\int_0^1 \Phi'(f_x)dx \leq 0$$

and

$$\int_0^1 \Phi'(f_x)f_{xxx}dx = -\int_0^1 \Phi''(f_x)f_{xx}^2 dx \leq 0,$$

since $\Phi$ is monotone increasing and convex. Further,

$$\int_0^1 ff_{xx}\Phi'(f_x)dx = -\int_0^1 f_x\Phi(f_x)dx.$$

Since $\Phi(u) = -(1+a/2)u\Phi'(u)$, we obtain

$$\int_0^1 \Phi(f_x(x,t))dx \leq \int_0^1 \Phi(f_x(x,0))dx.$$

By the same argument in section 2.1, we obtain

$$\text{(2.4)} \qquad \int_0^1 |f_x(x,t)|dx \leq c.$$

This inequality, however, is insufficient for our purpose. Accordingly, we return to (1.1): we multiply it by $f$ and integrate by parts to obtain

$$\frac{d}{dt}\|f_x(t)\|_2^2 = (2+a)\int_0^1 f_x^3 dx - 2\nu\|f_{xx}(t)\|_2^2 \leq c\|f_x(t)\|_1\|f_x(t)\|_\infty^2 - 2\nu\|f_{xx}(t)\|_2^2.$$

By the Gagliardo-Nirenberg theorem (see e.g., [2, 3]) it holds that

$$\|f_x(t)\|_2 \leq c\|f_x(t)\|_1^{2/3}\|f_{xx}(t)\|_2^{1/3}.$$

Also, the following inequality is well-known:

$$\|f_x(t)\|_\infty \leq c\|f_x(t)\|_2^{1/2}\|f_{xx}(t)\|_2^{1/2}.$$

By these inequalities we have

$$\|f_x(t)\|_\infty^2 \leq c\|f_{xx}(t)\|_2^{4/3}.$$

Here use has been made of (2.4). We therefore have

$$\frac{d}{dt}\|f_x(t)\|_2^2 \leq c\|f_{xx}(t)\|_2^{4/3} - 2\nu\|f_{xx}(t)\|_2^2 \leq c\left(\frac{\delta^p}{p}\|f_{xx}(t)\|_2^{4p/3} + \frac{1}{\delta^q q}\right) - 2\nu\|f_{xx}(t)\|_2^2,$$

where $\delta > 0$, $p > 1$, $q > 1$ and $1/p + 1/q = 1$. Taking $p = 3/2$ and $\delta^p = 2p\nu/c$, we obtain

$$\frac{d}{dt}\|f_x(t)\|_2^2 \leq c.$$

Consequently $\|f_x(t)\|_2$ is bounded in any bounded interval of $t$, which, together with the local existence theorem, gives us the global existence.

**3. Local existence.** Let

$$X = \left\{g \in L^2(0,1)\ ;\ \int_0^1 g(x)dx = 0\right\}.$$

Then we show in this section that the equation (1.1) has a unique solution such that $f_x \in C^0([0,T];X)$ for some $T > 0$.

We start with (2.3) for an arbitrary $a$. It can be written as

$$\text{(3.1)} \qquad u_t - \nu u_{xx} = -(fu)_x + \frac{3+a}{2}\left(u^2 - \|u(t)\|_2^2\right),$$

$$\text{(3.2)} \qquad f_x = u, \qquad f, u \in X,$$

where $u(t) \equiv u(\cdot,t)$ is viewed as a function : $[0,T] \to X$. By the Duhamel principle, we can rewrite (3.1) as follows:

$$\text{(3.3)} \qquad u(t) = e^{-tA}u(0) + \int_0^t e^{-(t-s)A}F(u(s))ds,$$

where we have defined $A$ as $-\nu(d^2/dx^2)$ with the periodic boundary condition, and $F$ is defined as

$$F(u) = -(fu)_x + \frac{3+a}{2}P(u^2)$$

with $Py(x) = y(x) - \int_0^1 y(\xi)d\xi$. We now prove that the integral equation (3.3) has a unique solution in $C^0([0,T];X)$ for a small $T > 0$. Note that "(3.1) & (3.2) $\Longleftrightarrow$ (3.3)" can be verified in a standard way (see [4, 5], or [8]).

In order to construct a solution of (3.3), we choose an arbitrary $g \in X$ and fix it. We then define an operator $K$ by

$$Ku(t) = e^{-tA}g + \int_0^t e^{-(t-s)A}F(u(s))ds.$$

The existence is proved by showing that the mapping $u \mapsto Ku$ is a contraction mapping of $u$ in $\mathbf{Y}$, where $\mathbf{Y}$ is a closed convex subset of $C^0([0,T];X)$ defined as

$$\mathbf{Y} = \Big\{ u \in C^0([0,T];X)\,; \qquad u(0) = g, \quad \max_{0\le t\le T} \|u(t)\|_2 \le 2\|g\|_2 \Big\}.$$

To this end we first recall that $-A$ is a generator of a contraction semigroup in $X$, i.e., $\|e^{-tA}u\|_2 \le \|u\|_2$. We then follow the standard argument such as in [4, 5, 8].

Suppose that we are given a $u \in \mathbf{Y}$. Then (3.2) defines $f \in C^0([0,T]; W^{1,2}(0,1) \cap X)$, where $W^{1,2}$ denotes the Sobolev space. It is easy to see that

$$\|A^{-1/2}(fu)_x\|_2 = \|fu\|_2 \le c\|u\|_2^2$$

and

$$\|A^{-1/2}P(u^2)\|_2 \le c\|u\|_2^2.$$

We now rewrite $Ku$ as

$$Ku(t) = e^{-tA}g + \int_0^t A^{1/2}e^{-(t-s)A}A^{-1/2}F(u(s))ds,$$

which gives us

$$\|Ku(t)\|_2 \le \|g\|_2 + c\int_0^t (t-s)^{-1/2}\|u(s)\|_2^2 ds. \tag{3.4}$$

This and a similar inequality for $Ku(t) - Ku(t')$ show that $Ku \in C^0([0,T];X)$. Note next that $u \in \mathbf{Y}$ and (3.4) imply that

$$\|Ku(t)\|_2 \le \|g\|_2 + 8c\|g\|_2^2\sqrt{t}.$$

Consequently, $K$ sends $\mathbf{Y}$ into itself if $8c\|g\|_2\sqrt{T} \le 1$. We now fix such a $T$.

Note finally that, for any $u \in \mathbf{Y}$ and $v \in \mathbf{Y}$, we have

$$\|A^{-1/2}\left(F(u) - F(v)\right)\|_2 \le c\left(\|u\|_2 + \|v\|_2\right)\|u - v\|_2.$$

This inequality yields

$$\|Ku(t) - Kv(t)\|_2 \le c\|g\|_2 \int_0^t (t-s)^{-1/2}\|u(s) - v(s)\|_2 ds$$

It then follows that $K$ is a contraction mapping from $\mathbf{Y}$ to itself, if $T$ is sufficiently small.

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