On certain generalizations of the Hardy inequality

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Abstract: In this paper, some new generalizations of Carleman's inequality and Hardy's inequalities on the constant e are considered.

Key words: Carleman's inequality; Hardy's inequality; monotonicity.

1. Introduction. It is well known that the constant e plays an important role in many areas of mathematics. It is involved in many inequalities, identities, series expansions, some special functions. The well known Hardy's inequality and Carleman's inequality are good examples of applications of approximation of e (see [2, 4, Theorem 334, 349]). Recently, there have been many results in strengthened the above mentioned two inequalities by using better approximations of e. In [7], Yang obtained the following result

$$\left(\frac{1+x}{x}\right)^x = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+x)^k}\right), \tag{1.1}$$

where $x > 0$, $b_k > 0$, $k = 1, 2, \dots$, and $\{b_k\}_{k=1}^{n}$ satisfy the following recursion formula: $b_1 = 1/2$, $b_{n+1} = 1/(n+1)[1/(n+2) - \sum_{j=1}^{n} b_j/(n+2-j)]$, $n = 1, 2, \dots$.

As an application of (1.1), he proved the following strengthened Hardy inequality and Carleman inequality:

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{1/\Lambda_n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{m}\frac{b_k}{(1+\Lambda_n/\lambda_n)^k}\right)\lambda_n a_n, \tag{1.2}$$

where $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n}\lambda_m$, $a_n \ge 0$, $(m, n \in N)$, $b_1 = 1/2$, $b_k > 0$, $k \ge 2$, and $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$.

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{1/n} < e\sum_{n=1}^{\infty}\left(1-\sum_{k=1}^{m}\frac{b_k}{(1+n)^k}\right)a_n, \tag{1.3}$$

where $a_n \ge 0$, $b_1 = 1/2$, $b_k > 0$, $k \ge 2$, and $0 < \sum_{n=1}^{\infty}\lambda_n a_n < \infty$ $(m, n \in N)$.

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In this paper, we give a extension of the strengthened Hardy inequality (1.2) and Carleman inequality (1.3) by using the strict monotonicity of the power mean of n distinct positive numbers.

2. Main results. For any positive values $a_1, a_2, \dots, a_n$ and positive weights $\alpha_1, \alpha_2 \dots, \alpha_n$, $\sum_{i=1}^{n}\alpha_i = 1$, and any real $p \ne 0$, we defined the power mean, or the mean of order p of the value a with weights α by

$$M_p(a;\alpha) = M_p(a_1, a_2, \dots, a_n; \alpha_1, \alpha_2, \dots, \alpha_n) = \left(\sum_{i=1}^{n}\alpha_i a_i^p\right)^{1/p}.$$

An easy application of L'Hospital's rule shows that

$$\lim_{p\to 0} M_p(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i},$$

the geometric mean. Accordingly, we define $M_0(a;\alpha) = \prod_{i=1}^{n} a_i^{\alpha_i}$. It is well known that $M_p(a;\alpha)$ is a nondecreasing function of p for $-\infty \le p \le \infty$, and is strictly increasing unless all the a_i are equal (cf. [1, 5]). Before we state and prove the main theorem, we need the following Lemmas:

Lemma 2.1. *Let $a_1, a_2, \dots, a_n$ $(n \ge 1)$ be a set of n nonnegative quantities, $x > 0$ and let $y \ge 0$, then*

$$\left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^{(x+y)} \le \left(\sum_{i=1}^{n}\alpha_i a_i^x\right)^{(x+y)/x} \le \sum_{i=1}^{n}\alpha_i a_i^{(x+y)} \tag{2.1}$$

with equality holding if and only if all a_i are same.

Proof. Observe that $M_p(A_1, A_2, \dots, A_n; \alpha_1, \alpha_2 \dots, \alpha_n)$ is a continuous strictly increasing function of p; that is

$$\prod_{i=1}^{n} A_i^{\alpha_i} \le \sum_{i=1}^{n} \alpha_i A_i \le \left(\sum_{i=1}^{n} \alpha_i A_i^p\right)^{1/p}$$

for $1 \le p$ with equality holding if and only if all A_i are the same. This is equivalent to

$$\left(\prod_{i=1}^{n} A_i^{\alpha_i}\right)^p \le \left(\sum_{i=1}^{n} \alpha_i A_i\right)^p \le \left(\sum_{i=1}^{n} \alpha_i A_i^p\right) \tag{2.2}$$

for $1 \le p$ with equality holding if and only if all A_i are the same. Let $x > 0$, $y \ge 0$. Then (2.2) can be restated as

$$\left(\prod_{i=1}^{n} A_i^{\alpha_i}\right)^{(x+y)/x} \le \left(\sum_{i=1}^{n} \alpha_i A_i\right)^{(x+y)/x} \le \sum_{i=1}^{n} \alpha_i A_i^{(x+y)/x} \tag{2.3}$$

for $1 \le (x+y)/x$ with all equalities holding if and only if all A_i are same. Let $a_1, a_2, \ldots, a_n \ge 0$, $x > 0$, then $a_i^x \ge 0$. In (2.3), substituting $A_i = a_i^x$, we obtain the inequalities (2.1) for $x > 0$ and $y \ge 0$. □

Lemma 2.2. *Let $a_1, a_2, \ldots, a_n$ $(n \ge 1)$ be a set of n nonnegative quantities, $0 < x$ and let $y \le -2x$, then*

$$\left(\prod_{i=1}^{n} a_i^{\alpha_i}\right)^{(x+y)} \le \left(\sum_{i=1}^{n} \alpha_i a_i^{-x}\right)^{-(x+y)/x} \le \sum_{i=1}^{n} a_i^{(x+y)}$$

with all equalities holding if and only if all a_i are same.

Proof. Observe that $M_p(A;\alpha)$ is a continuous strictly increasing function of p, we get

$$\left(\prod_{i=1}^{n} A_i^{\alpha_i}\right)^p \le \left(\sum_{i=1}^{n} \alpha_i A_i^{-1}\right)^{-p} \le \sum_{i=1}^{n} \alpha_i A_i^p$$

for $p \le -1$ with equality holding if and only if all a_i are the same. The rest of the proof can be completed by following the same steps as in the proof of Lemma 2.1 with suitable changes and hence we omit the details. □

Our main results are given in the following theorems.

Theorem 2.3. *If $0 < \lambda_{n+1} \le \lambda_n$, $\Lambda_n = \sum_{m=1}^{n} \lambda_m$ $(\Lambda_n \ge 1)$, $a_n \ge 0$, $c_n > 0$ $(n \in N)$, $x > 0$, $y \ge 0$, and $0 < \sum_{n=1}^{\infty} \lambda_n (a_n)^x < \infty$, then*

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{(x+y)/\Lambda_n} &< \frac{x+y}{x} \sum_{n=1}^{\infty} \left[e\left(1 - \sum_{k=1}^{s} \frac{b_k}{(1+\Lambda_n/\lambda_n)^k}\right)\right]^{x/(x+y)} \\ &\quad \times \lambda_n a_n^x \Lambda_n^{-y/(x+y)} \left(\sum_{t=1}^{n} \lambda_t (c_t a_t)^x\right)^{y/x},\end{aligned} \tag{2.4}$$

where $b_k > 0$, $k = 1, 2, \ldots, s$ $(s \in N)$ and $\{b_k\}_{k=1}^{s}$ satisfy the following recursion formula: $b_1 = 1/2$, $b_{n+1} = 1/(n+1)[1/(n+2) - \sum_{j=1}^{n} b_j/(n+2-j)]$, $n = 1, 2, \ldots$.

Proof. From the hypotheses, using Lemma 2.1, we have

$$(q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_n^{\alpha_n})^{x+y} \le \left(\sum_{m=1}^{n} \alpha_m q_m^x\right)^{(x+y)/x},$$

where $q_m \ge 0$, $\alpha_m > 0$, $(m = 1, 2, \ldots, n)$, $\sum_{m=1}^{n} \alpha_m = 1$. Setting $c_m > 0$, $q_m = c_m a_m$, and $\alpha_m = \lambda_m/\Lambda_n$, we obtain

$$\left((c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}\right)^{x+y} \le \left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^x\right)^{(x+y)/x}.$$

Using the above inequality, (for $\Lambda_n \ge 1$) we find that

$$\begin{aligned}&\sum_{n=1}^{\infty} \lambda_{n+1} (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{(x+y)/\Lambda_n} \\ &= \sum_{n=1}^{\infty} \lambda_{n+1} \times \left(\frac{(c_1 a_1)^{\lambda_1/\Lambda_n} (c_2 a_2)^{\lambda_2/\Lambda_n} \cdots (c_n a_n)^{\lambda_n/\Lambda_n}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{1/\Lambda_n}}\right)^{(x+y)} \\ &\le \sum_{n=1}^{\infty} \left[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{(x+y)/\Lambda_n}}\right] \times \left(\frac{1}{\Lambda_n} \sum_{m=1}^{n} \lambda_m (c_m a_m)^x\right)^{(x+y)/x} \\ &\le \sum_{n=1}^{\infty} \left[\frac{\lambda_{n+1}}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{(x+y)/\Lambda_n}}\right] \frac{1}{\Lambda_n} \times \left(\sum_{m=1}^{n} \lambda_m (c_m a_m)^x\right)^{(x+y)/x}.\end{aligned} \tag{2.5}$$

By using the following inequality (see [3, 6]),

$$\left(\sum_{m=1}^{n} z_m\right)^t \le t \sum_{m=1}^{n} z_m \left(\sum_{k=1}^{m} z_k\right)^{t-1},$$

where $t \ge 1$ is a constant and $z_m \ge 0$, $(m = 1, 2, \ldots)$, it is easy to observe that

$$(2.6)\quad \left(\sum_{m=1}^{n}\lambda_m(c_m a_m)^x\right)^{(x+y)/x} \le \frac{x+y}{x}\sum_{m=1}^{n}\lambda_m(c_m a_m)^x\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^x\right)^{y/x}.$$

Choosing $c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n} = (\Lambda_{n+1})^{\Lambda_n/(x+y)}$ $(n\in N)$, and setting $\Lambda_0 = 0$, because $\lambda_{n+1}\le\lambda_n$, we have

$$(2.7)\quad c_n = \left(1+\frac{\lambda_{n+1}}{\Lambda_n}\right)^{\Lambda_n/\lambda_n(x+y)}\cdot\Lambda_n^{1/(x+y)} \le \left(1+\frac{1}{\Lambda_n/\lambda_n}\right)^{\Lambda_n/\lambda_n(x+y)}\cdot\Lambda_n^{1/(x+y)}.$$

Then by (2.5), (2.6) and (2.7), we obtain that

$$\begin{aligned}(2.8)\quad &\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{(x+y)/\Lambda_n}\\ &\le \sum_{n=1}^{\infty}\frac{\lambda_{n+1}}{\Lambda_{n+1}}\frac{1}{\Lambda_n}\cdot\frac{x+y}{x}\times\sum_{m=1}^{n}\lambda_m(c_m a_m)^x\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^x\right)^{y/x}\\ &= \frac{x+y}{x}\sum_{m=1}^{\infty}\lambda_m(c_m a_m)^x\sum_{n=m}^{\infty}\frac{\lambda_{n+1}}{\Lambda_n\Lambda_{n+1}}\times\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^x\right)^{y/x}\\ &= \frac{x+y}{x}\sum_{m=1}^{\infty}\lambda_m(c_m a_m)^x\sum_{n=m}^{\infty}\left(\frac{1}{\Lambda_n}-\frac{1}{\Lambda_{n+1}}\right)\times\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^x\right)^{y/x}\\ &= \frac{x+y}{x}\sum_{m=1}^{\infty}\lambda_m(c_m a_m)^x\frac{1}{\Lambda_m}\times\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^x\right)^{y/x}\\ &\le \frac{x+y}{x}\sum_{m=1}^{\infty}\left[\left(1+\frac{1}{\Lambda_m/\lambda_m}\right)^{\Lambda_m/\lambda_m}\right]^{x/(x+y)}\times\lambda_m a_m^x\Lambda^{-y/(x+y)}\left(\sum_{k=1}^{m}\lambda_k(c_k a_k)^x\right)^{y/x}.\end{aligned}$$

From the equality (1.1), we observe that

$$(2.9)\quad \left(1+\frac{1}{\Lambda_m/\lambda_m}\right)^{\Lambda_m/\lambda_m} < e\left(1-\sum_{k=1}^{s}\frac{b_k}{(1+\Lambda_m/\lambda_m)^k}\right),$$

where $b_k>0$, $k=1,2,\dots,s(s\in N)$ and $\{b_k\}_{k=1}^{s}$ satisfy the following recursion formula: $b_1 = 1/2$ and

$$b_{n+1} = \frac{1}{n+1}\left(\frac{1}{n+2}-\sum_{j=1}^{n}\frac{b_j}{n+2-j}\right)$$

for $n=1,2,\dots$. Using (2.9) in (2.8), we get the required inequality (2.4). □

In Theorem 2.3, setting $y=0$, we have the strengthened Hardy's inequality (1.2) with a extension of the inequality (1.3). At the same time, we leads to the following some new inequality similar to the inequality given in Theorem 2.3 by using fairly elementary analysis.

Theorem 2.4. *If* $0<\lambda_{n+1}\le\lambda_n$, $\Lambda_n=\sum_{m=1}^{n}\lambda_m$ $(\Lambda_n\ge1)$, $a_n\ge0, c_n>0$ $(n\in N)$, $x>0$, $y\le -2x$, *and* $0<\sum_{n=1}^{\infty}\lambda_n(a_n)^x<\infty$, *then*

$$\sum_{n=1}^{\infty}\lambda_{n+1}(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{(x+y)/\Lambda_n} < \frac{x+y}{x}\sum_{n=1}^{\infty}\left[e\left(1-\sum_{k=1}^{s}\frac{b_k}{(1+\Lambda_n/\lambda_n)^k}\right)\right]^{-x/(x+y)} \times\lambda_n a_n^{-x}\Lambda^{-(2x+y)/(x+y)}\left(\sum_{t=1}^{n}\lambda_t(c_t a_t)^{-x}\right)^{-(2x+y)/x},$$

where $b_k>0$, $k=1,2,\dots,s$ $(s\in N)$ *and* $\{b_k\}_{k=1}^{s}$ *satisfy the following recursion formula*: $b_1=1/2$, $b_{n+1}=1/(n+1)[1/(n+2)-\sum_{j=1}^{n}b_j/(n+2-j)]$, $n=1,2,\dots$.

Proof. The details of the proof of Theorem 2.4 follows by an argument similar to that in the proof of Theorem 2.3 with suitable changes and hence we omit the details. □

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