"Hasse principle" for symmetric and alternating groups

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1. Notation and results. Extending the usage of language in Galois cohomology we can speak of the Hasse principle for any group G (cf. [1]). We know that the principle holds for $G =$ abelian, dihedral, quaternion, $PSL_2(\mathbf{Z})$, $PSL_2(\mathbf{F}_p)$ and free groups(cf. [1], [2]). The proof in [2] works as well for free groups generated by any set. In this paper, we prove the following

Theorem. *For any natural number n, the symmetric group S_n and the alternating group A_n enjoy the Hasse principle.*

We may assume that $n \geq 4$, since the case $n \leq 3$ are already settled. As is well known $G = S_n,\ A_n$ are generated by two elements: $G = \langle s, t \rangle$. To be more precise,

(1) for $G = S_n$, we have $s = (234\dots n),\ t = (12)$,

(2) for $G = A_n(n \text{ odd}),\ s = (345\dots n),\ t = (123)$,

(3) for $G = A_n(n \text{ even}),\ s = (234\dots n),\ t = (123)$.

Remark. In general, for any group G with two generators s, t let f be a cocycle on G which is normalized at s and locally trivial. The Hasse principle means that f is trivial. From the basic relation $f(st) = f(s)f(t)^s$ with $f(s) = 1$, $f(t) = a^{-1}a^t = a^{-1}tat^{-1}$, $f(st) = b^{-1}b^{st} = b^{-1}stbt^{-1}s^{-1}$, we infer that

$$st \sim sa^{-1}ta, \text{ (conjugacy in } G). \tag{4}$$

Then the Hasse principle will be proved for G if we find c in the centralizer of s so that $a^{-1}ta = c^{-1}tc$ using (4).

2. Proof of the Theorem.

2.1. $G = S_n$. From (1), we have

$$st = (23\dots n)(12) = (13\dots n2)$$

an n-cycle. Hence by (4), $sa^{-1}ta$ is also an n-cycle. If we write

$$a^{-1} = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ i_1 & i_2 & i_3 & \dots & i_n \end{pmatrix} \tag{5}$$

then $sa^{-1}ta = (23\dots n)(i_1 i_2)$. Since this is an n-cycle we have $a^{-1}ta = (i_1 i_2) = (1\ j)$, $j \geq 2$. On the other hand, if we take c so that $c^{-1} = s^{j-2}$, then one verifies easily that $c^{-1}tc = (1\ j)$. In view of the remark, this complete the Proof of the Theorem for $G = S_n$.

2.2. $G = A_n$ (n odd). From (2), we have

$$st = (345\dots n)(123) = (124\dots n3)$$

an n-cycle. Hence, by (4), $sa^{-1}ta$ is also an n-cycle. Write a^{-1} as in (5). Then $sa^{-1}ta = (34\dots n)(i_1 i_2 i_3)$. Since this must be an n-cycle, we have $a^{-1}ta = (i_1 i_2 i_3) = (12j)$ or $= (1j2)$, $j \geq 3$. Here, however, the second 3-cycle $(1j2)$ is impossible. In fact, if we had

$$st = (124\dots n3)$$
$$\sim (34\dots n)(1j2) = (21\,j+1\dots n3\dots j)$$

then we would have $u(st)u^{-1} = (21\,j+1\dots n3\dots j)$ with

$$u = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & n \\ 2 & 1 & j & j+1 & \dots & \dots \end{pmatrix}$$
$$= (12)s^{j-3} \notin A_n.$$

If $u_1(st)u_1^{-1} = u(st)u^{-1}$, then $(u^{-1}u_1)st(u^{-1}u_1)^{-1} = st$. From this equation, we infer that $u^{-1}u_1$ is a power of st. So u_1 is not in A_n. Therefore st and $(34\dots n)(1j2)$ cannot be conjugate in A_n, a contradiction. On the other hand, if we take c so that $c^{-1} = s^{j-3}$, then one verifies that $c^{-1}tc = (12j)$. In view of the remark, this proves the Theorem for A_n (n odd).

2.3. $G = A_n$ (n even). From (3), we have

$$st = (234\dots n)(123) = (13)(245\dots n).$$

If we write a^{-1} as in (5), then $a^{-1}ta = (i_1 i_2 i_3)$ and, by (4), st is conjugate to $sa^{-1}ta = (234\dots n)(i_1 i_2 i_3)$. Since st has no fixed points, we may assume that $(i_1 i_2 i_3) = (1ij)$.

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Similarly from $f(t^2) = f(t) \cdot f(t)^t = a^{-1}t^2at^{-2}$ we have, replacing t by t^2 in the proof of (4)

$$\begin{aligned} st^2 &= (14\ldots n2)(3) \sim sa^{-1}t^2a = (23\ldots n)(1ji) \\ &= (\ldots i \ldots j)(1ji) = (1\ldots i)(j\ldots). \end{aligned}$$

From this, we infer that $(j\ldots)$ must be one cycle and either $j = i+1$ or $i = n,\ j = 2$. On the other hand, if we take c so that $c^{-1} = s^{i-2}$ then one verifies that $c^{-1}tc = (1ij)$. In view of the remark, this proves the Theorem for A_n (n even).

References

[1] T. Ono: "Hasse principle" for $PSL_2(\mathbf{Z})$ and $PSL_2(\mathbf{F}_p)$. Proc. Japan Acad., **74A**, 130–131 (1998).

[2] T. Ono and H. Wada: "Hasse principle" for free groups. Proc. Japan Acad., **75A**, 1–2 (1999).