Valuation of default swap with affine-type hazard rate

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1. Introduction. The aim in this paper is to give implements applicable to valuing "default swap", a kind of financial commodity called credit derivative. Davis and Marvoidis [2], under the assumption that the hazard rate is a Gaussian and independent of the riskless spot rate, evaluated the value of the swap by using forward measure approach and the integral approximation. The Gaussian hazard rate model has, however, an undesirable property that it may become negative, hence, the probability of not-default at some time may be over one. So the author is motivated by the idea that CIR term structure model, for example, must be effective for modeling hazard rate.

The main result concerns the formula which computes the expectation of the special functional of the hazard rate, under the assumption that the hazard rate process follows the so-called affine type model including CIR model. It is proved by using Itô formula and usual calculus.

2. The result. Let $(\Omega, \mathcal{F}, P)$ be a probability space.

Denote by B a one dimensional standard Brownian motion on the above space.

Theorem 1. *Let $T \in (0,\infty)$.*

Let h_t satisfy the following stochastic differential equation (*called the affine-type model*) *on* $[0,T]$.

$$dh_t = m(h_t,t)dt + \sigma(h_t,t)dB_t, \quad h_0 > 0, \tag{1}$$

where m and σ are deterministic functions of the following form:

$$m(x,t) = m_1(t) + m_2(t)x, \sigma(x,t)^2 = \sigma_1(t) + \sigma_2(t)x$$

for deterministic functions $m_i(t), \sigma_i(t)$ $(i = 1,2)$ with $\sigma_2 \neq 0$ and

$$m_1(t) - m_2(t)\sigma_1(t)\sigma_2(t)^{-1} \geq 0, t \in [0,T]. \tag{2}$$

Let β be a nonnegative real number and $\kappa(t)$ be a strictly positive deterministic continuously differentiable function.

Then the following equality holds: *for $t \in [0,T]$,*

$$E[\exp(-\int_0^t \kappa(s)h_s ds - \beta h_t)h_t] = \Phi(t)\frac{G(t) + J(t)h_0}{K(t)}$$

where

$$\Phi(t) = \exp(-a(t) - b(t)h_0),$$
$$G(t) = -\frac{1}{2}\sigma_1(t)b(t)^2 + (b(t) - \beta)m_1(t),$$
$$J(t) = -\frac{1}{2}\sigma_2(t)b(t)^2 + m_2(t)b(t) + \kappa(t),$$
$$K(t) = \kappa(t) + \beta m_2(t) - \frac{1}{2}\beta^2\sigma_2(t),$$

and $a(t), b(t)$ are solutions to the following differential equations:

$$\begin{cases} b'(t) = -\frac{1}{2}\sigma_2(t)b(t)^2 + m_2(t)b(t) + \kappa(t) \\ a'(t) = -\frac{1}{2}\sigma_1(t)b(t)^2 + m_1(t)b(t) \\ a(0) = 0, \ b(0) = \beta. \end{cases} \tag{3}$$

Remark. The condition (2) guarantees the existence of a solution h to the SDE (1) with $h_t \geq -\sigma_1(t)\sigma_2(t)^{-1}$ for all $t \in [0,T]$. (See Duffie [3].) In particular, by assuming $\sigma_1 = 0$, the positive solution is achieved.

To begin with, we state the following crucial proposition without proof.

Proposition 2. *Assume h_t satisfies* (1) *in Theorem* 1.

For any non-negative β and strictly positive function $\kappa(t)$, we have

$$E[\exp(-\beta h_t - \int_0^t \kappa(s)h_s ds)] = \exp(-a(t) - b(t)h_0),$$

where $a(t)$ and $b(t)$ are solutions of (3).

It goes without saying that if $a(t)$ and $b(t)$ have an explicit form as a function of β (see Example 3), the result in the theorem can be easily achieved by differentiating $\exp(-a(t) - b(t)h_0)$ in β. Now we give the proof of Theorem 1 applicable to other general cases.

Proof. Let a and b be solutions to (3). Now

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we define $\Phi(t)$ by

$$\Phi(t) = E[\exp(-\int_0^t \kappa(s)h_s ds - \beta h_t)]$$
$$= \exp(-a(t) - b(t)h_0),$$

Thanks to Itô's formula and usual calculus,

$$\int_0^T (1 + \frac{1}{\kappa(t)}[\beta m_2(t) - \frac{1}{2}\beta^2\sigma_2(t)])I(t)dt$$
$$= \frac{e^{-\beta(0)h_0}}{\kappa(0)} - \frac{1}{\kappa(T)}\Phi(T)$$
$$+ \int_0^T \frac{1}{\kappa(t)^2}[\kappa(t)\{\frac{1}{2}\beta^2\sigma_1(t) - \beta m_1(t)\} - \kappa'(t)]\Phi(t)dt,$$

where $I(t) = E[\exp(-\int_0^t \kappa(s)h_s ds - \beta h_t)h_t]$. By differentiating the both sides in T and noting that $\Phi'(t) = \Phi(t)(-a'(t) - b'(t)h_0)$, we complete the proof. □

Example 3. For a financial application, we give as an example the CIR type:

$$dh_t = (a - bh_t)dt + \sigma\sqrt{h_t}dB_t, \quad h_0 > 0.$$

Let β be a nonnegative real number and κ be a constant with $0 < \kappa < 1$.

Since, in this case, the equations (3) have explicit analytic solutions, it follows that

$$E[\exp(-\int_0^t \kappa h_s ds - \beta(t)h_t)] = \Phi(t),$$
$$E[\exp(-\int_0^t \kappa h_s ds - \beta(t)h_t)h_t]$$
$$= \Phi(t)\Big(\frac{2a(e^{\gamma_\kappa t} - 1)}{\Psi(t)} + \frac{4h_0\gamma_\kappa^2 e^{\gamma_\kappa t}}{\Psi(t)^2}\Big),$$

where

$$\Phi(t) = \exp(-a\phi(t) - h_0\psi(t)),$$
$$\phi(t) = -\frac{2}{\sigma^2}\log\frac{2\gamma_\kappa e^{\frac{\gamma_\kappa + b}{2}t}}{\Psi(t)},$$
$$\psi(t) = \frac{\beta(t)(\gamma_\kappa + b + e^{\gamma_\kappa t}(\gamma_\kappa - b)) + 2\kappa(e^{\gamma_\kappa t} - 1)}{\Psi(t)},$$
$$\Psi(t) = \sigma^2\beta(t)(e^{\gamma_\kappa t} - 1) + \gamma_\kappa - b + e^{\gamma_\kappa t}(\gamma_\kappa + b),$$
$$\gamma_\kappa = \sqrt{b^2 + 2\sigma^2\kappa}.$$

3. Default swap valuation.

3.1. Mathematical model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $0 < T < \infty$ be a finite time horizon.

In view of finance, we regard P as a risk neutral measure, meaning that all the discounted asset prices are martingales under the probability measure P.

Let $(\mathcal{G}_t)_{t\in[0,T]}$ be a two dimensional Brownian filtration.

We fix 2-dimentional $(P, (\mathcal{G}_t)_{t\in[0,T]})$-Brownian motion $(B, \tilde{B})$, that is, $(\mathcal{G}_t)_{t\in[0,T]}$ is the smallest complete, right-continuous filtration to which both B and $\tilde{B}$ are adapted.

Denote by $\tau : \Omega \longrightarrow [0,\infty]$ a random time with continuous distribution. We will regard it as the default time of the issuer of the underlying bond for a swap.

Set $\mathcal{F}_t = \bigcap_{s>t}(\mathcal{G}_s \vee \sigma\{\tau \wedge s\})$.

Then $(\mathcal{F}_t)$ is a right-continuous filtration and τ is an $(\mathcal{F}_t)$-stopping time.

Let r be a nonnegative $(\mathcal{G}_t)$- progressively measurable process.

We assume that r is independent of B, that is, r depends upon only $\tilde{B}$. r is considered as riskless spot rate process. Then for $t, s \in [0,T]$ with $t \le s$,

$$Z(t,s) = E[\exp(-\int_t^s r_u du)|\mathcal{G}_t]$$

is regarded as the price at time t of s-maturity riskless zero-coupon bond.

We assume that there is a nonnegative $(\mathcal{G}_t)$- progressively measurable process h that makes

$$M_t = 1_{\{t\ge\tau\}} - \int_0^t h_s 1_{\{\tau>s\}} ds$$

a $(P, (\mathcal{F}_t))$-martingale. We call this process h the hazard rate process.

Set $H(t) = \exp(-\int_0^t h_s ds)$.

Furthermore we assume that each $(P, (\mathcal{G}_t)_{t\in[0,T]})$-martingale is also $(P, (\mathcal{F}_t)_{t\in[0,T]})$-martingale.

Under the above settings, Duffie et al.([4]) and Kusuoka([6]) show that the following result holds.

Proposition 4. *If $0 \le t < s \le T$ and Z is a $\mathcal{G}_s$-measurable bounded random variable, then*

$$E[Z1_{\{\tau>s\}}|\mathcal{F}_t] = 1_{\{\tau>t\}}E[Z\exp(-\int_t^s h_u du)|\mathcal{G}_t].$$

We will also utilize the next corollaries of the last proposition.

Corollary 5. *For all $t \in [0,T]$,*

$$P(\tau > t) = E[H(t)].$$

Corollary 6. *For any bounded $(\mathcal{G})_{[0,T]}$-*

predictable process Z,

$$E[Z_\tau 1_{\{\tau \le T\}}] = E[\int_0^T Z_t h_t H(t)dt],$$

Moreover we suppose that h_t satisfies the affine type SDE (1): $h_0 > 0$ and

$$dh_t = m(h_t, t)dt + \sigma(h_t, t)dB_t,$$

where B is the Brownian motion independent of $\tilde{B}$, fixed before.

Remark. We immediately see that r and h are independent.

3.2. Formal definition of a default swap. Now we consider the valuation of default swap following the scheme proposed by Davis and Mavroidis [2].

We begin with confirmation of the rule of default swap. It sounds so complicated, but it is indeed not so complicated. It is a contract made between two parties — one is a firm which holds a defaultable bond (called "A") and the other is a bank (called "B"), for example. (We will call "C" the issuer of the bond held by "A".)

a) T is the contract termination date. If the default of "C" occurs before T, the contract is then stopped.
b) "A" have to pay for "B" a fixed premium c_i ($i = 1, \cdots, n$) at each fixed time t_i ($i = 1, \cdots, n$, $0 \le t_1 < \cdots < t_n \le T$). (The fixed side)
c) "B" pays the differential between the notional (we set 1) and some recovered amount from "C" for "A" if the default of "C" occurs before T. Here we define the recovered amount from "C" by $L\times$ (the immediate pre-default value of the bond), where L is a constant with $0 < L < 1$. (The recovery side)
d) The underlying bond that "A" holds allows a fixed coupon b_j ($j = 1, 2, \cdots$) at each fixed time u_j ($j = 1, 2, \cdots$, $0 \le u_1 < u_2 < \cdots$) unless the default happens.

Remark. The ways of defining the recovered amount from the bond issuer are variously considered. Here we follow the way Duffie et al. uses.

3.3. The value of fixed side. The current value of the fixed side is defined by

$$E[\sum_{i=1}^n c_i \exp(-\int_0^{t_i} r_u du) 1_{\{t_i < \tau\}}].$$

By using Corollary 5,

$$\begin{aligned} E[\sum_{i=1}^n c_i \exp(-\int_0^{t_i} r_u du) 1_{\{t_i < \tau\}}] \\ = \sum_{i=1}^n c_i Z(0, t_i) E[H(t)]. \end{aligned}$$

The expectation in right-hand can be calculated by using Proposition 2.

3.4. The value of recovery side. We define

$$Y(t) = E[\sum_{u_i \ge t} b_i \exp(-\int_t^{u_i} \{r_s + (1-L)h_s\}ds) | \mathcal{F}_t].$$

Indeed $Y(t)$ is naturally allowed to be regarded as the cum-coupon value of the underlying defaultable bond. It is thought that the fixed side can recover $LY(\tau)$, where τ is seen as the default time of the bond issuer.

Hence we define the value of the recovery side by

$$\begin{aligned} &E[\exp(-\int_0^\tau r_s ds)(1 - LY(\tau))1_{\{\tau \le T\}}] \\ &= E[\exp(-\int_0^\tau r_s ds)1_{\{\tau \le T\}}] \\ &\quad - LE[\exp(-\int_0^\tau r_s ds) Y(\tau) 1_{\{\tau \le T\}}]. \end{aligned}$$

Using Corollary 6,

$$\begin{aligned} &E[\exp(-\int_0^\tau r_s ds)1_{\{\tau \le T\}}] \\ &= \int_0^T Z(0,t) E[h_t H(t)]dt. \end{aligned} \tag{4}$$

On the other hand, using Corollary 6 once again,

$$\begin{aligned} &E[\exp(-\int_0^\tau r_s ds) Y(\tau) 1_{\{\tau \le T\}}] \\ &= E[\int_0^T \exp(-\int_0^t r_s ds) Y(t) h_t H(t) dt] \\ &= \sum_j b_j Z(0, u_j) \int_0^{T \wedge u_j} F(t, u_j) dt, \end{aligned}$$

where $F(t, u_j)$ stands for

$$\begin{aligned} &E[h_t H(t) \exp(-\int_t^{u_j} h_s ds)^{(1-L)}] \\ &= E[h_t H(t) E[\exp(-\int_t^{u_j} (1-L) h_s ds) | \mathcal{G}_t]] \\ &= E[h_t H(t) \exp(-\tilde{a}(u_j - t) - \tilde{b}(u_j - t) h_t)]. \end{aligned} \tag{5}$$

The second equality is due to the Markov property of h. $\tilde{a}(t)$ and $\tilde{b}(t)$ are solutions of equations similar to (3).

When $\tilde{b}$ is positive on $[0, T]$, it follows that each expectation part in (4) and (5) can be calculated along Theorem 1.

As a result, these lead to the explicit formula for the current values of both the fixed and the recovery side of the default swap in the sense that it only remains to solve some ordinary differential equations. We avoid the full description because of its complication.

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