# On the topology of the moduli space of negative constant scalar curvature metrics on a Haken manifold

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**1. Introduction.** The topology of the space of positive scalar curvature metrics on a closed manifold $M$ has been studied by several authors ([6]). It turns out that the topology of this space is very complicated, and the moduli space of positive scalar curvature metrics quotient by the diffeomorphism group of $M$ can have infinitely many connected components. By contrast, the topology of the space of negative scalar curvature metrics is very simple ([7]).

Let $M$ be a closed connected manifold. Denote by $\mathcal{M}_{-1}(M)$ the set of all Riemannian metrics with scalar curvature $-1$. The diffeomorphism group acts on $\mathcal{M}_{-1}(M)$ by pull-back. In this paper, we will report the topological structure of the moduli space $\mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$, the space of Riemannian metrics with scalar curvature $-1$ devided by the group $\mathrm{Diff}_0(M)$ of diffeomorphisms which are isotopic to the identity map. The result gives a fact that if $M$ is a closed connected Haken manifold with no nontrivial symmetry, then the moduli space $\mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$ is a contractible manifold. Note that this result is an analogue to the contractibility of the Teichmüller space on an oriented surface with negative Euler number ([3], [10]). It seems that there are similarities between Haken manifolds and oriented surfaces with non-positive Euler number.

**2. The space of negative constant scalar curvature metrics.** Let $M$ be a closed $n$-manifold, and $\mathcal{M}(M)$ be the space of all Riemannain metrics on $M$. For $g \in \mathcal{M}(M)$, let $R_g$ denote the scalar curvature of $g$, and $\mathcal{M}_{-1}(M)$ denote the space of Riemannian metrics with scalar curvature $-1$. It is known that if $M$ is a closed $n$-manifold, $n \geq 3$, then $M$ admits a Riemannian metric with scalar curvature $-1$, i.e., $\mathcal{M}_{-1}(M)$ is a non-empty set if $\dim M \geq 3$. We denote by $L^2_k\mathcal{M}(M)$ the space of all $L^2_k$-metrics, where $L^2_k$ is a Sobolev space whose derivatives of order less than or equal to $k$ are all in $L^2$. Then the space $L^2_k\mathcal{M}(M)$ is a Hilbert manifold for $2k > n$. It is known that the space $\mathcal{M}(M)$ is an ILH-manifold in the sense of the inverse limit of Hilbert manifolds: $\mathcal{M}(M) = \lim_{\leftarrow} L^2_k\mathcal{M}(M)$ ([8]).

For $2k > n+2$, let $\mathcal{R} : L^2_k\mathcal{M}(M) \to L^2_{k-2}(M)$ defined by $\mathcal{R}(g) := R_g$ denote the scalar curvature map. The tangent space at $g \in L^2_k\mathcal{M}(M)$ can be identified with the space $L^2_k(M; S^2T^*M)$ of symmetric $(0,2)$-tensor fields of class $L^2_k$. We denote its differential at $g \in L^2_k\mathcal{M}(M)$ by $\beta_g := d\mathcal{R}_g : L^2_k(M; S^2T^*M) \to L^2_{k-2}(M)$.

**Lemma 2.1.** *The differential $\beta_g$ of the scalar curvature map is given by*

$$\beta_g(h) = -\Delta_g(\mathrm{tr}_g h) + \delta_g\delta_g h - (h, \mathrm{Ric}_g),$$

*where $\delta_g$ is the formal adjoint of the covariant derivative of $g$ and $\mathrm{Ric}_g$ is the Ricci curvature of $g$.*

**Theorem 2.2** ([2]). *Let $g \in L^2_k\mathcal{M}(M)$, $2k > n+2$, with $R_g = -1$. Then $\beta_g$ is surjective.*

**Theorem 2.3.** *$\mathcal{M}_{-1}(M)$ is a smooth contractible ILH-submanifold of $\mathcal{M}(M)$ with tangent space $T_g\mathcal{M}_{-1}(M)$ at $g \in \mathcal{M}_{-1}(M)$ given as $\mathrm{Ker}\,\beta_g$ the kernel of the differential of the scalar curvature map.*

**3. Some results on Haken manifolds.** A compact connected orientable 3-manifold $M$ is said to be *irreducible* if every 2-sphere $S^2$ in $M$ bounds a 3-ball $B^3$.

Let $M$ be a compact connected orientable 3-manifold. Let $S$ be a compact connected orientable surface, and let $i : S \to M$ be an embedding of $S$ into $M$. Then $i$ induces a homomorphisms on the homotopy groups $i_* : \pi_k(S) \to \pi_k(M)$ for $k \geq 1$. The embedded surface $i(S)$ is *incompressible* if the induced homomorphism $i_*$ is injective on the fundamental group $\pi_1(S)$. A 3-manifold is *sufficiently large* if it contains an incompressible surface of genus greater than zero.

**Definition 3.1.** *A Haken manifold $M$ is an irreducible compact connected orientable sufficiently large 3-manifold.*

**Remark 3.2.** A connected manifold $M$ is called a *$K(\pi,1)$-manifold* if the fundamental group

$\pi_1(M)$ of $M$ is isomorphic to $\pi$, and the $k$-th homotopy group $\pi_k(M) = \{0\}$ for $k \geq 2$. A Haken manifold must be an irreducible $K(\pi, 1)$-manifold, and the fundamental group is infinite and not isomorphic to $\boldsymbol{Z}$. Moreover, it is known that a Haken manifold can not admit a positive scalar curvature metric ([6]). Therefore, by normalization of volume, the constant scalar curvature of a metric on a Haken manifold may be 0 or $-1$.

We denote by $L^2_k\mathrm{Diff}\,(M)$ the space of all $L^2_k$-diffeomorphisms. We know that the group $\mathrm{Diff}\,(M)$ of all diffeomorphisms of $M$ is an ILH-Lie group in the sence that $\mathrm{Diff}\,(M) = \lim_{\leftarrow} L^2_k\mathrm{Diff}\,(M)$ ([8]). Let $\mathrm{Diff}_0(M)$ denote the group of diffeomorphisms which are isotopic to the identity map.

Let $G$ be a group. We denote the group of automorphisms of $G$ by $\mathrm{Aut}\,(G)$. Let $\mathrm{Inn}\,(G)$ denote its normal subgroup of inner autmorphisms, and let $\mathrm{Out}\,(G) := \mathrm{Aut}\,(G)/\mathrm{Inn}\,(G)$ denote the quotient group of outer automorphisms. We denote the center of $G$ by $C(G)$.

**Theorem 3.3.** *Let $M$ be a Haken manifold with fundamental group $\pi_1(M) \cong G$. Then the homotopy type of the diffeomorphism group is given by the followings*:

$$\pi_0(\mathrm{Diff}\,(M)) \cong \mathrm{Diff}\,(M)/\mathrm{Diff}_0(M) \cong \mathrm{Out}\,(G),$$
$$\pi_1(\mathrm{Diff}\,(M)) \cong \pi_1(\mathrm{Diff}_0(M)) \cong C(G),$$
$$\pi_k(\mathrm{Diff}\,(M)) \cong \pi_k(\mathrm{Diff}_0(M)) = \{0\}$$

*for $k \geq 2$.*

**Remark 3.4.** A proof of Theorem 3.3 is due to the results of Hatcher (See [4], [5]). The important fact is that a Haken manifold can be reduced to a ball with the use of incompressible surfaces. Let $S$ be an incompressible surface in a Haken manifold $M$, consider the fibration $\mathrm{Diff}\,(M - S) \to \mathrm{Diff}\,(M) \to \mathrm{Emb}\,(S, M)$, where $\mathrm{Emb}\,(S, M)$ is the space of smooth embeddings of $S$ into $M$. If $\pi_k(\mathrm{Emb}\,(S, M)) = \{0\}$, then $\pi_k(\mathrm{Diff}\,(M)) \cong \pi_k(\mathrm{Diff}\,(M - S))$. Now from the assumption, $M$ can be reduced to a ball by cutting operations with the use of incompressible surfaces, hence for $k \geq 2$, $\pi_k(\mathrm{Diff}\,(M)) \cong \pi_k(\mathrm{Diff}\,(B^3)) \cong \{0\}$. In fact, we know that $\pi_k(\mathrm{Emb}\,(S, M)) \cong \pi_{k-1}(\mathrm{Diff}\,(S \times [0, 1])) \cong \{0\}$ in this case.

Let $\mathrm{Isom}\,(M, g)$ denote the isometry group of a Riemannian manifold $(M, g)$. For a connected $n$-manifold $M$, define the *degree* of $M$ by

$$\deg\,(M) := \max\{\dim \mathrm{Isom}\,(M, g) \mid g \in \mathcal{M}(M)\}$$

**Theorem 3.5.** *Let $M$ be a Haken manifold with* $\deg\,(M) = 0$. *Then* $\mathrm{Diff}_0(M)$ *is a contractible ILH-Lie group.*

**4. Contractibility of the moduli space on a Haken manifold.** For a metric $g \in \mathcal{M}(M)$, the Lie derivative gives us a mapping $\alpha_g : \mathcal{X}(M) \to C^\infty(M; S^2T^*M)$ defined as $\alpha_g(X) := \mathcal{L}_X g$, where $\mathcal{X}(M)$ denotes the space of vector fields on $M$. We use the Riemannian metric $g$ to identify the tangent bundle and cotangent bundle of $M$. The formal adjoint operator $\alpha_g^*$ of $\alpha_g$ is given by $\alpha_g^*(h) = 2\delta_g h$.

**Theorem 4.1** ([2])**.** *Let $M$ be a closed manifold, and $g \in \mathcal{M}(M)$ be a Riemannian metric on $M$ with scalar curvature $-1$. Then $\mathrm{Im}\,\alpha_g \subset \mathrm{Ker}\,\beta_g$, so we have the following splitting of the tangent space $T_g\mathcal{M}(M)$ at $g$*:

$$T_g\mathcal{M}(M) = \mathrm{Im}\,\beta_g^* \oplus \mathrm{Im}\,\alpha_g \oplus (\mathrm{Ker}\,\alpha_g^* \cap \mathrm{Ker}\,\beta_g).$$

**Proposition 4.2.** *Let $M$ be a closed Haken manifold with* $\deg\,(M) = 0$. *Then the action of* $\mathrm{Diff}_0(M)$ *on* $\mathcal{M}_{-1}(M)$ *is smooth, proper and free.*

**Theorem 4.3.** *Let $M$ be a closed connected Haken manifold with* $\deg\,(M) = 0$. *Then the moduli space $\mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$ is a smooth contractible ILH-manifold with tangent space $T_{[g]}(\mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M))$ at $[g] \in \mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$ isomorphic to the space* $\mathrm{Ker}\,\beta_g/\,\mathrm{Im}\,\alpha_g \cong \mathrm{Ker}\,\alpha_g^* \cap \mathrm{Ker}\,\beta_g$.

**Remark 4.4.** Let $M$ be a closed connected oriented surface. Denote by $\mathcal{C}(M)$ the set of all complex structures on $M$. The quotient space $\mathcal{T}(M) := \mathcal{C}(M)/\mathrm{Diff}_0(M)$ is called the *Teichmüller space.* There is a $\mathrm{Diff}_0(M)$-invariant diffeomorphism $\Psi : \mathcal{C}(M) \to \mathcal{M}_{-1}(M)$ and thus $\Psi$ induces a diffeomorphism of the moduli space $\mathcal{T}(M) \cong \mathcal{M}_{-1}(M)/\mathrm{Diff}_0(M)$. It is known that if the Euler number $\chi(M)$ of $M$ is negative, then $\mathcal{T}(M)$ is a cell of real dimension $-3\chi(M)$, and hence it is contractible. This diffeomorphism also becomes an isometry between the Weil-Petersson metric on $\mathcal{T}(M)$ and the $L^2$-metric on $\mathcal{M}_{-1}/\mathrm{Diff}_0(M)$. Formore detail, see [3], [10]. We will also discuss properties of the $L^2$-metric on the moduli space on a Haken manifold in forthcoming paper.

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