A Note on the Iwasawa λ-invariants of Real Quadratic Fields

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§1. Introduction For a number field k and a prime number p, denote respectively by $\lambda_p(k)$ and $\mu_p(k)$ the Iwasawa λ-invariant and the μ-invariant associated to the ideal class group of the cyclotomic $\boldsymbol{Z}_p$-extension over k. It is conjectured that $\lambda_p(k) = \mu_p(k) = 0$ for any totally real number field k and any p ([11, p. 316], [7]), which is often called Greenberg's conjecture. As for μ-invariants, we know that $\mu_p(k) = 0$ when k is an abelian field ([6]). The conjecture is still open even for real quadratic fields in spite of efforts of several authors (see Remark 2(2), Remark 3).

Let p be a fixed odd prime number and $k = \boldsymbol{Q}(\sqrt{d})$ a real quadratic field. Denote by χ the primitive Dirichlet character associated to k. Let $\lambda_p^*(k)$ be the λ-invariant of the power series associated to the p-adic L-function $L_p(s, \chi)$ (cf. [21, Thm. 7.10]). We have $\lambda_p(k) \leq \lambda_p^*(k)$ by the Iwasawa main conjecture (proved in [15]). So, $\lambda_p(k) = 0$ if $\lambda_p^*(k) = 0$. But, there are several examples with $\lambda_p^*(k) \geq 1$ (cf. [7, p. 266], [3]). Thus, it is natural to consider the following weak conjecture:

$$\lambda_p(k) \leq \max\{0, \lambda_p^*(k) - 1\}?$$

Let χ^* be the primitive Dirichlet character associated to $\omega\chi^{-1}$, where ω denotes the Teichmüller character $\boldsymbol{Z}/p\boldsymbol{Z} \to \boldsymbol{Z}_p$. When $\chi^*(p) = 1$, it is known that $\lambda_p^*(k) \geq 1$ and the weak conjecture is valid (see e.g. [10]).

The purpose of this note is to give some families of infinitely many real quadratic fields k with $\chi^*(p) \neq 1$ for which $\lambda_p^*(k) \geq 1$ and the weak conjecture is valid.

§2. Result/Remarks. Fix an odd prime number p and a square free natural number r with $\left(\frac{r}{p}\right) = -1$, where $\left(\frac{*}{p}\right)$ denotes the quadratic residue symbol. For each natural number m, we put

$$d_m^{(1)} = p^4 r^2 m^2 + r,\ d_m^{(2)} = p^4 m^2 + p.$$

Denote by $k_m^{(i)}$ the real quadratic field $\boldsymbol{Q}(\sqrt{d_m^{(i)}})$ $(i = 1,2)$. The prime p remains prime in $k_m^{(1)}$, and ramifies in $k_m^{(2)}$. Further, we have $\chi^*(p) \neq 1$ for these real quadratic fields. We prove the following

Proposition. *If $d_m^{(i)}$ is square free, then, $\lambda_p^*(k_m^{(i)}) \geq 1$ and the weak conjecture is valid for $k_m^{(i)}$ $(i = 1,2)$.*

Remark 1. Since the polynomial $p^4 r^2 X^2 + r$ (resp. $p^4 X^2 + p$) in X is irreducible in $\boldsymbol{Z}[X]$, there exist infinitely many m's for which $d_m^{(1)}$ (resp. $d_m^{(2)}$) is square free ([16], [17]).

Remark 2. (1) It is well-known that $\lambda_p(k) = 0$ for any quadratic field k such that $\left(\frac{k}{p}\right) \neq 1$ and $p \nmid h(k)$, $h(k)$ being the class number of k ([21, Thm. 10.4]). Let $p = 3$ and $r = 2$. Then, the family $\{k_m^{(1)}\}$ is "nontrivial" in the sense that we have several m satisfying the assumption of Proposition and $3 \mid h(k_m^{(1)})$, for example, $m = 1,3$. On the other hand, there are examples with $3 \nmid h(k_m^{(1)})$ such as $m = 2,4$. The family $\{k_m^{(1)}\}$ for $(p, r) = (5,2)$ and the family $\{k_m^{(2)}\}$ for $p = 3,5$ are also nontrivial. The author does not know, for $p \geq 7$, whether or not, the families given in Proposition are nontrivial. **(2)** It is proved that there exist infinitely many real quadratic fields k such that $\left(\frac{k}{3}\right) \neq 1$ and $3 \nmid h(k)$ ([18]). So, we have infinitely many real quadratic fields k with $\lambda_3(k) = 0$.

Remark 3. Several authors have given some criterions for the validity of Greenberg's conjecture or the weak conjecture (e.g. [4], [8], [9], [10], [12], [13], [14], [19], [20]). Using them, they have shown by some computation that $\lambda_3(k) = 0$ for many real quadratic fields k with "small" discriminants. The key lemma (Lemma 2) we use in the proof is one of the existing criterions.

§3. Proof of Proposition. Let k be a real quadratic field with a fundamental unit ε and χ the associated Dirichlet character. We need the following two lemmas.

Lemma 1. *If* $\varepsilon^{p^2-1} \equiv 1 \bmod (\xi_p - 1)^p$, *then* $\lambda_p^*(k) \geq 1$. *Here,* ξ_p *denotes a primitive* p*-th root of unity.*

Proof. Put $K = k(\mu_p)$ and $\Delta = \mathrm{Gal}(K/\boldsymbol{Q})$. Let K_∞/K be the cyclotomic $\boldsymbol{Z}_p$-extension with its n-th layer $K_n (n \geq 0)$. Denote by A_n the Sylow p-subgroup of the ideal class group of K_n and by $A_\infty = \varprojlim A_n$ the projective limit w.r.t. the relative norms. Let ψ be any $\boldsymbol{Q}_p$-valued character of Δ. For a module M over $\boldsymbol{Z}_p[\Delta]$ (e.g., $M = A_\infty, A_n$), we denote by $M(\psi)$ its ψ-component. We regard $A_\infty(\psi)$ as a module over $\Lambda = \boldsymbol{Z}_p[[T]]$ by letting $1 + T$ act as a (fixed) topological generator of $\mathrm{Gal}(K_\infty/K)$. Then, $A_\infty(\psi)$ is finitely generated and torsion over Λ by [11, Thm.5]. We regard χ and χ^* as $\boldsymbol{Q}_p$-valued characters of Δ. The λ-invariant $\lambda(A_\infty(\chi^*))$ of the torsion Λ module $A_\infty(\chi^*)$ equals to $\lambda_p^*(k)$ by the Iwasawa main conjecture (proved in [15]). On the other hand, $\lambda(A_\infty(\chi^*)) \geq 1$ if $A_0(\chi^*) \neq \{1\}$ since χ^* is an odd character (cf. [21, Cor. 13.29]). So, it suffices to show that $A_0(\chi^*) \neq \{1\}$. Let L/K be the maximal unramified abelian extension whose Galois group $G = \mathrm{Gal}(L/K)$ is of exponent p. Then, Δ acts on G in a natural way. By class field theory, we have a canonical isomorphism $G \simeq A_0/A_0^p$ compatible with the Δ-action. Let V be the subgroup of $K^\times/K^{\times p}$ such that

$$L = K(\alpha^{1/p} \mid [\alpha] \in V).$$

From the Kummer pairing

$$G \times V \longrightarrow \mu_p,$$

we obtain the following isomorphism (cf. [21, Chap. 10]):

$$((A_0/A_0^p)(\chi^*) \simeq)\ G(\chi^*) \simeq \mathrm{Hom}(V(\chi), \mu_p).$$

Since ε^{p^2-1} is congruent to 1 modulo $(\xi_p - 1)^p$, we see that the cyclic extension $K(\varepsilon^{1/p})/K$ of degree p is unramified (cf. [21, p. 183]). Hence, $([1] \neq) [\varepsilon] \in V(\chi)$. Therefore, we get $A_0(\chi^*) \neq \{1\}$ from the above isomorphism. □

Lemma 2 (cf. [19, §4]). *Assume that* $\left(\frac{k}{p}\right) \neq 1$ *and that* $\varepsilon^{p^2-1} \equiv 1 \bmod \mathfrak{p}^2$, *here* $\mathfrak{p}$ *denotes the prime ideal of* k *over* p. *Then, we have* $\lambda_p(k) \leq \max\{0, \lambda_p^*(k) - 1\}$.

Proof of Proposition. The real quadratic fields given in Proposition are of "Richaud-Degert types". We have a simple explicit formulas for a fundamental unit of a real quadratic field of such types (e.g. [1, Lemma 3]). Using it, (since $d_m^{(i)}$ is square free,) we see that

$$\varepsilon = (2p^4rm^2 + 1) + 2p^2m\sqrt{d_m^{(1)}}$$
$$(\text{resp. } \varepsilon = (2p^3m^2 + 1) + 2pm\sqrt{d_m^{(2)}})$$

is a fundamental unit of $k_m^{(1)}$ (resp. $k_m^{(2)}$). Now, our assertion follows from this and lemmas. □

Remark 4. In [2], a family of real quadratic fields for which a fundamental unit satisfies the assumptions of Lemmas 1 and 2 with $p = 3$ is given in connection with a normal integral basis problem.

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